Review: Linear differential equations.

Theorem (Variable coefficients)

Given continuous functions $a, b : (t_1, t_2) \rightarrow \mathbb{R}$, with $t_2 > t_1$, and given constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, the IVP

$$y' = -a(t) y + b(t), \quad y(t_0) = y_0,$$

has the unique solution $y : (t_1, t_2) \rightarrow \mathbb{R}$ given by

$$y(t) = \frac{1}{\mu(t)} \left[ y_0 + \int_{t_0}^{t} \mu(s) b(s) \, ds \right], \quad (1)$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \quad A(t) = \int_{t_0}^{t} a(s) \, ds.$$

Proof: Based on the integration factor method.
Review: Linear differential equations.

Remarks:
- The Theorem above assumes that the coefficients $a, b$, are continuous in $(t_1, t_2) \subset \mathbb{R}$.
- The Theorem above implies:
  - (a) There is an explicit expression for the solutions of a linear IVP, given in Eq. (1).
  - (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  - (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
- Non-linear differential equations.
- The Picard-Lindelöf Theorem.
- Properties of solutions to non-linear ODE.
- Direction Fields.
Non-linear differential equations.

Definition
An ordinary differential equation $y'(t) = f(t, y(t))$ is called non-linear iff the function $f$ is non-linear in the second argument.

Example
(a) The differential equation $y'(t) = \frac{t^2}{y^3(t)}$ is non-linear, since the function $f(t, u) = \frac{t^2}{u^3}$ is non-linear in the second argument.
(b) The differential equation $y'(t) = 2ty(t) + \ln(y(t))$ is non-linear, since the function $f(t, u) = 2tu + \ln(u)$ is non-linear in the second argument, due to the term $\ln(u)$.
(c) The differential equation $\frac{y'(t)}{y(t)} = 2t^2$ is linear, since the function $f(t, u) = 2t^2u$ is linear in the second argument.

On linear and non-linear equations. (Sect. 1.6).

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The Picard-Lindelöf Theorem.

Theorem (Picard-Lindelöf)

Consider the initial value problem

\[ y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \]

If \( f : S \to \mathbb{R} \) is continuous on the square

\[ S = [t_0 - a, t_0 + a] \times [y_0 - a, y_0 + a] \subset \mathbb{R}^2, \]

for some \( a > 0 \), and satisfies the Lipschitz condition that there exists \( k > 0 \) such that

\[ |f(t, y_2) - f(t, y_1)| < k|y_2 - y_1|, \]

for all \((t, y_2), (t, y_1) \in S\), then there exists a positive \( b < a \) such that there exists a unique solution \( y : [t_0 - b, t_0 + b] \to \mathbb{R} \) to the IVP above.

On linear and non-linear equations. (Sect. 1.6).

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Properties of solutions to non-linear ODE.

Recall: The non-linear initial value problem

\[ y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \]

has a unique solution in a region small enough near the initial data.

Remarks:

(i) There is no general explicit expression for the solution \( y(t) \) to a non-linear ODE.

(ii) Non-uniqueness of solution to the IVP above may happen at points \((t, u) \in \mathbb{R}^2\) where \( \partial_u f \) is not continuous.

(iii) Changing the initial data \( y_0 \) may change the domain on the variable \( t \) where the solution \( y(t) \) is defined.

Properties of solutions to non-linear ODE.

Example

Given non-zero constants \( a_1, a_2, a_3, a_4 \), find every solution \( y \) of

\[
y'(t) = \frac{t^2}{(y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1)}.
\]

Solution: The ODE is separable. So first, rewrite the equation as

\[
(y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1) y' = t^2,
\]

then we integrate in \( t \) on both sides of the equation,

\[
\int (y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1) \, y' \, dt = \int t^2 \, dt + C.
\]

Introduce the substitution \( u = y(t) \), so \( du = y'(t) \, dt \),

\[
\int (u^4 + a_4 u^3 + a_3 u^2 + a_2 u + a_1) \, du = \int t^2 \, dt + C.
\]
Properties of solutions to non-linear ODE.

Example
Given non-zero constants $a_1, a_2, a_3, a_4$, find every solution $y$ of

$$y' = \frac{t^2}{(y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1)}.$$  

Solution:
Recall: $\int (u^4 + a_4 u^3 + a_3 u^2 + a_2 u + a_1) \, du = \int t^2 \, dt + c$.  
Integrate, and in the result substitute back the function $y$:

$$\frac{1}{5} y^5(t) + \frac{a_4}{4} y^4(t) + \frac{a_3}{3} y^3(t) + \frac{a_2}{2} y^2(t) + a_1 y(t) = \frac{t^3}{3} + c.$$  

The solution is in implicit form. It is the root of a polynomial degree five. There is no formula for the roots of a general polynomial degree five or bigger.  

There is no explicit expression for solutions $y$ of the ODE.  

Properties of solutions to non-linear ODE.

Example
Find every solution $y$ of the initial value problem

$$y'(t) = y^{1/3}(t), \quad y(0) = 0.$$  

Remark: The equation above is non-linear, separable, and the function $f(t, u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3} \frac{1}{u^{2/3}},$$  

so $\partial_u f$ is not continuous at $u = 0$.  

The initial condition above is precisely where $f$ is not continuous.  

Solution: There are two solutions to the IVP above:  
The first solution is

$$y_1(t) = 0.$$  

The second solution is

$$y_2(t) = \left( \frac{3}{2} \right)^{1/3} \left( \frac{2}{3} t^3 + c \right)^{1/3}.$$  

where $c$ is a constant.
Properties of solutions to non-linear ODE.

Example
Find every solution $y$ of the initial value problem

$$y'(t) = y^{1/3}(t), \quad y(0) = 0.$$ 

Solution: The second solution is obtained as follows:

$$\int [y(t)]^{-1/3} y'(t) \, dt = \int dt + c.$$ 

Then, the substitution $u = y(t)$, with $du = y'(t) \, dt$, implies that

$$\int u^{-1/3} \, du = \int dt + c \quad \Rightarrow \quad \frac{3}{2} [y(t)]^{2/3} = t + c,$$

$$y(t) = \left[\frac{2}{3} (t + c)\right]^{3/2} \Rightarrow 0 = y(0) = \left(\frac{2}{3} c\right)^{3/2} \Rightarrow c = 0.$$ 

So, the second solution is: $y_2(t) = \left(\frac{2}{3} t\right)^{3/2}$. Recall $y_1(t) = 0$. <

Properties of solutions to non-linear ODE.

Example
Find the solution $y$ to the initial value problem

$$y'(t) = y^2(t), \quad y(0) = y_0.$$ 

Solution: This is a separable equation. So,

$$\int \frac{y'}{y^2} \, dt = \int dt + c \quad \Rightarrow \quad -\frac{1}{y} = t + c \quad \Rightarrow \quad y(t) = -\frac{1}{t + c}.$$ 

Using the initial condition in the expression above,

$$y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \quad \Rightarrow \quad y(t) = \frac{1}{\frac{1}{y_0} - t}.$$ 

This solution diverges at $t = 1/y_0$, so its domain is $\mathbb{R} - \{y_0\}$. 

The solution domain depends on the values of the initial data $y_0$. <
Properties of solutions to non-linear ODE.

Summary:

- **Linear ODE:**
  1. There is an explicit expression for the solution of a linear IVP.
  2. For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  3. The domain of the solution of a linear IVP is defined for every initial condition $y_0 \in \mathbb{R}$.

- **Non-linear ODE:**
  1. There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
  2. Non-uniqueness of solution to a non-linear IVP may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  3. Changing the initial data $y_0$ may change the domain on the variable $t$ where the solution $y(t)$ is defined.

On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
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- Properties of solutions to non-linear ODE.
- **Direction Fields.**
Direction Fields.

Remarks:
- One does not need to solve a differential equation \( y'(t) = f(t, y(t)) \) to have a qualitative idea of the solution.
- Recall that \( y'(t) \) represents the slope of the tangent line to the graph of function \( y \) at the point \( (t, y(t)) \).
- A differential equation provides these slopes, \( f(t, y(t)) \), for every point \( (t, y(t)) \).
- **Key idea:** Graph the function \( f(t, y) \) on the \( yt \)-plane, not as points, but as slopes of small segments.

Definition
A *Direction Field* for the differential equation \( y'(t) = f(t, y(t)) \) is the graph on the \( yt \)-plane of the values \( f(t, y) \) as slopes of a small segments.

Direction Fields.

Example
We know that the solution of \( y' = y \) are the exponentials \( y(t) = y_0 e^t \). The graph of these solution is simple. So is the direction field:
Direction Fields.

Example
The solution of \( y' = \sin(y) \) is simple to compute. The equation is separable. After some calculations the implicit solution are

\[
\ln \left| \csc(y_0) + \cot(y) \right| = t.
\]

for \( y_0 \in \mathbb{R} \). The graph of these solution is not simple to do. But the direction field is simple to plot:

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Direction Fields.

Example
The solution of \( y' = \frac{(1 + y^3)}{(1 + t^2)} \) could be hard to compute. But the direction field is simple to plot: