Separable differential equations (Sect. 1.3).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Euler homogeneous equations.

Separable ODE.

Definition
A separable differential equation on the function \( y \) has the form

\[
    h(y) y'(t) = g(t),
\]

where \( h, g \) are given scalar functions.

Remark:
A differential equation \( y'(t) = f(t, y(t)) \) is separable iff

\[
    y' = \frac{g(t)}{h(y)} \iff f(t, y) = \frac{g(t)}{h(y)}.
\]

Example

\[
    y'(t) = \frac{t^2}{1 - y^2(t)}, \quad y'(t) + y^2(t) \cos(2t) = 0.
\]
Separable ODE.

Example
Determine whether the differential equation below is separable,

\[ y'(t) = \frac{t^2}{1 - y^2(t)}. \]

Solution: The differential equation is separable, since it is equivalent to

\[ (1 - y^2) y' = t^2 \implies \begin{cases} g(t) = t^2, \\ h(y) = 1 - y^2. \end{cases} \]

Remark: The functions \( g \) and \( h \) are not uniquely defined. Another choice here is:

\[ g(t) = c t^2, \quad h(y) = c (1 - y^2), \quad c \in \mathbb{R}. \]

Separable ODE.

Example
Determine whether the differential equation below is separable,

\[ y'(t) + y^2(t) \cos(2t) = 0 \]

Solution: The differential equation is separable, since it is equivalent to

\[ \frac{1}{y^2} y' = -\cos(2t) \implies \begin{cases} g(t) = -\cos(2t), \\ h(y) = \frac{1}{y^2}. \end{cases} \]

Remark: The functions \( g \) and \( h \) are not uniquely defined. Another choice here is:

\[ g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}. \]
Remark: Not every first order ODE is separable.

Example
- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
- The linear differential equation $y'(t) = -\frac{2}{t} y(t) + 4t$ is not separable.
- The linear differential equation $y'(t) = -a(t) y(t) + b(t)$, with $b(t)$ non-constant, is not separable.
Solutions to separable ODE.

Theorem (Separable equations)
If the functions \( h, g \) are continuous, \( h \neq 0 \), then the separable ODE
\[
h(y) y' = g(t)
\]
has infinitely many solutions \( y \) satisfying the algebraic equation
\[
H(y(t)) = G(t) + c,
\]
where \( c \in \mathbb{R} \) is arbitrary, and where \( H \) and \( G \) are primitives (antiderivatives) of \( h \) and \( g \), respectively; that is,
\[
H'(u) = h(u), \quad G'(t) = g(t).
\]

Remark: Given functions \( g, h \), find their primitives \( G, H \).

Solutions to separable ODE.

Example
Find all solutions \( y \) to the equation \( y'(t) = \frac{t^2}{1 - y^2(t)} \).

Solution: The equation is equivalent to
\[
(1 - y^2) y'(t) = t^2 \quad \Rightarrow \quad g(t) = t^2, \quad h(y) = 1 - y^2.
\]
Integrate on both sides of the equation,
\[
\int [1 - y^2(t)] y'(t) \, dt = \int t^2 \, dt + c.
\]
The substitution \( u = y(t) \), \( du = y'(t) \, dt \), implies that
\[
\int (1 - u^2) \, du = \int t^2 \, dt + c \quad \Leftrightarrow \quad \left( u - \frac{u^3}{3} \right) = \frac{t^3}{3} + c.
\]
Example

Find all solutions $y$ to the equation $y'(t) = \frac{t^2}{1 - y^2(t)}$.

Solution: Recall: \[(u - \frac{u^3}{3}) = \frac{t^3}{3} + c.\]

Substitute the unknown function $y$ back in the equation above,
\[
\left( y - \frac{y^3}{3} \right) = \frac{t^3}{3} + c, \quad c \in \mathbb{R}.
\]

Remark: Recall the notation in the Theorem:

$g(t) = t^2 \Rightarrow G(t) = \frac{t^3}{3},$

$h(y) = 1 - y^2 \Rightarrow H(y) = y - \frac{y^3}{3}.$

Hence we recover the Theorem expression: $H(y(t)) = G(t) + c$.

Remarks:

- The equation $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is algebraic in $y$, since there is no $y'$ in the equation.
- Every function $y$ satisfying the algebraic equation

\[
y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c;
\]

is a solution of the differential equation above.
- We now verify the previous statement: Differentiate on both sides with respect to $t$, that is,

\[
y'(t) - 3 \left( \frac{y^2(t)}{3} \right) y'(t) = 3 \frac{t^2}{3} \Rightarrow (1 - y^2) y' = t^2.
\]
Example
Find all solutions $y$ to the equation $y'(t) + y^2(t) \cos(2t) = 0$.

Solution: The differential equation is separable,

$$\frac{y'(t)}{y^2(t)} = -\cos(2t) \implies g(t) = -\cos(2t), \quad h(y) = \frac{1}{y^2}.$$

Integrate on both sides of the equation,

$$\int \frac{y'(t)}{y^2(t)} \, dt = - \int \cos(2t) \, dt + c.$$

The substitution $u = y(t)$, $du = y'(t) \, dt$, implies that

$$\int \frac{du}{u^2} = - \int \cos(2t) \, dt + c \iff -\frac{1}{u} = -\frac{1}{2} \sin(2t) + c.$$

Remark: Recall the notation in the Theorem:

$$g(t) = -\cos(2t) \implies G(t) = -\frac{1}{2} \sin(2t).$$

$$h(y) = \frac{1}{y^2} \implies H(y) = -\frac{1}{y}.$$

Hence we recover the Theorem expression: $H(y(t)) = G(t) + c$. 
Separable differential equations (Sect. 1.3).

- Separable ODE.
- Solutions to separable ODE.
- **Explicit and implicit solutions.**
- Euler homogeneous equations.

**Explicit and implicit solutions.**

**Definition**
Assume the notation in the Theorem above. The solution $y$ of a separable ODE is given in **implicit form** iff function $y$ is given by

$$H(y(t)) = G(t) + c,$$

The solution is given in **explicit form** iff function $H$ is invertible and

$$y(t) = H^{-1}(G(t) + c).$$

**Example**

(a) $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is in implicit form.

(b) $-\frac{1}{y(t)} = -\frac{1}{2} \sin(2t) + c$ is in implicit form.

(c) $y(t) = \frac{2}{\sin(2t) - 2c}$ is in explicit form.
Separable differential equations (Sect. 1.3).

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Euler homogeneous equations.

Definition
The first order ODE \( y'(t) = f(t, y(t)) \) is called Euler homogeneous iff for every numbers \( c, t, u \in \mathbb{R} \) the function \( f \) satisfies
\[
f(ct, cu) = f(t, u).
\]

Remark:
- The function \( f \) is invariant under the change of scale of its arguments.
- If \( f(t, u) \) has the property above, it must depend only on \( u/t \).
- Therefore, there exists \( F : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(t, u) = F\left(\frac{u}{t}\right) \).
- So, a first order ODE is Euler homogeneous iff it has the form
\[
y'(t) = F\left(\frac{y(t)}{t}\right).
\]
Euler homogeneous equations.

Example
Show that the equation below is Euler homogeneous,

\[(t - y) y' - 2y + 3t + \frac{y^2}{t} = 0.\]

Solution: Rewrite the equation in the standard form

\[(t - y) y' = 2y - 3t - \frac{y^2}{t} \implies y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{(t - y)}.\]

Divide numerator and denominator by \(t\). We get,

\[y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{(t - y)} \cdot \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)} \implies y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{\left[1 - \left(\frac{y}{t}\right)\right]}.\]

We conclude that the ODE is Euler homogeneous, because the right-hand side of the equation above depends only on \(y/t\).

Indeed, in our case:

\[f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y}, \quad F(x) = \frac{2x - 3 - x^2}{1 - x},\]

and \(f(t, y) = F(y/t)\). \(\triangleq\)
Example
Determine whether the equation below is Euler homogeneous,

\[ y' = \frac{t^2}{1 - y^3}. \]

Solution:
Divide numerator and denominator by \( t^3 \), we obtain

\[
y' = \frac{t^2}{(1 - y^3)} \left( \frac{1}{t^3} \right) \Rightarrow y' = \frac{\left( \frac{1}{t} \right)}{\left( \frac{1}{t^3} \right) - \left( \frac{y}{t} \right)^3}.
\]

Then, the differential equation is not Euler homogeneous.

\[ \square \]

Euler homogeneous equations.

Theorem
If the equation \( y'(t) = f(t, y(t)) \) is Euler homogeneous, then the differential equation for the unknown \( v(t) = \frac{y(t)}{t} \) is separable.

Remark: Euler homogeneous equations can be transformed into separable equations.

Proof: If \( y' = f(t, y) \) is Euler homogeneous, then it can be written as \( y' = F(y/t) \) for some function \( F \). Introducing \( v = y/t \),

\[
y(t) = t \ v(t) \quad \Rightarrow \quad y'(t) = v(t) + t \ v'(t).
\]

Introduce all these changes into the ODE, then

\[
v + t \ v' = F(v) \quad \Rightarrow \quad v' = \frac{(F(v) - v)}{t}.
\]

This last equation is separable.

\[ \square \]
Euler homogeneous equations.

Example
Find all solutions $y$ of the equation $y' = \frac{t^2 + 3y^2}{2ty}$.

Solution: The equation is Euler homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \cdot \frac{1}{t^2} \Rightarrow y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$ 

Therefore, we introduce the change of unknown $v = y/t$, so $y = tv$ and $y' = v + tv'$. Hence

$$v + tv' = \frac{1 + 3v^2}{2v} \Rightarrow tv' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

We obtain the separable equation $v' = \frac{1}{t} \left(\frac{1 + v^2}{2v}\right)$. 

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Euler homogeneous equations.

Example
Find all solutions $y$ of the equation $y' = \frac{t^2 + 3y^2}{2ty}$.

Solution: Recall: $v' = \frac{1}{t} \left(\frac{1 + v^2}{2v}\right)$. We rewrite and integrate it,

$$\frac{2v}{1 + v^2} v' = \frac{1}{t} \Rightarrow \int \frac{2v}{1 + v^2} v' \, dt = \int \frac{1}{t} \, dt + c_0.$$ 

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\, v'(t) \, dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \Rightarrow \ln(u) = \ln(t) + c_0 \Rightarrow u = e^{\ln(t) + c_0}.$$ 

But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \Rightarrow 1 + \left(\frac{y}{t}\right)^2 = c_1 t \Rightarrow y(t) = \pm t\sqrt{c_1 t - 1}.$$