Linear Variable coefficient equations (Sect. 1.2)

- Review: Linear constant coefficient equations.
- The Initial Value Problem.
- Linear variable coefficients equations.
- The Bernoulli equation: A nonlinear equation.

Review: Linear constant coefficient equations

Definition

Given functions a, b, a *first order, linear ODE* in the unknown function y is the equation

$$y'(t) = a(t)\, y(t) + b(t).$$

Theorem (Constant coefficients)

Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

$$y'(t) = a\, y(t) + b$$

has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

$$y(t) = c\, e^{at} - \frac{b}{a}.$$

Review: Linear constant coefficient equations

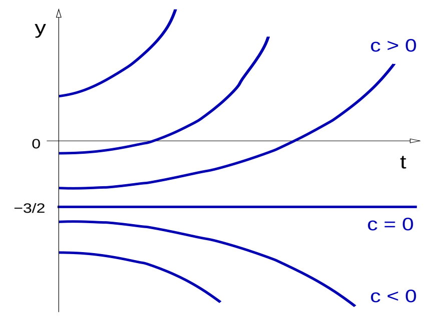
Example

Find all functions y solution of the ODE $y' = 2y + 3$.

Solution:

We concluded that the ODE has infinitely many solutions, given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$



Since we did one integration, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.

Verification: $y' = 2c\, e^{2t}$, but we know that $2c\, e^{2t} = 2y + 3$, therefore we conclude that y satisfies the ODE $y' = 2y + 3$. $\lhd$

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The Initial Value Problem

Definition

The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions a, b and constants t_0, y_0, find a function y solving

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0.$$

Remark: The initial condition selects one solution of the ODE.

Theorem (Constant coefficients)

Given constants a, b, t_0, y_0, with $a \neq 0$, the initial value problem

$$y' = a\, y + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \Big(y_0 + \frac{b}{a}\Big) e^{a(t-t_0)} - \frac{b}{a}.$$

The Initial Value Problem

Example

Find the solution to the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1.$$

Solution: Every solution of the ODE above is given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$

We conclude that $y(t) = \dfrac{5}{2}\, e^{2t} - \dfrac{3}{2}$. $\lhd$

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Linear variable coefficients equations

Theorem (Variable coefficients)

Given continuous functions a, b and constants t_0, y_0, all solutions to the equation

$$y' = a(t)y + b(t)$$

are given by

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)dt,$$

where we have introduced the function $A(t) = \int a(t)dt$.

Remarks:

(a) The function $\mu(t) = e^{-A(t)}$ is called an integrating factor.

(b) See the proof in the Lecture Notes.

Linear variable coefficients equations

Example

Find all solutions y to the equation

$$t\,y' = -2y + 4t^2,$$

Solution: We first express the equation as in the Theorem,

$$y' = -\frac{2}{t}\,y + 4t \quad \Rightarrow \quad y' + \frac{2}{t}\,y = 4.$$

$$e^{f(t)}\,y' + \frac{2}{t}\,e^{f(t)}\,y = 4t\,e^{f(t)}, \quad f'(t) = \frac{2}{t}.$$

This function $\mu = e^{f(t)}$ is an integrating factor.

$$f(t) = \int \frac{2}{t}\,dt = 2\,\ln(t) = \ln\bigl(t^2\bigr)$$

Therefore, $\mu(t) = e^{f(t)} = t^2$.

Linear variable coefficients equations

Example

Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2\Bigl(y' + \frac{2}{t}\,y\Bigr) = t^2(4t) \quad \Leftrightarrow \quad t^2\,y' + 2t\,y = 4t^3$$

$$\bigl(t^2 y\bigr)' = 4t^3 \quad \Leftrightarrow \quad t^2 y = t^4 + c \quad \Leftrightarrow \quad y = t^2 + \frac{c}{t^2}.$$

We conclude $y(t) = t^2 + \dfrac{c}{t^2}$ for any $c \in \mathbb{R}$. $\triangleleft$

Linear variable coefficients equations

Theorem (Variable coefficients)

Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ and given constants $t_0, y_0 \in \mathbb{R}$, the IVP

$$y' = a(t)y + b(t) \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = e^{A(t)} \left(y_0 + \int_{t_0}^{t} e^{-A(s)}\, b(s)ds \right),$$

where we have introduced the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)ds$.

Remarks:

(a) The function $\mu(t) = e^{-A(t)}$ is called an integrating factor.

(b) See the proof in the Lecture Notes.

Linear variable coefficients equations

Example

Find the solution y to the IVP

$$t\, y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: We already know that all solutions to the differential equation are

$$y = t^2 + \frac{c}{t^2}.$$

The initial condition implies

$$2 = y(1) = 1 + c, \quad \Rightarrow \quad c = 1.$$

We conclude that $y(t) = t^2 + \dfrac{1}{t^2}$. $\triangleleft$

Linear Variable coefficient equations (Sect. 1.2)

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The Bernoulli equation

Remark: The Bernoulli equation is a non-linear differential equation that can be transformed into a linear differential equation.

Definition
Given functions $p, q : \mathbb{R} \to \mathbb{R}$ and a real number n, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y' + p(t)\, y = q(t)\, y^n$$

is called the *Bernoulli equation*.

Theorem
The function $y : \mathbb{R} \to \mathbb{R}$ is a solution of the Bernoulli equation for

$$y' + p(t)\, y = q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' - (n-1)p(t)\, v = -(n-1)q(t).$$

The Bernoulli equation

Example

Given arbitrary constants $a \neq 0$ and b, find every solution of the differential equation

$$y' = a\,y + b\,y^3.$$

Solution: This is a Bernoulli equation. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a}{y^2} + b.$$

Introduce the function $v = 1/y^2$, with derivative $v' = -2\Big(\frac{y'}{y^3}\Big)$, into the differential equation above,

$$-\frac{v'}{2} = a\,v + b \quad \Rightarrow \quad v' = -2a\,v - 2b \quad \Rightarrow \quad v' + 2a\,v = -2b.$$

The Bernoulli equation

Example

Given arbitrary constants $a \neq 0$ and b, find every solution of the differential equation

$$y' = a\,y + b\,y^3.$$

Solution: Recall: $v' + 2a\,v = -2b$.

The last equation is a linear differential equation for v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{2at}$,

$$\big(e^{2at}v\big)' = -2b\,e^{2at} \quad \Rightarrow \quad e^{2at}v = -\frac{b}{a}\,e^{2at} + c$$

We obtain that $v = c\,e^{-2at} - \dfrac{b}{a}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c\,e^{-2at} - \frac{b}{a} \quad \Rightarrow \quad y(t) = \pm\frac{1}{\big(c\,e^{-2at} - \frac{b}{a}\big)^{1/2}}. \qquad \lhd$$