Name: Key. Please, Verify. ID Number: ____________

MTH 415
Exam 1
July 20, 2012
50 minutes
Sects: 1.2, 1.4, 2.1-2.6
3.1, 3.2.

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded. If you say something wrong and it is not crossed over, we take points off.
Show all your work. $\boxed{\textit{Box your answers.}}$

1. (14 points) Find every value of k such that the system $\begin{aligned} kx_1 + x_2 &= 1 \\ x_1 + kx_2 &= 1. \end{aligned}$ satisfies that:

(a) It has no solution.

(b) It has only one solution (also find that solution x_1, x_2).

(c) It has infinitely many solutions (also find all those solutions x_1, x_2).

$$\left[\begin{array}{cc|c} k & 1 & 1 \\ 1 & k & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right] = \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & (1-k)(1+k) & (1-k) \end{array}\right]$$

a) $\boxed{k=-1}$ No Solution, since $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$

b) $\boxed{k \neq 1,\ k \neq -1}$ $\Rightarrow \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & (1+k) & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1}{(1+k)} \end{array}\right]$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 - \frac{k}{(1+k)} \\ 0 & 1 & \frac{1}{(1+k)} \end{array}\right] \Rightarrow x_1 = \frac{1+k-k}{1+k} \Rightarrow \boxed{x_1 = \frac{1}{(1+k)}}$$

$$\boxed{x_2 = \frac{1}{(1+k)}}$$

c) $\boxed{k=1}$ $\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{x_1 = 1 - x_2}$ $\boxed{x_2 : \text{free.}}$ $\boxed{\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2}$

2. (20 points)

(a) Find every solution vector x of the linear system $\mathsf{Ax} = \mathsf{b}$, where

$$\mathsf{A} = \begin{bmatrix} 1 & 3 & -9 \\ 2 & -2 & 6 \\ 3 & 1 & -3 \end{bmatrix}, \qquad \mathsf{b} = \begin{bmatrix} 12 \\ -8 \\ 4 \end{bmatrix}.$$

(b) Find $N(\mathsf{A})$, the Null space of A, and state the matrix rank.

(c) Sketch a graph in $\mathbb{R}^3$ representing the $N(\mathsf{A})$ and the set of all solutions found in (a).

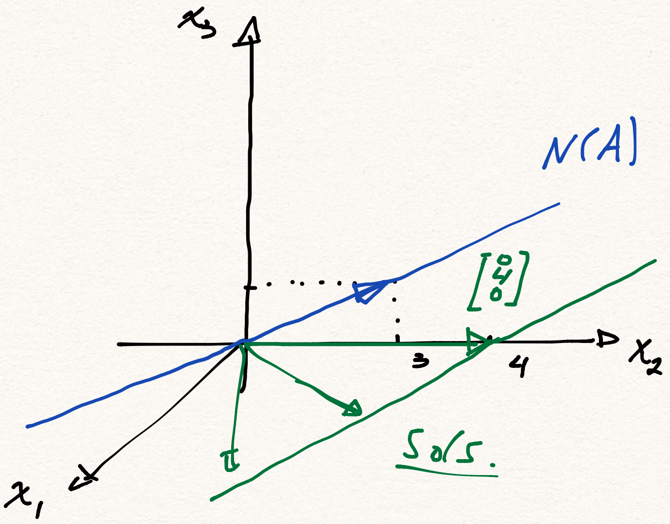
a)

$$\left[\begin{array}{ccc|c} 1 & 3 & -9 & 12 \\ 2 & -2 & 6 & -8 \\ 3 & 1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -9 & 12 \\ 0 & -8 & 24 & -32 \\ 0 & -8 & 24 & -32 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -9 & 12 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 4 + 3x_3 \\ x_3 : \text{free} \end{array}} \qquad \boxed{\underline{x} = \begin{bmatrix} 0 \\ 4+3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} x_3 .}$$

b) $\boxed{N(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}}$

c)



3. (16 points) Determine which of the following functions $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ is linear:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 - x_2)^2 \\ x_2 \end{bmatrix}, \qquad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_2.$$

If a function is linear, give a proof; if a function is not linear, show it with an example.

$\boxed{\text{T is Not linear}}$

$$\left.\begin{array}{l} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \end{array}\right\} \Rightarrow T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) \neq 3\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$\boxed{\text{S is linear}}$

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \boxed{S(\underline{x}) = A\underline{x}}$$

Since $A(a\underline{x} + b\underline{y}) = aA\underline{x} + bA\underline{y} \Rightarrow \boxed{S(a\underline{x} + b\underline{y}) = a\,S(\underline{x}) + b\,S(\underline{y})}$

4. (20 points) Verify whether $W_1 = W_2$ or not, and justify your answer, where

$$W_1 = \text{Span}\left(\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}5\\6\\7\end{bmatrix}\right\}\right), \qquad W_2 = \text{Span}\left(\left\{\begin{bmatrix}1\\4\\0\end{bmatrix}, \begin{bmatrix}-4\\-8\\-4\end{bmatrix}, \begin{bmatrix}4\\6\\5\end{bmatrix}\right\}\right).$$

$$A = \begin{bmatrix}1 & 1 & 5\\2 & 0 & 6\\1 & 2 & 7\end{bmatrix} \qquad B = \begin{bmatrix}1 & -4 & 4\\4 & -8 & 6\\0 & -4 & 5\end{bmatrix} \qquad R(A) = R(B) \iff E_{(A^T)} = E_{(B^T)}$$

$$A^T = \begin{bmatrix}1 & 2 & 1\\1 & 0 & 2\\5 & 6 & 7\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1\\0 & -2 & 1\\0 & -4 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1/2\\0 & 0 & 0\end{bmatrix} = E_{(A^T)}$$

$$B^T = \begin{bmatrix}1 & 4 & 0\\-4 & -8 & -4\\4 & 6 & 5\end{bmatrix} \to \begin{bmatrix}1 & 4 & 0\\0 & 8 & -4\\0 & -10 & 5\end{bmatrix} \to \begin{bmatrix}1 & 4 & 0\\0 & 1 & -1/2\\0 & 1 & -1/2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1/2\\0 & 0 & 0\end{bmatrix} = E_{(B^T)}$$

$$\therefore \boxed{W_1 = R(A) = R(B) = W_2}$$

5. (20 points) Use the floating point numbers $\mathbb{F}_{3,10,6}$ (three digits arithmetic) to find all solutions x_1, x_2, to the linear system

$$7x_1 + 9x_2 = 34,$$
$$12x_1 + x_2 = 15.$$

Do not use partial pivoting or complete pivoting. Then use the real numbers set $\mathbb{R}$ to find the exact solution of the linear system.

$$7x_1 + 9x_2 = 34$$
$$12x_1 + x_2 = 15$$

$$\left[\begin{array}{cc|c} 7 & 9 & 34 \\ 12 & 1 & 15 \end{array}\right] \leftarrow \text{divide by } 7 \quad \left[\begin{array}{cc|c} 1 & 1.29 & 4.86 \\ 12 & 1 & 15 \end{array}\right] \xrightarrow{-12} \left[\begin{array}{cc|c} 1 & 1.29 & 4.86 \\ 0 & -14.5 & -43.3 \end{array}\right]$$

$$fl\left(\frac{9}{7}\right) = fl(1.2857\ldots) = 1.29$$

$$fl\left(\frac{34}{7}\right) = fl(4.8571\ldots) = 4.86$$

$$fl(1 - fl(15.48)) = fl(1 - 15.5) = -14.5$$

$$fl(15 - fl(58.32)) = fl(15 - 58.3) = -43.3$$

$$fl\left(\frac{43.3}{14.5}\right) = fl(2.9862\ldots) = 2.99$$

$$\left[\begin{array}{cc|c} 1 & 1.29 & 4.86 \\ 0 & 1 & 2.99 \end{array}\right]$$

$$fl(4.86 - fl((1.29)(2.99))) =$$
$$= fl(4.86 - fl(3.8571))$$
$$= fl(4.86 - 3.86) = 1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2.99 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 2.99 \end{array}}$$

6. (10 points) Find the LU-factorization of the matrix $\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 8 & 3 \\ -3 & -3 & -6 & -1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 8 & 3 \\ -3 & -3 & -6 & -1 \end{bmatrix} \xrightarrow{-2,\ 3} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 3 & 2 \end{bmatrix} \xrightarrow{-1} \boxed{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = U}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}}$$

Check: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 8 & 3 \\ -3 & -3 & -6 & -1 \end{bmatrix} = A$.

#	Pts	Score
1	14	
2	20	
3	16	
4	20	
5	20	
6	10	
Σ	100	