Review: Linear constant coefficient equations

**Definition**
Given functions $a$, $b$, a *first order, linear ODE* in the unknown function $y$ is the equation

$$y'(t) = a(t) y(t) + b(t).$$

**Theorem (Constant coefficients)**

*Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

$$y'(t) = a y(t) + b$$

has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

$$y(t) = c e^{at} - \frac{b}{a}.$"
Review: Linear constant coefficient equations

Example
Find all functions $y$ solution of the ODE $y' = 2y + 3$.

Solution:
We concluded that the ODE has infinitely many solutions, given by
$$y(t) = ce^{2t} - \frac{3}{2}, \quad c \in \mathbb{R}.$$ 

Since we did one integration, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.

Verification: $y' = 2ce^{2t}$, but we know that $2ce^{2t} = 2y + 3$, therefore we conclude that $y$ satisfies the ODE $y' = 2y + 3$.  \[\Box\]

Linear Variable coefficient equations (Sect. 1.2)

- Review: Linear constant coefficient equations.
- **The Initial Value Problem**.
- Linear variable coefficients equations.
- The Bernoulli equation: A nonlinear equation.
The Initial Value Problem

**Definition**
The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions $a, b$ and constants $t_0, y_0$, find a function $y$ solving

$$y' = a(t) y + b(t), \quad y(t_0) = y_0.$$ 

**Remark:** The initial condition selects one solution of the ODE.

**Theorem (Constant coefficients)**
Given constants $a, b, t_0, y_0$, with $a \neq 0$, the initial value problem

$$y' = ay + b, \quad y(t_0) = y_0$$

has the unique solution

$$y(t) = \left(y_0 + \frac{b}{a}\right)e^{a(t-t_0)} - \frac{b}{a}.$$

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The Initial Value Problem

**Example**
Find the solution to the initial value problem

$$y' = 2y + 3, \quad y(0) = 1.$$ 

**Solution:** Every solution of the ODE above is given by

$$y(t) = c e^{2t} - \frac{3}{2}, \quad c \in \mathbb{R}.$$ 

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$  

We conclude that $y(t) = \frac{5}{2} e^{2t} - \frac{3}{2}$. \ }

Linear Variable coefficient equations (Sect. 1.2)

- Review: Linear constant coefficient equations.
- The Initial Value Problem.
- Linear variable coefficients equations.
- The Bernoulli equation: A nonlinear equation.

Theorem (Variable coefficients)

Given continuous functions $a, b$ and constants $t_0, y_0$, all solutions to the equation

$$y' = a(t)y + b(t)$$

are given by

$$y(t) = c e^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t) dt,$$

where we have introduced the function $A(t) = \int a(t) dt$.

Remarks:
(a) The function $\mu(t) = e^{-A(t)}$ is called an integrating factor.
(b) See the proof in the Lecture Notes.
Linear variable coefficients equations

Example
Find all solutions $y$ to the equation

$$ty' = -2y + 4t^2,$$

**Solution:** We first express the equation as in the Theorem,

$$y' = -\frac{2}{t}y + 4t \quad \Rightarrow \quad y' + \frac{2}{t}y = 4.$$

$$e^{f(t)}y' + \frac{2}{t}e^{f(t)}y = 4e^{f(t)}, \quad f'(t) = \frac{2}{t}.$$

This function $\mu = e^{f(t)}$ is an integrating factor.

$$f(t) = \int \frac{2}{t} dt = 2 \ln(t) = \ln(t^2)$$

Therefore, $\mu(t) = e^{f(t)} = t^2$.

Linear variable coefficients equations

Example
Find the solution $y$ to the IVP

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

**Solution:** The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2\left(y' + \frac{2}{t}y\right) = t^2(4t) \quad \Leftrightarrow \quad t^2y' + 2ty = 4t^3$$

$$(t^2y)' = 4t^3 \quad \Leftrightarrow \quad t^2y = t^4 + c \quad \Leftrightarrow \quad y = t^2 + \frac{c}{t^2}.$$  

We conclude $y(t) = t^2 + \frac{c}{t^2}$ for any $c \in \mathbb{R}$.  \(\triangleright\)
Linear variable coefficients equations

**Theorem (Variable coefficients)**

*Given continuous functions* $a, b : \mathbb{R} \to \mathbb{R}$ *and given constants* $t_0, y_0 \in \mathbb{R}$, *the IVP*

$$y' = a(t)y + b(t) \quad y(t_0) = y_0$$

*has the unique solution*

$$y(t) = e^{A(t)} \left( y_0 + \int_{t_0}^{t} e^{-A(s)} b(s) ds \right),$$

*where we have introduced the function* $A(t) = \int_{t_0}^{t} a(s) ds$.

**Remarks:**

(a) The function $\mu(t) = e^{-A(t)}$ is called an *integrating factor*.

(b) See the proof in the Lecture Notes.

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Linear variable coefficients equations

**Example**

Find the solution $y$ to the IVP

$$t \, y' + 2y = 4t^2, \quad y(1) = 2.$$

**Solution:** We already know that all solutions to the differential equation are

$$y = t^2 + \frac{c}{t^2}.$$

The initial condition implies

$$2 = y(1) = 1 + c, \quad \Rightarrow \quad c = 1.$$

We conclude that $y(t) = t^2 + \frac{1}{t^2}$. ◯
The Bernoulli equation

Remark: The Bernoulli equation is a non-linear differential equation that can be transformed into a linear differential equation.

Definition
Given functions $p, q : \mathbb{R} \to \mathbb{R}$ and a real number $n$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y' + p(t) y = q(t) y^n$$

is called the Bernoulli equation.

Theorem
The function $y : \mathbb{R} \to \mathbb{R}$ is a solution of the Bernoulli equation for

$$y' + p(t) y = q(t) y^n, \quad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' - (n - 1)p(t) v = -(n - 1)q(t).$$
The Bernoulli equation

Example
Given arbitrary constants $a \neq 0$ and $b$, find every solution of the differential equation

$$y' = ay + by^3.$$ 

Solution: This is a Bernoulli equation. Divide the equation by $y^3$,

$$\frac{y'}{y^3} = \frac{a}{y^2} + b.$$

Introduce the function $v = 1/y^2$, with derivative $v' = -2\left(\frac{y'}{y^3}\right)$, into the differential equation above,

$$-\frac{v'}{2} = a v + b \quad \Rightarrow \quad v' = -2a v - 2b \quad \Rightarrow \quad v' + 2a v = -2b.$$

The Bernoulli equation

Example
Given arbitrary constants $a \neq 0$ and $b$, find every solution of the differential equation

$$y' = ay + by^3.$$ 

Solution: Recall: $v' + 2av = -2b$.

The last equation is a linear differential equation for $v$. This equation can be solved using the integrating factor method.

Multiply the equation by $\mu(t) = e^{2at}$,

$$(e^{2at}v)' = -2b e^{2at} \quad \Rightarrow \quad e^{2at}v = -\frac{b}{a} e^{2at} + c$$

We obtain that $v = c e^{-2at} - \frac{b}{a}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c e^{-2at} - \frac{b}{a} \quad \Rightarrow \quad y(t) = \pm \frac{1}{\left(c e^{-2at} - \frac{b}{a}\right)^{1/2}}.$$