Name: Key, Please Verify. ID Number: ____________

MTH 415
Exam 2
August 3, 2012
50 minutes
Sects: 4.1-4.4,
5.1-5.5, 6.1, 6.2.

One notes page, handwritten. No Calculator.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded. If you say something wrong and it is not crossed over, we take points off.
Show all your work. *Box your answers.*

1. (20 points) Determine whether the subsets W_1 and W_2 and are subspaces of the vector space of polynomials $\mathbb{P}_2$. If your answer is "yes", then find a basis for the subspace.

(a) $W_1 = \left\{ \boldsymbol{p} \in \mathbb{P}_2 \ : \ \int_0^1 \boldsymbol{p}(x)\, dx \geqslant 1 \right\}$;

(b) $W_2 = \left\{ \boldsymbol{p} \in \mathbb{P}_2 \ : \ \int_0^1 \boldsymbol{p}(x)\, dx = 0 \right\}$.

a) $\boxed{W_1 \text{ is Not a Subspace.}}$

If $\underline{p} \in W_1$, then $-\underline{p} \notin W_1$, since $\int_0^1 -\underline{p}(x)\, dx \leq -1$.

b) $\boxed{W_2 \text{ is a Subspace.}}$

Let $\underline{p}, \underline{q} \in W_2 \Rightarrow \int_0^1 \underline{p}(x)\, dx = 0, \quad \int_0^1 \underline{q}(x)\, dx = 0.$

Then for all $a, b \in \mathbb{F}$ holds

$$\int_0^1 (a\, p(x) + b\, q(x))\, dx = a \int_0^1 p(x)\, dx + b \int_0^1 q(x)\, dx = 0 + 0.$$

<u>basis of W_2</u>

$$\underline{p}(x) = a_0 + a_1 x + a_2 x^2 \in W_2 \iff 0 = \int_0^1 (a_0 + a_1 x + a_2 x^2)\, dx$$

$$0 = a_0\, x\Big|_0^1 + \frac{a_1}{2} x^2\Big|_0^1 + \frac{a_2}{3} x^3\Big|_0^1 \Rightarrow 0 = a_0 + \frac{a_1}{2} + \frac{a_2}{3} \Rightarrow$$

$$\Rightarrow a_0 = -\frac{a_1}{2} - \frac{a_2}{3} \Rightarrow p = -\frac{a_1}{2} - \frac{a_2}{3} + a_1 x + a_2 x^2 = a_1\left(-\frac{1}{2} + x\right) + a_2\left(-\frac{1}{3} + x^2\right)$$

$$\boxed{\text{basis of } W_2 : \left\{ \left(-\frac{1}{2} + x\right), \left(-\frac{1}{3} + x^2\right) \right\}}$$

2. (20 points) Consider the matrix $\mathsf{A} = \begin{bmatrix} 1 & 3 & 5 & 8 \\ 2 & 2 & 6 & 4 \\ 3 & 1 & 7 & 0 \end{bmatrix}$.

(a) Verify that the vector $\mathsf{v} = \begin{bmatrix} -2 \\ -8 \\ 2 \\ 2 \end{bmatrix}$ belongs to the null space of A.

(b) Extend the set $\{\mathsf{v}\}$ to a basis of the null space of A.

a) $$A\underline{v} = \begin{bmatrix} 1 & 3 & 5 & 8 \\ 2 & 2 & 6 & 4 \\ 3 & 1 & 7 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 - 24 + 10 + 16 \\ -4 - 16 + 12 + 8 \\ -6 - 8 + 14 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) First, find a basis of $N(A)$.

$$\begin{bmatrix} 1 & 3 & 5 & 8 \\ 2 & 2 & 6 & 4 \\ 3 & 1 & 7 & 0 \end{bmatrix} \begin{matrix} \cdot 2) \; -3) \\ \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 3 & 5 & 8 \\ 0 & -4 & -4 & -12 \\ 0 & -8 & -8 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} -2x_3 + x_4 \\ -x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_4.$$

$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ basis of $N(A)$.

$$\therefore \boxed{\left\{ \begin{bmatrix} -2 \\ -8 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}} \quad \text{or} \quad \boxed{\left\{ \begin{bmatrix} -2 \\ -8 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

3. (20 points) Consider the operator $(\boldsymbol{I} - \boldsymbol{D}^2) : \mathbb{P}_3 \to \mathbb{P}_3$, where $\boldsymbol{D} : \mathbb{P}_3 \to \mathbb{P}_3$ is the differentiation operator, $\boldsymbol{D}(\boldsymbol{p}) = \boldsymbol{p}'$, and $\boldsymbol{I} : \mathbb{P}_3 \to \mathbb{P}_3$ is the identity operator. Denote by $\mathcal{S} = (1,\ x,\ x^2,\ x^3)$ the standard ordered basis of $\mathbb{P}_3$.

(a) Find the matrices $[\boldsymbol{D}]_{ss}$ and $[\boldsymbol{I} - \boldsymbol{D}^2]_{ss}$.

(b) Determine whether the operator $(\boldsymbol{I} - \boldsymbol{D}^2)$ is invertible or not. If your answer is "yes", find the inverse operator. Justify your answers.

a) $[D]_{ss} = \left[\, [D(1)]_s \,,\, [D(x)]_s \,,\, [D(x^2)]_s \,,\, [D(x^3)]_s \,\right]$

$$\boxed{[D]_{ss} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \Rightarrow [D^2]_{ss} = [D]_{ss}^2 \Rightarrow [D^2]_{ss} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[I - D^2]_{ss} = I_4 - [D^2]_{ss} \Rightarrow \boxed{[I - D^2]_{ss} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

b)

$$\det(I - D^2) = 1^4 = 1 \neq 0 \Rightarrow \boxed{I - D^2 \text{ is invertible}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{[I - D^2]^{-1} = I + D^2}$$

<u>check</u>: $D^4 = 0$, therefore $(I + D^2)(I - D^2) = I + D^2 - D^2 - D^4 = I$

4. (20 points) Consider the ordered bases of $\mathbb{R}^2$, $\mathcal{B} = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $\mathcal{C} = (\boldsymbol{c}_1, \boldsymbol{c}_2)$ which are relate by the equations

$$\boldsymbol{c}_1 = 3\boldsymbol{b}_1 + 2\boldsymbol{b}_2, \quad \boldsymbol{c}_2 = 2\boldsymbol{b}_1 + 3\boldsymbol{b}_2.$$

(a) Find both $[\boldsymbol{x}]_b$ and $[\boldsymbol{x}]_c$ for the vector $\boldsymbol{x} = \boldsymbol{b}_1 - 2\boldsymbol{b}_2$.

(b) Find $[\boldsymbol{T}]_{bb}$ for the matrix $[\boldsymbol{T}]_{cc} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

a) $\underline{x} = \underline{b}_1 - 2\underline{b}_2 \Rightarrow \boxed{[x]_b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

$[x]_c = P^{-1}[x]_b \quad ; \quad P = [I]_{cb} = \left[[\underline{c}_1]_b , [\underline{c}_2]_b \right] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$P^{-1} = \frac{1}{(9-4)} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \Rightarrow [\underline{x}]_c = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 7 \\ -8 \end{bmatrix}$

$$\boxed{[x]_c = \frac{1}{5} \begin{bmatrix} 7 \\ -8 \end{bmatrix}}$$

b) $[T]_{bb} = \tilde{Q}^{-1} [T]_{cc} \tilde{P}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{array}{ccc} c & \quad & c \to [T]_{cc} \\ \tilde{P} = [I]_{bc} \downarrow & & \downarrow \tilde{Q} = \tilde{P} \\ b & & b \to [T]_{bb} \end{array}$$

$\tilde{P} = [I]_{bc} = P^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \quad ; \quad \tilde{P}^{-1} = P = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$[T]_{bb} = \tilde{P}^{-1} [T]_{cc} \tilde{P} = P [T]_{cc} P^{-1}$

$[T]_{bb} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

$$\boxed{[T]_{bb} = \frac{1}{5} \begin{bmatrix} 17 & -13 \\ 13 & -7 \end{bmatrix}}$$

5. (20 points) Determine whether the function $\langle\,,\rangle_M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defines an inner product in the vector space $\mathbb{R}^2$, where $\langle \mathsf{x}, \mathsf{y}\rangle_M = \mathsf{x}^T \mathsf{M} \mathsf{y}$, with $\mathsf{M} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Justify your answer.

Symmetry and linearity should be simple to prove.
We leave that for later, if needed.
Positivity may fail, so we start with the positivity part.

Show that $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \geq 0$ for all $\underset{\sim}{x} \in \mathbb{R}^2$

and $= 0$ iff $\underset{\sim}{x} = \underset{\sim}{0}$.

Now: $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = [x_1, x_2] \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, x_2] \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$

$$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = (x_1)^2 + 4x_1 x_2 + 4(x_2)^2$$

Consider $\underset{\sim}{x} = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} \Rightarrow \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = (-1)^2 + 4(-1)\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)^2$

$= 1 - 2 + 1$

$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 0$ and $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\boxed{\text{Positivity Fails.}}$

$\boxed{\text{Not an inner Product.}}$

#	Pts	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Σ	100	