Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.

Radioactive decay

Remarks:
(a) Radioactive substances randomly emit protons, electors, radiation, and they are transformed in another substance.

(b) It can be seen that the time rate of change of the amount $N$ of a radioactive substances is proportional to the negative amount of radioactive substance.

$$N'(t) = -a \, N(t), \quad N(0) = N_0, \quad a > 0.$$ 

(c) The integrating factor method implies $N(t) = N_0 \, e^{-at}$.

(d) The half-life is the time $\tau$ needed to get $N(\tau) = N_0/2$.

$$N_0 \, e^{-a\tau} = \frac{N_0}{2} \quad \Rightarrow \quad -a\tau = \ln\left(\frac{1}{2}\right) \quad \Rightarrow \quad \tau = \frac{\ln(2)}{a}.$$ 

(e) Using the half-life, we get $N(t) = N_0 \, 2^{-t/\tau}$.
Radioactive decay

Example
Remains containing 14% of the original amount of Carbon-14 are found. Knowing that Carbon-14 half-live is $\tau = 5730$ years, date the remains.

Solution: Set $t = 0$ when the organism dies. Since the amount $N$ of Carbon-14 only decays after the organism dies,

$$N(t) = N_0 \times 2^{-t/\tau}, \quad \tau = 5730 \text{ years.}$$

The remains contain 14% of the original amount at the time $t$,

$$\frac{N(t)}{N_0} = \frac{14}{100} \implies 2^{-t/\tau} = \frac{14}{100}$$

$$-\frac{t}{\tau} = \log_2(14/100) \implies t = \tau \log_2(100/14).$$

The organism died 16,253 years ago.

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Salt in a water tank.

Problem: Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

Main ideas of the test:
- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- The salt in the tank also depends on the water rates coming in and going out of the tank.
- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

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The experimental device.

Definitions:
- \( r_i(t), r_o(t) \): Rates in and out of water entering and leaving the tank at the time \( t \).
- \( q_i(t), q_o(t) \): Salt concentration of the water entering and leaving the tank at the time \( t \).
- \( V(t) \): Water volume in the tank at the time \( t \).
- \( Q(t) \): Salt mass in the tank at the time \( t \).

Units:
\[
[r_i(t)] = [r_o(t)] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i(t)] = [q_o(t)] = \frac{\text{Mass}}{\text{Volume}}.
\]
\[
[V(t)] = \text{Volume}, \quad [Q(t)] = \text{Mass}.
\]
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**The main equations.**

Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

\[
\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)
\]

\[
\frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \quad (2)
\]

\[
q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \quad (3)
\]

\[
 r_i, \ r_o : \ \text{Constants.} \quad (4)
\]
The main equations.

Remarks:

\[ \frac{dV}{dt} = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o], \]

\[ \frac{dQ}{dt} = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o], \]

\[ [r_i q_i - r_o q_o] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}. \]

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Analysis of the mathematical model.

Eqs. (??) and (??) imply

\[ V(t) = (r_i - r_o) t + V_0, \]  
\[ (5) \]

where \( V(0) = V_0 \) is the initial volume of water in the tank.

Eqs. (??) and (??) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \]  
\[ (6) \]

Eqs. (??) and (??) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]  
\[ (7) \]

Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \)

Notation: \( a(t) = -\frac{r_o}{(r_i - r_o) t + V_0}, \) and \( b(t) = r_i q_i(t). \)

The main equation of the description is given by

\[ Q'(t) = a(t) Q(t) + b(t). \]

Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = e^{A(t)} \left[ Q_0 + \int_0^t e^{-A(s)} b(s) \, ds \right] \]

with \( Q(0) = Q_0 \), and \( A(t) = \int_0^t a(s) \, ds. \)
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Predictions for particular situations.

**Example**
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

**Solution:** Always holds \( Q'(t) = a(t) Q(t) + b(t) \).

In this case:

\[
a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0.
\]

\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = r q_i = b_0.
\]

We need to solve the IVP:

\[
Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
\]

\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right]. = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]

But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0 \), and \( a_0 = \frac{r}{V_0} \). We conclude:

\[
Q(t) = \left( Q_0 - q_i V_0 \right) e^{-rt/V_0} + q_i V_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_o/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous
Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[ Q(t) = Q_0 e^{-rt/V_0}. \]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,

\[ \frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \Rightarrow e^{-rt_1/V_0} = \frac{1}{100}. \]

Predictions for particular situations.

Example
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find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: Recall: \( e^{-rt_1/V_0} = \frac{1}{100} \). Then,

\[-\frac{r}{V_0} t_1 = \ln \left( \frac{1}{100} \right) = -\ln(100) \Rightarrow \frac{r}{V_0} t_1 = \ln(100).\]

We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \).

In this case: \( t_1 = 100 \ln(100) \). \( \triangleright \)
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) are constants. If \( r = 5 \times 10^6 \) gal/year, \( q_i(t) = 2 + \sin(2t) \) grams/gal, \( V_0 = 10^6 \) gal, \( Q_0 = 0 \), find \( Q(t) \).

Solution: Recall: \( Q'(t) = a(t) Q(t) + b(t) \). In this case:

\[
a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0,
\]

\[
b(t) = r_i q_i(t) \Rightarrow b(t) = r [2 + \sin(2t)].
\]

We need to solve the IVP: \( Q'(t) = -a_0 Q(t) + b(t) \), \( Q(0) = 0 \).

\[
e^{a_0 t} Q(t) = \int_0^t e^{a_0 s} b(s) \, ds.
\]

We conclude: \( Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] \, ds.\)