Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- Even-periodic, odd-periodic extensions of functions.
Even, odd functions.

Definition
A function \( f : [-L, L] \rightarrow \mathbb{R} \) is **even** iff for all \( x \in [-L, L] \) holds

\[
f(-x) = f(x).
\]

A function \( f : [-L, L] \rightarrow \mathbb{R} \) is **odd** iff for all \( x \in [-L, L] \) holds

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Remarks:
- The only function that is both odd and even is $f = 0$. 
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Remarks:
- The only function that is both odd and even is \( f = 0 \).
- Most functions are neither odd nor even.
Even, odd functions.

Example
Show that the function $f(x) = x^2$ is even on $[-L, L]$. 

Solution:
The function is even, since $f(-x) = (-x)^2 = x^2 = f(x)$. 

$f(x)f(-x)$
Even, odd functions.

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Example
Show that the function $f(x) = x^3$ is odd on $[-L, L]$. 

Solution: The function is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$. 

$f(-x)$  
$f(x)$  
$-x$  
y  
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Even, odd functions.

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;
Even, odd functions.

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Even, odd functions.

Example

1. The function $f(x) = \cos(ax)$ is even on $[-L, L]$;
2. The function $f(x) = \sin(ax)$ is odd on $[-L, L]$;
3. The functions $f(x) = e^x$ and $f(x) = (x - 2)^2$ are neither even nor odd.
Even, odd functions.

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- **Main properties of even, odd functions.**
- Sine and cosine series.
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Main properties of even, odd functions.

Theorem

1. A linear combination of even (odd) functions is even (odd).
2. The product of two odd functions is even.
3. The product of two even functions is even.
4. The product of an even function by an odd function is odd.
Main properties of even, odd functions.

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Proof:
(1) Let $f$ and $g$ be even,
Main properties of even, odd functions.

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**Proof:**

(1) Let $f$ and $g$ be even, that is, $f(-x) = f(x), g(-x) = g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$$(af + bg)(-x)$$
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Case ”odd” is similar.
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Proof:
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Cases (3), (4) are similar.
Main properties of even, odd functions.

**Theorem**

If $f: [-L, L] \rightarrow \mathbb{R}$ is even, then

$$\int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx.$$

If $f: [-L, L] \rightarrow \mathbb{R}$ is odd, then

$$\int_{-L}^{L} f(x) \, dx = 0.$$
Main properties of even, odd functions.

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*If $f: [-L, L] \rightarrow \mathbb{R}$ is even, then* \[ \int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx. \]

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Main properties of even, odd functions.

Proof:

\[ I = \int_{-L}^{L} f(x) \, dx \]
Main properties of even, odd functions.

Proof:

\[ I = \int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx \]

Even case:

\[ f(-y) = f(y), \quad \text{therefore,} \quad I = \int_{0}^{L} f(y) \, dy + \int_{0}^{L} f(x) \, dx \Rightarrow \int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx. \]

Odd case:

\[ f(-y) = -f(y), \quad \text{therefore,} \quad I = -\int_{0}^{L} f(y) \, dy + \int_{0}^{L} f(x) \, dx \Rightarrow \int_{-L}^{L} f(x) \, dx = 0. \]
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Proof:

\[ I = \int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx \quad y = -x, \ dy = -dx. \]
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Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
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- *Sine and cosine series.*
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Sine and cosine series.

**Theorem (Cosine and Sine Series)**

Consider the function $f : [-L, L] \rightarrow \mathbb{R}$ with Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$

(1) If $f$ is even, then $b_n = 0$ for $n = 1, 2, \cdots$, and the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

is called a **Cosine Series**.

(2) If $f$ is odd, then $a_n = 0$ for $n = 0, 1, \cdots$, and the Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

is called a **Sine Series**.
Sine and cosine series.

Proof:
If $f$ is even, and since the Sine function is odd,
Sine and cosine series.

Proof:
If $f$ is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0,$$
Sine and cosine series.

Proof:
If \( f \) is even, and since the Sine function is odd, then

\[
 b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0,
\]

since we are integrating an odd function on \([-L, L]\).
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Proof:
If \( f \) is even, and since the Sine function is odd, then

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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0,
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If \( f \) is odd, and since the Cosine function is even,
Sine and cosine series.

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If \( f \) is even, and since the Sine function is odd, then
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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0,
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since we are integrating an odd function on \([-L, L]\).

If \( f \) is odd, and since the Cosine function is even, then
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a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx = 0,
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Sine and cosine series.

Proof:
If $f$ is even, and since the Sine function is odd, then

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If $f$ is odd, and since the Cosine function is even, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx = 0,$$

since we are integrating an odd function on $[-L, L]$. \qed
Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- **Even-periodic, odd-periodic extensions of functions.**
Even-periodic, odd-periodic extensions of functions.

(1) Even-periodic case:
A function $f : [0, L] \rightarrow \mathbb{R}$ can be extended as an even function $f : [-L, L] \rightarrow \mathbb{R}$ requiring for $x \in [0, L]$ that

$$f(-x) = f(x).$$
(1) Even-periodic case:
A function \( f : [0, L] \to \mathbb{R} \) can be extended as an even function \( f : [-L, L] \to \mathbb{R} \) requiring for \( x \in [0, L] \) that
\[
    f(-x) = f(x).
\]
This function \( f : [-L, L] \to \mathbb{R} \) can be further extended as a periodic function \( f : \mathbb{R} \to \mathbb{R} \) requiring for \( x \in [-L, L] \) that
\[
    f(x + 2nL) = f(x).
\]
Even-periodic, odd-periodic extensions of functions.

Example
Sketch the graph of the even-periodic extension of \( f(x) = x^5 \), with \( x \in [0, 1] \).
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(2) Odd-periodic case:
A function \( f : (0, L) \to \mathbb{R} \) can be extended as an odd function \( f : (-L, L) \to \mathbb{R} \) requiring for \( x \in (0, L) \) that
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Even-periodic, odd-periodic extensions of functions.

Example
Sketch the graph of the odd-periodic extension of \( f(x) = x^5 \), with \( x \in (0, 1) \).
Even-periodic, odd-periodic extensions of functions.

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Sketch the graph of the even-periodic extension of \( f(x) = x \), with \( x \in [0, 1] \), and then find its Fourier Series.
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Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

Solution: Since $f$ is even and periodic, then the Fourier Series is a Cosine Series,
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a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx
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\begin{align*}
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos\left( \frac{n\pi x}{L} \right) \, dx, \\
a_n &= 2 \int_{0}^{1} x \cos(n\pi x) \, dx
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Example
Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0, 1)$, and then find its Fourier Series.

Solution: Recall: $a_n = 0$, and $b_n = \frac{2 (-1)^{n+1}}{n\pi}$.
Even-periodic, odd-periodic extensions of functions.

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Solution: Recall: \( a_n = 0 \), and \( b_n = \frac{2(-1)^{n+1}}{n\pi} \). Therefore,

\[
f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x).
\]

△
Solving the Heat Equation (Sect. 10.5).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

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Solving the Heat Equation (Sect. 10.5).

- **The Heat Equation.**
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.
The Heat Equation.

Remarks:

- The unknown of the problem is $u(t, x)$, the temperature of the bar at the time $t$ and position $x$. 

- The temperature does not depend on $y$ or $z$.

- The one-dimensional Heat Equation is:
  \[ \frac{\partial u}{\partial t}(t, x) = k \frac{\partial^2 u}{\partial x^2}(t, x), \]
  where $k > 0$ is the heat conductivity, units: $[k] = \text{(distance)}^2 \text{/(time)}$. 

- The Heat Equation is a Partial Differential Equation, PDE.
Remarks:

- The unknown of the problem is $u(t, x)$, the temperature of the bar at the time $t$ and position $x$.
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▶ The Heat Equation is a Partial Differential Equation, PDE.
Solving the Heat Equation (Sect. 10.5).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.
The Initial-Boundary Value Problem.

Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

\[ \partial_t u(t, x) = k \partial_x^2 u(t, x), \]

I.C.:

\[ u(0, x) = f(x), \]

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\[ \partial_t u(t, x) = k \partial_x^2 u(t, x), \quad u(0, x) = f(x), \quad u(t, 0) = 0, \quad u(t, L) = 0. \]
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The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x). \]
The separation of variables method.

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\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x). \]

where

- ▶ \( v_n \): Solution of an IVP.

Remark: The separation of variables method does not work for every PDE.
Summary: IBVP for the Heat Equation.

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- \( v_n \): Solution of an IVP.
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Summary: IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \psi_n(t) \phi_n(x). \]

where

- \( \psi_n \): Solution of an IVP.
- \( \phi_n \): Solution of a BVP, an eigenvalue-eigenfunction problem.
- \( c_n \): Fourier Series coefficients.

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Propose:

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One looks for solutions $u$ given by an infinite series of simpler functions, $u_n$. 
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One looks for solutions $u$ given by an infinite series of simpler functions, $u_n$, that is,

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Here $c_n$ are constants, $n = 1, 2, \cdots$. 
The separation of variables method.

Step 2:
Introduce the series expansion for $u$ into the Heat Equation,

$$\partial_t u - k \partial_x^2 u = 0$$

A sufficient condition for the equation above is:

To find $u_n$, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k \partial_x^2 u_n = 0$$

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Find $u_n(t, x) = v_n(t) w_n(x)$ solution of the IBVP

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$$\partial_t u - k \partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[ \partial_t u_n - k \partial_x^2 u_n \right] = 0.$$
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$\text{I.C.}: u_n(0, x) = w_n(x), \text{B.C.}: u_n(t, 0) = 0, u_n(t, L) = 0.$
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**Step 4:** (Key step.)
Transform the IBVP for \( u_n \) into:

\[
\partial_t u_n(t, x) = \partial^2_x u_n(t, x) = k v_n(t) w_n(x) \frac{d^2 w_n(x)}{dx^2}.
\]

Depends only on \( t \).
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$$

$$
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Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$
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Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$ is given by

$$w_n(x) \frac{dv_n}{dt}(t) = k v_n(t) \frac{d^2 w_n}{dx^2}(x).$$
The separation of variables method.

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Transform the IBVP for $u_n$ into: (a) IVP for $v_n$; (b) BVP for $w_n$.

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\[ w_n(x) \frac{dv_n}{dt}(t) = k v_n(t) \frac{d^2 w_n}{dx^2}(x) \]

\[ \frac{1}{k v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x). \]
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Transform the IBVP for \( u_n \) into: (a) IVP for \( v_n \); (b) BVP for \( w_n \).

Notice:

\[
\partial_t u_n(t, x) = \partial_t \left[ v_n(t) w_n(x) \right] = w_n(x) \frac{dv_n}{dt}(t).
\]

\[
\partial_x^2 u_n(t, x) = \partial_x^2 \left[ v_n(t) w_n(x) \right] = v_n(t) \frac{d^2 w_n}{dx^2}(x).
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w_n(x) \frac{dv_n}{dt}(t) = k v_n(t) \frac{d^2 w_n}{dx^2}(x)
\]

\[
\frac{1}{k v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x).
\]

Depends only on \( t \)
The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for $u_n$ into: (a) IVP for $v_n$; (b) BVP for $w_n$.

Notice:

$$\partial_t u_n(t, x) = \partial_t [v_n(t) w_n(x)] = w_n(x) \frac{dv_n}{dt}(t).$$

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Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$ is given by

$$w_n(x) \frac{dv_n}{dt}(t) = k v_n(t) \frac{d^2w_n}{dx^2}(x)$$

$$\frac{1}{k v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2w_n}{dx^2}(x).$$

Depends only on $t$ = Depends only on $x$. 
The separation of variables method.

Recall:

\[
\frac{1}{k v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2w_n}{dx^2}(x).
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Depends only on \( t \) \quad = \quad Depends only on \( x \).
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Recall:

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Depends only on \( t \) = Depends only on \( x \).

- The Heat Equation has the following property:
  The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).
The separation of variables method.

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Depends only on \( t \) \quad \Rightarrow \quad \text{Depends only on} \ x.

- The Heat Equation has the following property:
The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).

- When this happens in a PDE, one can use the separation of variables method on that PDE.
The separation of variables method.

Recall: \[-\frac{1}{k} \frac{d v_n(t)}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n(x)}{dx^2}(x) .\]

Depends only on \(t\) = Depends only on \(x\).

- The Heat Equation has the following property:
  The left-hand side depends only on \(t\), while the right-hand side depends only on \(x\).

- When this happens in a PDE, one can use the separation of variables method on that PDE.

- We conclude that for appropriate constants \(\lambda_m\) holds
  \[-\frac{1}{k} \frac{d v_n(t)}{dt}(t) = -\lambda_n,\]
The separation of variables method.

Recall:
\[
\frac{1}{k \, v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x).
\]

Depends only on \( t \) \quad \text{Depends only on} \ x.

- The Heat Equation has the following property: The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants \( \lambda_m \) holds
\[
\frac{1}{k \, v_n(t)} \frac{dv_n}{dt}(t) = -\lambda_n, \quad \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x) = -\lambda_n.
\]
The separation of variables method.

Recall:
\[
\frac{1}{k \, \nu_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x).
\]

The left-hand side depends only on \( t \) \( = \) the right-hand side depends only on \( x \).

- The Heat Equation has the following property:
The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).

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- We conclude that for appropriate constants \( \lambda_m \) holds
\[
\frac{1}{k \, \nu_n(t)} \frac{dv_n}{dt}(t) = -\lambda_n, \quad \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x) = -\lambda_n.
\]

- We have transformed the original PDE into infinitely many ODEs parametrized by \( n \), positive integer.
The separation of variables method.

Summary Step 4: The original IBVP for the Heat Equation, PDE, can be transformed into:

(a) We choose to solve the following IVP for $v_n(t)$:

$$\frac{dv_n}{dt}(t) = -\lambda_n,$$

I.C.: $v_n(0) = 1$.

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for $w_n(x)$:

$$\frac{d^2w_n}{dx^2}(x) = -\lambda_n,$$

B.C.: $w_n(0) = 0$, $w_n(L) = 0$.

Step 5:

(a) Solve the IVP for $v_n$.

(b) Solve the BVP for $w_n$. 
The separation of variables method.

**Summary Step 4:** The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for $v_n$,

$$
\frac{1}{k \, v_n(t)} \frac{d v_n}{d t}(t) = -\lambda_n,
$$

(b) The BVP for $w_n$,

$$
\frac{d^2 w_n}{d x^2}(x) = -\lambda_n,
$$

I.C.:

$$
v_n(0) = 1.
$$

Remark: This choice of I.C. simplifies the problem.

Step 5:

(a) Solve the IVP for $v_n$.

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The separation of variables method.

**Summary Step 4:** The original IBVP for the Heat Equation, PDE, can be transformed into:

(a) We choose to solve the following IVP for $v_n$,

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(b) The BVP for $w_n$,

$$1 \frac{d^2w_n}{dx^2}(x) = -\lambda_n, \quad \text{B.C.: } w_n(0) = 0, \quad w_n(L) = 0.$$
The separation of variables method.

**Summary Step 4:** The original *IBVP* for the Heat Equation, PDE, can be transformed into:

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**Step 5:**

(a) Solve the IVP for $v_n$.

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The separation of variables method.

**Step 5(a):** Solving the IVP for $v_n$. 

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The integrating factor method implies that \( \mu(t) = e^{k\lambda_n t} \).

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The real-valued general solution is

$$w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).$$
The separation of variables method.

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Choosing \( c_2 = 1 \), we get \( w_n(x) = \sin\left( \frac{n\pi x}{L} \right) \).
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We conclude that: \( u_n(t, x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots \).
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Step 6: Recall: \( u_n(t, x) = e^{-k \left( \frac{n \pi}{L} \right)^2 t} \sin \left( \frac{n \pi x}{L} \right) \).
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Compute the solution to the IBVP for the Heat Equation,

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, u_n(t, x). \]
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\begin{align*}
  u(t, x) &= \sum_{n=1}^{\infty} c_n u_n(t, x).
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By construction, this solution satisfies the boundary conditions,

\( u(t, 0) = 0 \), \( u(t, L) = 0 \).

Given a function \( f \) with \( f(0) = f(L) = 0 \), the solution \( u \) above satisfies the initial condition \( f(x) = u(0, x) \) iff holds

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This is a Sine Series for \( f \). The coefficients \( c_n \) are computed in the usual way.
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\[
\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases}
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Multiply the equation for \( u \) by \( \sin \left( \frac{m\pi x}{L} \right) \) and integrate,

\[
\sum_{n=1}^{\infty} c_n \int_{0}^{L} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx = \int_{0}^{L} f(x) \sin \left( \frac{m\pi x}{L} \right) \, dx.
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\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right). \]

This is a Sine Series for \( f \). The coefficients \( c_n \) are computed in the usual way. Recall the orthogonality relation

\[ \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases} \]

Multiply the equation for \( u \) by \( \sin\left(\frac{m\pi x}{L}\right) \) and integrate,

\[ \sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) \, dx. \]

\[ c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \]
The separation of variables method.

**Summary:** IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x). \]
The separation of variables method.

**Summary:** IBVP for the Heat Equation.

**Propose:**

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x). \]

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- \( v_n \): Solution of an IVP.
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- \( v_n \): Solution of an IVP.
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Remark: The separation of variables method does not work for every PDE.
The separation of variables method.

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Remark:
The separation of variables method does not work for every PDE.
Solving the Heat Equation (Sect. 10.5).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.
An example of separation of variables.

Example
Find the solution to the IBVP  

\[ 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]
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Solution: Let \( u_n(t, x) = v_n(t) w_n(x) \).
An example of separation of variables.

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Solution: Let \( u_n(t, x) = v_n(t) w_n(x) \). Then

\[ 4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2w}{dx^2}(x). \]
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Solution: Let \( u_n(t, x) = v_n(t) w_n(x) \). Then

\[ 4 w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v'_n(t)}{v_n(t)} = \frac{w''_n(x)}{w_n(x)} = -\lambda_n. \]
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Find the solution to the IBVP

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Solution: Let $u_n(t, x) = v_n(t) w_n(x)$. Then

$$4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v'_n(t)}{v_n(t)} = \frac{w''_n(x)}{w_n(x)} = -\lambda_n.$$ 

The equations for $v_n$ and $w_n$ are

$$v'_n(t) + \frac{\lambda_n}{4} v_n(t) = 0,$$
An example of separation of variables.

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\[ 4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n. \]
The equations for \( v_n \) and \( w_n \) are
\[ v_n'(t) + \frac{\lambda_n}{4} v_n(t) = 0, \quad w_n''(x) + \lambda_n w_n(x) = 0. \]
Example

Find the solution to the IBVP \( 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)
\( u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \)

Solution: Let \( u_n(t, x) = v_n(t) w_n(x). \) Then

\[
4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.
\]

The equations for \( v_n \) and \( w_n \) are

\[
v_n'(t) + \frac{\lambda_n}{4} v_n(t) = 0, \quad w_n''(x) + \lambda_n w_n(x) = 0.
\]

We solve for \( v_n \) with the initial condition \( v_n(0) = 1. \)

\[
e^{\frac{\lambda_n}{4} t} v_n'(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0
\]
An example of separation of variables.

**Example**

Find the solution to the IBVP

\[ 4 \frac{\partial_t u}{t} = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

**Solution:** Let \( u_n(t, x) = v_n(t) w_n(x) \). Then

\[ 4 w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4 v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n. \]

The equations for \( v_n \) and \( w_n \) are

\[ v_n'(t) + \frac{\lambda_n}{4} v_n(t) = 0, \quad w_n''(x) + \lambda_n w_n(x) = 0. \]

We solve for \( v_n \) with the initial condition \( v_n(0) = 1 \).

\[ e^{\frac{\lambda_n}{4} t} v_n'(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0 \quad \Rightarrow \quad \left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0. \]
An example of separation of variables.

Example
Find the solution to the IBVP

\[4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],\]

\[u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.\]

Solution: Recall: \[e^{\frac{\lambda_n t}{4}} v_n(t)\] ' = 0.
An example of separation of variables.

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\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
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Solution: Recall: \[ [e^{\lambda_n^2 t} \nu_n(t)]' = 0. \] Therefore,
\[ \nu_n(t) = c e^{-\lambda_n t}, \]
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\[ v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) \]
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Example
Find the solution to the IBVP $4 \partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$,
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u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.
\]
Solution: Recall: $[e^{\lambda_n t} v_n(t)]' = 0$. Therefore,
\[
\nu_n(t) = c e^{-\frac{\lambda_n t}{4}}, \quad 1 = \nu_n(0) = c \quad \Rightarrow \quad \nu_n(t) = e^{-\frac{\lambda_n t}{4}}.
\]
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\[v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}.\]

Next the BVP: \(w_n''(x) + \lambda_n w_n(x) = 0,\) with \(w_n(0) = w_n(L) = 0.\)
An example of separation of variables.

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Find the solution to the IBVP \( 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)

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Since \( \lambda_n > 0, \) introduce \( \lambda_n = \mu_n^2. \)
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\[ p(r) = r^2 + \mu_n^2 \]
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\[ 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
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Solution: Recall: \[ \left[ e^{\frac{\lambda_n}{4} t} \, v_n(t) \right]' = 0. \] Therefore,

\[ v_n(t) = c \, e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}. \]

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Since \( \lambda_n > 0 \), introduce \( \lambda_n = \mu_n^2 \). The characteristic polynomial is

\[ p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i. \]
An example of separation of variables.

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Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$

Solution: Recall: $[e^{\lambda_n t} v_n(t)]' = 0.$ Therefore,

$v_n(t) = c e^{-\lambda_n t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\lambda_n t}.$

Next the BVP: $w''_n(x) + \lambda_n w_n(x) = 0,$ with $w_n(0) = w_n(L) = 0.$

Since $\lambda_n > 0,$ introduce $\lambda_n = \mu_n^2.$ The characteristic polynomial is

$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i.$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).$
An example of separation of variables.

Example

Find the solution to the IBVP
\[ 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:
\[ \left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0. \]
Therefore,
\[ v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}. \]

Next the BVP:
\[ w_n''(x) + \lambda_n w_n(x) = 0, \quad \text{with} \quad w_n(0) = w_n(L) = 0. \]
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The boundary conditions imply
\[ 0 = w_n(0) = c_1, \]
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4 \partial_t u = \partial^2_x u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \[ \left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0. \] Therefore,

\[ v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}. \]

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The general solution, \( w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x) \).

The boundary conditions imply

\[ 0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x). \]
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \( v_n(t) = e^{-\frac{\lambda_n}{4} t} \), and \( w_n(x) = c_2 \sin(\mu_n x) \).
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Example

Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \( v_n(t) = e^{-\lambda_n t} \), and \( w_n(x) = c_2 \sin(\mu_n x) \).

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \]
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Solution: Recall: \( v_n(t) = e^{-\frac{\lambda_n}{4} t}, \) and \( w_n(x) = c_2 \sin(\mu_n x). \)

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \]
An example of separation of variables.

Example
Find the solution to the IBVP

$$4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

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Solution: Recall: \(v_n(t) = e^{-\frac{\lambda_n}{4} t}\), and \(w_n(x) = c_2 \sin(\mu_n x)\).

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin \left( \frac{\pi x}{2} \right), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:
\[ v_n(t) = e^{-\frac{\lambda_n}{4} t}, \quad \text{and} \quad w_n(x) = c_2 \sin(\mu_n x). \]

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \( \mu_n 2 = n\pi \),
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin\left(\frac{\pi x}{2}\right), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \( v_n(t) = e^{-\frac{\lambda n}{4} t} \), and \( w_n(x) = c_2 \sin(\mu_n x) \).

\[
0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.
\]

Then, \( \mu_n 2 = n\pi \), that is, \( \mu_n = \frac{n\pi}{2} \).
An example of separation of variables.

Example
Find the solution to the IBVP

$$4 \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2],$$

$$u(0, x) = 3 \sin(\frac{\pi x}{2}), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$ 

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4} t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$ 

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \frac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$
An example of separation of variables.

Example
Find the solution to the IBVP

$$4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

$$u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$ 

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4} t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$ 

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \frac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

$$u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right).$$
An example of separation of variables.

Example

Find the solution to the IBVP: $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],
\quad u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n e^{-(n\pi / 4)^2 t} \sin\left(\frac{n\pi x}{2}\right).$
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\frac{\pi x}{2}), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:
\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is
\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]
An example of separation of variables.

Example
Find the solution to the IBVP \[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
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The orthogonality of the sine functions implies
\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n \pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) \, dx. \]
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin\left(\frac{\pi x}{2}\right), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_{0}^{2} \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_{0}^{2} \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2}, \)
An example of separation of variables.

Example
Find the solution to the IBVP

\[ 4 \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n \pi}{4}\right)^2 t} \sin\left(\frac{n \pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n \pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \).
An example of separation of variables.

Example
Find the solution to the IBVP

\[ 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \). Therefore,

\[ 3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \]
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4 \frac{\partial t u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \). Therefore,

\[ 3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3. \]
An example of separation of variables.

Example
Find the solution to the IBVP

$$4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

$$u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$ 

Solution: We conclude that

$$u(t, x) = 3 e^{-\left(\frac{\pi}{4}\right)^2 t} \sin\left(\frac{\pi x}{2}\right).$$
Review Exam 4.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Overview of linear differential systems (7.1).
  - Review of Linear Algebra (7.2, 7.3).
  - Basic Theory of first order systems (7.4).
  - Homogeneous constant coefficients systems:
    - Real and different eigenvalues (7.5).
    - Complex eigenvalues (7.6).
    - Real and repeated eigenvalues (7.8).
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]
Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):

\[ p(\lambda) = \begin{vmatrix} (1 - \lambda) & 2 \\ -2 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0 \]

\[ \lambda = 1 \pm 2i. \]
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 2 \\ -2 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0
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Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 2 \\ -2 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0
\]

\[(\lambda - 1)^2 = -4\]
Example
Find the real-valued general solution of
\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):
\[ p(\lambda) = \begin{vmatrix} (1 - \lambda) & 2 \\ -2 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0 \]
\[ (\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda_\pm = 1 \pm 2i. \]
Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} 1 - \lambda & 2 \\ -2 & 1 - \lambda \end{vmatrix} = (\lambda - 1)^2 + 4 = 0
\]

\[(\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda_{\pm} = 1 \pm 2i.\]

Eigenvector for \( \lambda_+ \).

\[
(A - \lambda_+ I) = \begin{bmatrix} 1 - (1 + 2i) & 2 \\ -2 & 1 - (1 + 2i) \end{bmatrix}
\]
Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} 1 - \lambda & 2 \\ -2 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)^2 + 4 = 0
\]

\((\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda_\pm = 1 \pm 2i.\)

Eigenvector for \(\lambda_+\).

\[
(A - \lambda_+ I) = \begin{bmatrix} 1 - (1 + 2i) & 2 \\ -2 & 1 - (1 + 2i) \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}.
\]
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_\pm = 1 \pm 2i \), \( (A - \lambda_\pm I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \).
Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad (A - \lambda_+ I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}. \)

\[ \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \]
Example

Find the real-valued general solution of

\[ x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), \( (A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}. \)

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix}
\]
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), \((A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}.\)

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}
\]
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), \( (A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}. \)

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -iv_2.
\]
Example

Find the real-valued general solution of \( x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad (A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}. \)

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -iv_2.
\]

Choosing \( v_2 = 1, \)
Example

Find the real-valued general solution of

$$x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$  

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), \( (A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \).

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -iv_2.
\]

Choosing \( v_2 = 1 \), we get \( v_1 = -i \),
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad (A - \lambda_{\pm} I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}. \)

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -iv_2.
\]

Choosing \( v_2 = 1, \) we get \( v_1 = -i, \) that is,

\[ v^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}. \]
Example

Find the real-valued general solution of

\[ x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), so \((A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \).

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -iv_2.
\]

Choosing \( v_2 = 1 \), we get \( v_1 = -i \), that is,

\[ v^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow v^{(-)} = \overline{v^{(+)}} \]
Example

Find the real-valued general solution of

\[
x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad (A - \lambda_{\pm}I) = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \).

\[
\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \to \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -iv_2.
\]

Choosing \( v_2 = 1 \), we get \( v_1 = -i \), that is,

\[
v^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \Rightarrow \quad v^{(-)} = \overline{v^{(+)}} = \begin{bmatrix} i \\ 1 \end{bmatrix}.
\]
Example

Find the real-valued general sol. \( x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \).

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad \text{and} \quad v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \).
Example

Find the real-valued general sol. \( x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \) and \( v^{(\pm)} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i. \)

Also recalling: If \( \lambda_{\pm} = \alpha \pm \beta i \) and \( v^{(\pm)} = a \pm bi, \)
Example

Find the real-valued general sol. \( x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \quad \text{and} \quad v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i. \)

Also recalling: If \( \lambda_{\pm} = \alpha \pm \beta i \) and \( v^{(\pm)} = a \pm bi, \) then

\[ x^{(1)}(t) = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right], \]
Exam: November 11, 2008. Problem 4

Example

Find the real-valued general sol. $x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = 1 \pm 2i$, and $v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

Also recalling: If $\lambda_{\pm} = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, then

$$x^{(1)}(t) = e^{\alpha t} \left[a \cos(\beta t) - b \sin(\beta t)\right],$$

$$x^{(2)}(t) = e^{\alpha t} \left[a \sin(\beta t) + b \cos(\beta t)\right].$$
Exam: November 11, 2008. Problem 4

Example

Find the real-valued general sol. \( x'(t) = A x(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i, \) and \( v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i. \)

Also recalling: If \( \lambda_{\pm} = \alpha \pm \beta i \) and \( v^{(\pm)} = a \pm bi, \) then

\[
x^{(1)}(t) = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right],
\]

\[
x^{(2)}(t) = e^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].
\]

\[
x^{(1)} = e^t \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t) \right)
\]
Example

Find the real-valued general sol. \( x'(t) = Ax(t), \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \).

Solution: Recall: \( \lambda_{\pm} = 1 \pm 2i \), and \( \mathbf{v}^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \).

Also recalling: If \( \lambda_{\pm} = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = a \pm bi \), then

\[
\begin{align*}
x^{(1)}(t) &= e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right], \\
x^{(2)}(t) &= e^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].
\end{align*}
\]

\[
\begin{align*}
x^{(1)} &= e^t \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t) \right) \Rightarrow x^{(1)} = e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.
\end{align*}
\]
Example

Find the real-valued general sol. $x'(t) = Ax(t), \ A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = 1 \pm 2i$, and $v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

Also recalling: If $\lambda_{\pm} = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, then

$$x^{(1)}(t) = e^{\alpha t} \begin{bmatrix} a \\ -b \end{bmatrix} \cos(\beta t) - b \sin(\beta t),$$
$$x^{(2)}(t) = e^{\alpha t} \begin{bmatrix} a \\ -b \end{bmatrix} \sin(\beta t) + b \cos(\beta t).$$

$$x^{(1)} = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t) \Rightarrow x^{(1)} = e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.$$

$$x^{(2)} = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(2t).$$
Example

Find the real-valued general sol. \(x' (t) = Ax (t)\), \(A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\).

Solution: Recall: \(\lambda_{\pm} = 1 \pm 2i\), and \(v^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i\).

Also recalling: If \(\lambda_{\pm} = \alpha \pm \beta i\) and \(v^{(\pm)} = a \pm bi\), then

\[
x^{(1)}(t) = e^{\alpha t} \begin{bmatrix} a \\ b \end{bmatrix} \cos(\beta t) - \begin{bmatrix} a \\ b \end{bmatrix} \sin(\beta t),
\]

\[
x^{(2)}(t) = e^{\alpha t} \begin{bmatrix} a \\ b \end{bmatrix} \sin(\beta t) + \begin{bmatrix} a \\ b \end{bmatrix} \cos(\beta t).
\]

\[
x^{(1)} = e^{t} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t) \right) \Rightarrow x^{(1)} = e^{t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.
\]

\[
x^{(2)} = e^{t} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(2t) \right) \Rightarrow x^{(2)} = e^{t} \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}.
\]
Hint to remember formulas for $x^{(1)}$ and $x^{(2)}$.

Remark: The formulas for

$$x^{(1)}(t) = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right],$$
Hint to remember formulas for $x^{(1)}$ and $x^{(2)}$.

Remark: The formulas for

$$x^{(1)}(t) = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right],$$

$$x^{(2)}(t) = e^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].$$
Hint to remember formulas for $x^{(1)}$ and $x^{(2)}$.

Remark: The formulas for

$$x^{(1)}(t) = e^{\alpha t} [a \cos(\beta t) - b \sin(\beta t)],$$

$$x^{(2)}(t) = e^{\alpha t} [a \sin(\beta t) + b \cos(\beta t)].$$

are the real and imaginary part of $\tilde{x}^{(+)} = (a + bi) e^{(\alpha + \beta i)t}$. 
Hints to remember formulas for $x^{(1)}$ and $x^{(2)}$.

**Remark:** The formulas for

$$x^{(1)}(t) = e^{\alpha t} [a \cos(\beta t) - b \sin(\beta t)],$$

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are the real and imaginary part of $\tilde{x}^{(+)} = (a + bi) e^{(\alpha + \beta i)t}$. Indeed,

$$\tilde{x}^{(+)} = (a + bi) \left[ \cos(\beta t) + i \sin(\beta t) \right] e^{\alpha t}.$$
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Remark: The formulas for

$$x^{(1)}(t) = e^{\alpha t}\left[a \cos(\beta t) - b \sin(\beta t)\right],$$

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$$\tilde{x}^{(+)} = \left[a \cos(\beta t) - b \sin(\beta t)\right]e^{\alpha t} + i\left[a \sin(\beta t) + b \cos(\beta t)\right]e^{\alpha t}.$$
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.
Example

Find the general solution of $\mathbf{x}' = A \mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

Example

Find the general solution of \( \mathbf{x}' = A \mathbf{x} \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} (-3 - \lambda) & \sqrt{2} \\ \sqrt{2} & (-2 - \lambda) \end{vmatrix}
\]

Hence \( \lambda_{\pm} = \frac{\pm \sqrt{25 - 16}}{2} = \frac{\pm 3}{2} \).

Eigenvector for \( \lambda_{\pm} \):

\[
(A + I) \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Choosing \( \mathbf{v}_1 = \sqrt{2} \) and \( \mathbf{v}_2 = 2 \), we get

\[
\mathbf{v}(\pm) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}.
\]
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$
Example

Find the general solution of $\mathbf{x}' = A \mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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$$\lambda^2 + 5\lambda + 4 = 0$$
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}[-5 \pm \sqrt{25 - 16}]$$
Example

Find the general solution of \( \mathbf{x}' = A \mathbf{x} \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0
\]

\[
\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -5 \pm \sqrt{25 - 16} \right] = \frac{1}{2} \left[ -5 \pm 3 \right]
\]
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \implies \lambda_{\pm} = \frac{1}{2}[-5 \pm \sqrt{25 - 16}] = \frac{1}{2}[-5 \pm 3]$$

Hence $\lambda_+ = -1$, 

$\lambda_{\pm} = \frac{1}{2}[-5 \pm \sqrt{25 - 16}] = \frac{1}{2}[-5 \pm 3]$
Example

Find the general solution of $x' = A x$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0
\]

\[
\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -5 \pm \sqrt{25 - 16} \right] = \frac{1}{2} \left[ -5 \pm 3 \right]
\]

Hence $\lambda_+ = -1$, $\lambda_- = -4$. 
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$

$solve for \lambda$:

$\lambda^2 + 5\lambda + 4 = 0 \implies \lambda_{\pm} = \frac{1}{2}[-5 \pm \sqrt{25 - 16}] = \frac{1}{2}[-5 \pm 3]$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for $\lambda_+$.

$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -5 \pm \sqrt{25 - 16} \right] = \frac{1}{2} \left[ -5 \pm 3 \right]$$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for $\lambda_+$.

$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix}$$
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \Rightarrow \lambda_{\pm} = \frac{1}{2} \left[ -5 \pm \sqrt{25 - 16} \right] = \frac{1}{2} \left[ -5 \pm 3 \right]$$

Hence $\lambda_+ = -1, \quad \lambda_- = -4$. Eigenvector for $\lambda_+$.

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Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

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Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for $\lambda_+$.

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$$2v_1 = \sqrt{2} v_2.$$

Example

Find the general solution of \( x' = Ax \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0
\]

\[
\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -5 \pm \sqrt{25 - 16} \right] = \frac{1}{2} \left[ -5 \pm 3 \right]
\]

Hence \( \lambda_+ = -1 \), \( \lambda_- = -4 \). Eigenvector for \( \lambda_+ \).

\[
(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.
\]

\( 2v_1 = \sqrt{2} v_2 \). Choosing \( v_1 = \sqrt{2} \).
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2} [-5 \pm \sqrt{25 - 16}] = \frac{1}{2} [-5 \pm 3]$$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for $\lambda_+$.

$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$ 

$2v_1 = \sqrt{2}v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, 
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of $A$:

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$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$2v_1 = \sqrt{2} v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$. 
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.
Example

Find the general solution of $\mathbf{x}' = A \mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$. 
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Eigenvector for $\lambda_-$. 

\[
(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}
\]

Example

Find the general solution of \( x' = Ax \), where

\[
A = \begin{bmatrix}
-3 & \sqrt{2} \\
\sqrt{2} & -2
\end{bmatrix}.
\]

Solution: Recall: \( \lambda_+ = -1 \), \( \lambda_- = -4 \), and \( v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \).

Eigenvector for \( \lambda_- \).

\[
(A + 4I) = \begin{bmatrix}
1 & \sqrt{2} \\
\sqrt{2} & 2
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}
\]
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Eigenvector for $\lambda_-$. 

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Eigenvector for $\lambda_-$. 

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$  

$v_1 = -\sqrt{2} v_2$. 
Example

Find the general solution of \( x' = Ax \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Recall: \( \lambda_+ = -1 \), \( \lambda_- = -4 \), and \( \mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \).

Eigenvector for \( \lambda_- \).

\[
(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.
\]

\( \mathbf{v}_1 = -\sqrt{2} \mathbf{v}_2 \). Choosing \( \mathbf{v}_1 = -\sqrt{2} \).
Example

Find the general solution of \( x' = Ax \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Recall: \( \lambda_+ = -1 \), \( \lambda_- = -4 \), and \( v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \).

Eigenvector for \( \lambda_- \).

\[
(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.
\]

\( v_1 = -\sqrt{2} \, v_2 \). Choosing \( v_1 = -\sqrt{2} \) and \( v_2 = 1 \),
Example

Find the general solution of \( x' = A x \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Recall: \( \lambda_+ = -1 \), \( \lambda_- = -4 \), and \( v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \).

Eigenvector for \( \lambda_- \).

\[
(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.
\]

\( v_1 = -\sqrt{2} v_2 \). Choosing \( v_1 = -\sqrt{2} \) and \( v_2 = 1 \), so, \( v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \).
Example

Find the general solution of $\mathbf{x}' = A \mathbf{x}$, where

$$A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}.$$

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Eigenvector for $\lambda_-$. 

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$\mathbf{v}_1 = -\sqrt{2} \mathbf{v}_2$. Choosing $\mathbf{v}_1 = -\sqrt{2}$ and $\mathbf{v}_2 = 1$, so, $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t},$
Example

Find the general solution of $x' = Ax$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Eigenvector for $\lambda_-$. 

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$ 

$v_1 = -\sqrt{2} v_2$. Choosing $v_1 = -\sqrt{2}$ and $v_2 = 1$, so, $v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Fundamental solutions: $x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. 

◁
Example

Find the general solution of \( x' = Ax \), where \( A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \).

Solution: Recall: \( \lambda_+ = -1 \), \( \lambda_- = -4 \), and \( \mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \). Eigenvector for \( \lambda_- \).

\[
\begin{align*}
(A + 4I) &= \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

\( \mathbf{v}_1 = -\sqrt{2} \mathbf{v}_2 \). Choosing \( \mathbf{v}_1 = -\sqrt{2} \) and \( \mathbf{v}_2 = 1 \), so, \( \mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \).

Fundamental solutions: \( \mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t} \).

General solution: \( \mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \)
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:

We start plotting the vectors

\[ v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \]

\[ v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \]
Example
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \left[ \frac{\sqrt{2}}{2} \right] e^{-t}, \quad x^{(-)} = \left[ -\frac{\sqrt{2}}{1} \right] e^{-4t}. \]

Solution:
We start plotting the vectors

\[ v^{(+)} = \left[ \frac{\sqrt{2}}{2} \right], \quad v^{(-)} = \left[ -\frac{\sqrt{2}}{1} \right]. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:
We plot the solutions

\[ x^{(+)}, \quad -x^{(+)}, \]

\[ x^{(-)}, \quad -x^{(-)}. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:

We plot the solutions

\[ x^{(+)}, \quad -x^{(+)}, \]
\[ x^{(-)}, \quad -x^{(-)}. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ \mathbf{x}(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:
Recall: \( \lambda_- < \lambda_+ < 0 \). We plot the solutions

\[ \mathbf{x} = \mathbf{x}(+) + \mathbf{x}(-), \]

that is,

\[ \mathbf{x} = \mathbf{v}(+) e^{-t} + \mathbf{v}(-) e^{-4t}. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:

Recall: \( \lambda_- < \lambda_+ < 0 \). We plot the solutions

\[ x = x^{(+)} + x^{(-)}, \]

that is,

\[ x = v^{(+)} e^{-t} + v^{(-)} e^{-4t}. \]
Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[ x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Solution:
We plot the solutions

\[ x = c_1 x^{(+)} + c_2 x^{(-)}, \]

for different values of \( c_1 \) and \( c_2 \).
Example
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

\[
x^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad x^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.
\]

Solution:
We plot the solutions

\[ x = c_1 x^{(+)} + c_2 x^{(-)}, \]

for different values of \( c_1 \) and \( c_2 \).
Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = \mathbf{v}^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = \mathbf{v}^{(-)} e^{\lambda_- t}$,
Example

Let \( \lambda_+ = 4, \ \lambda_- = 1, \ v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \) and \( v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \)

Plot the phase portrait of several linear combinations of the fundamental solutions \( x^{(+)} = v^{(+)} e^{\lambda_+ t}, \ x^{(-)} = v^{(-)} e^{\lambda_- t}, \)

Solution:
Here \( \lambda_+ > \lambda_- > 0. \) We plot the solutions

\( x^{(+)}, \ -x^{(+)} , \)

\( x^{(-)}, \ -x^{(-)}. \)
Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = \mathbf{v}^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = \mathbf{v}^{(-)} e^{\lambda_- t}$.

Solution:
Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

$\mathbf{x}^{(+)}$, $-\mathbf{x}^{(+)}$,

$\mathbf{x}^{(-)}$, $-\mathbf{x}^{(-)}$. 
Example

Let \( \lambda_+ = 4 \), \( \lambda_- = 1 \), \( \mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \), and \( \mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \).

Plot the phase portrait of several linear combinations of the fundamental solutions \( \mathbf{x}^{(+)} = \mathbf{v}^{(+)} e^{\lambda_+ t} \), \( \mathbf{x}^{(-)} = \mathbf{v}^{(-)} e^{\lambda_- t} \),

Solution:
Recall: \( \lambda_+ > \lambda_- > 0 \). We plot the solutions

\[
\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},
\]

that is,

\[
\mathbf{x} = \mathbf{v}^{(+)} e^{4t} + \mathbf{v}^{(-)} e^{t}.
\]
Example
Let \( \lambda_+ = 4, \ \lambda_- = 1, \ v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \) and \( v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \)

Plot the phase portrait of several linear combinations of the fundamental solutions \( x^{(+)} = v^{(+)} e^{\lambda_+ t}, \ x^{(-)} = v^{(-)} e^{\lambda_- t}, \)

Solution:
Recall: \( \lambda_+ > \lambda_- > 0. \) We plot the solutions
\[
x = x^{(+)} + x^{(-)},
\]
that is,
\[
x = v^{(+)} e^{4t} + v^{(-)} e^{t}.
\]

Example
Let \( \lambda_+ = 4, \lambda_- = 1, \mathbf{v}(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \) and \( \mathbf{v}(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \)

Plot the phase portrait of several linear combinations of the fundamental solutions \( \mathbf{x}(+) = \mathbf{v}(+) e^{\lambda_+ t}, \mathbf{x}(-) = \mathbf{v}(-) e^{\lambda_- t}, \)

Solution:
Recall: \( \lambda_+ > \lambda_- > 0. \) We plot the solutions

\[
\mathbf{x} = c_1 \mathbf{x}(+) + c_2 \mathbf{x}(-),
\]

for different values of \( c_1 \)
and \( c_2. \)
Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $v^+(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $v^(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $x^+(+) = v^+(+) e^{\lambda_+ t}$, $x^<>(-) = v^<>(-) e^{\lambda_- t}$,

Solution:
Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$x = c_1 x^+(+) + c_2 x^<>(-),$$

for different values of $c_1$ and $c_2$. 
Example

Let \( \lambda_+ = 4, \lambda_- = -1, \ v(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \) and \( v(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \)

Plot the phase portrait of several linear combinations of the fundamental solutions \( x(+) = v(+) e^{\lambda_+ t}, \ x(-) = v(-) e^{\lambda_- t}, \)
Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $x^{(+) = v^{(+) e^{\lambda_+ t}}$, $x^{(-)} = v^{(-) e^{\lambda_- t}}$.

Solution:
Here $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$x^{(+)}$, $-x^{(+)},$

$x^{(-)}$, $-x^{(-)}$. 
Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $v^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $x^{(+)} = v^{(+)} e^{\lambda_+ t}$, $x^{(-)} = v^{(-)} e^{\lambda_- t}$,

Solution:

Here $\lambda_+ > 0 > \lambda_-$. We plot the solutions $x^{(+)}$, $-x^{(+)}$, $x^{(-)}$, $-x^{(-)}$. 

![Phase portrait diagram]
Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = \mathbf{v}^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = \mathbf{v}^{(-)} e^{\lambda_- t}$, $\mathbf{x} = \mathbf{v}^{(+)} e^{4t} + \mathbf{v}^{(-)} e^{-t}$.

Solution:
Recall: $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)}$,

that is,

$\mathbf{x} = \mathbf{v}^{(+)} e^{4t} + \mathbf{v}^{(-)} e^{-t}$.
Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = \mathbf{v}^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = \mathbf{v}^{(-)} e^{\lambda_- t}$.

Solution:
Recall: $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)} e^{4t} + \mathbf{v}^{(-)} e^{-t}.$$
Example

Let \( \lambda_+ = 4, \lambda_- = -1 \), \( \mathbf{v}(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \), and \( \mathbf{v}(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \).

Plot the phase portrait of several linear combinations of the fundamental solutions \( \mathbf{x}(+) = \mathbf{v}(+) e^{\lambda_+ t} \), \( \mathbf{x}(-) = \mathbf{v}(-) e^{\lambda_- t} \),

Solution:
Recall: \( \lambda_+ > 0 > \lambda_- \). We plot the solutions

\[ \mathbf{x} = c_1 \mathbf{x}(+) + c_2 \mathbf{x}(-), \]

for different values of \( c_1 \) and \( c_2 \).
Example

Let \( \lambda_+ = 4, \lambda_- = -1, \ v(+) = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \) and \( v(-) = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}. \)

Plot the phase portrait of several linear combinations of the fundamental solutions \( x(+) = v(+) e^{\lambda_+ t}, \ x(-) = v(-) e^{\lambda_- t}, \)

Solution:
Recall: \( \lambda_+ > 0 > \lambda_- \). We plot the solutions

\[ x = c_1 x(+) + c_2 x(-), \]

for different values of \( c_1 \) and \( c_2. \)
Extra problem.

Example

Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]
Extra problem.

Example
Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of $A$:
Example
Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = \frac{-1}{2}, \frac{-5}{2}.$$ 

Eigenvector for $\lambda = \frac{-1}{2}$:

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ 

$v_1 = 2v_2$.

Choosing $v_1 = 2$ and $v_2 = 1$, we get $v(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. 

Extra problem.
Example

Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$  

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$
Example

Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}. $$

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (-3 - \lambda) & 4 \\ -1 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$
Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (-3 - \lambda) & 4 \\ -1 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}[-2 \pm \sqrt{4 - 4}]$$
Extra problem.

Example
Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}[-2 \pm \sqrt{4 - 4}] = -1.$$
Extra problem.

Example
Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0
\]

\[
\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} [-2 \pm \sqrt{4 - 4}] = -1.
\]

Hence \( \lambda_+ = \lambda_- = -1 \).
Extra problem.

Example
Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}. $$

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}[-2 \pm \sqrt{4 - 4}] = -1.$$ 

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_{\pm}$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$
Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} (-3 - \lambda) & 4 \\ -1 & (1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 4} \right] = -1.$$ 

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_{\pm}$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}.$$
Extra problem.

**Example**

Find \( \mathbf{x} \) solution of the IVP

\[
\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

**Solution:** Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0
\]

\[
\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} [-2 \pm \sqrt{4 - 4}] = -1.
\]

Hence \( \lambda_{+} = \lambda_{-} = -1 \). Eigenvector for \( \lambda_{\pm} \).

\[
(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.
\]
Extra problem.

Example

Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}[-2 \pm \sqrt{4 - 4}] = -1.$$ 

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_{\pm}$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$ 

$v_1 = 2 v_2.$
Extra problem.

Example
Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Eigenvalues of $A$:

$$p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} [-2 \pm \sqrt{4 - 4}] = -1.$$ 

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_{\pm}$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$ 

$v_1 = 2 v_2$. Choosing $v_1 = 2$
Example

Find \( x \) solution of the IVP

\[
x' = A x, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0
\]

\[
\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 4} \right] = -1.
\]

Hence \( \lambda_+ = \lambda_- = -1 \). Eigenvector for \( \lambda_{\pm} \).

\[
(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.
\]

\( v_1 = 2 v_2 \). Choosing \( v_1 = 2 \) and \( v_2 = 1 \),
Example

Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Eigenvalues of \( A \):

\[
p(\lambda) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0
\]

\[
\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2} [-2 \pm \sqrt{4 - 4}] = -1.
\]

Hence \( \lambda_+ = \lambda_- = -1 \). Eigenvector for \( \lambda_{\pm} \).

\[
(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.
\]

\( \nu_1 = 2 \nu_2 \). Choosing \( \nu_1 = 2 \) and \( \nu_2 = 1 \), we get \( \nu^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).
Extra problem.

Example
Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Recall: $\lambda_\pm = -1$, and $v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$
Example
Find $x$ solution of the IVP
\[ x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}. \]

Solution: Recall: $\lambda_{\pm} = -1$, and $v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $w$ solution of $(A + I)w = v$. 
\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]
Extra problem.

Example

Find \( \mathbf{x} \) solution of the IVP

\[
\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = -1 \), and \( \mathbf{v}(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Find \( \mathbf{w} \) solution of \((A + I)\mathbf{w} = \mathbf{v}\).

\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]
Extra problem.

Example

Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = -1 \), and \( v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Find \( w \) solution of \( (A + I)w = v \).

\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]
Extra problem.

Example
Find \( \mathbf{x} \) solution of the IVP

\[
\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_\pm = -1 \), and \( \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Find \( \mathbf{w} \) solution of \((A + I)\mathbf{w} = \mathbf{v}\).

\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Hence \( w_1 = 2w_2 - 1 \),
Example
Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = -1 \), and \( v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Find \( w \) solution of \((A + I)w = v\).

\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Hence \( w_1 = 2w_2 - 1 \), that is, \( w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).
Example
Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = -1 \), and \( v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Find \( w \) solution of \((A + I)w = v\).

\[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Hence \( w_1 = 2w_2 - 1 \), that is, \( w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).

Choose \( w_2 = 0 \), so \( w = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).
Extra problem.

Example
Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Recall: $\lambda_{\pm} = -1$, $v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. 

Example

Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Recall: $\lambda_{\pm} = -1$, $v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t},$
Extra problem.

Example

Find \( x \) solution of the IVP

\[
x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \( \lambda_{\pm} = -1 \), \( \mathbf{v}(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).

Fundamental sol: \( x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \quad x^{(2)} = \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}. \)
Example

Find \(x\) solution of the IVP

\[
x' = Ax,
\]

\[
x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.
\]

Solution: Recall: \(\lambda_{\pm} = -1\), \(v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\) and \(w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\).

Fundamental sol: \(x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}\), \(x^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{-t}\).

General sol: \(x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{-t}\).
Extra problem.

Example

Find $x$ solution of the IVP

$$x' = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$ 

Solution: Recall: $x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$
Extra problem.

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Initial condition: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$
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Example
Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$  

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that is,  

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$
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that is, \( \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \) also, \( \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \)
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that is, $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, also, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. 
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Find $x$ solution of the IVP

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The solution is $x = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$. \hfill ∎
Extra problem.

Example

Let $\lambda = -1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm \mathbf{x}^{(1)} = \pm \mathbf{v} e^{-t}$ and $\pm \mathbf{x}^{(2)} = \pm (\mathbf{v} t + \mathbf{w}) e^{-t}$.
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Plot $\pm \mathbf{x}^{(1)} = \pm \mathbf{v} e^{-t}$ and $\pm \mathbf{x}^{(2)} = \pm (\mathbf{v} t + \mathbf{w}) e^{-t}$.

Solution:
Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm x^{(1)} = \pm \mathbf{v} e^t$ and $\pm x^{(2)} = \pm (\mathbf{v} t + \mathbf{w}) e^t$. 
Extra problem.

Example

Let $\lambda = 1$ with $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm x^{(1)} = \pm ve^t$ and $\pm x^{(2)} = \pm (vt + w)e^t$.

Solution:
Example
Given any vectors \( \mathbf{a} \) and \( \mathbf{b} \), sketch qualitative phase portraits of

\[
x^{(1)} = \left[ a \cos(\beta t) - b \sin(\beta t) \right] e^{\alpha t}, \quad x^{(2)} = \left[ a \sin(\beta t) + b \cos(\beta t) \right] e^{\alpha t}.
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for the cases \( \alpha = 0 \), and \( \alpha > 0 \), where \( \beta > 0 \).
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Given any vectors $a$ and $b$, sketch qualitative phase portraits of
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Given any vectors $\mathbf{a}$ and $\mathbf{b}$, sketch qualitative phase portraits of
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Solution:
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\begin{align*}
\mathbf{x}^{(1)} &= [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t}, \\
\mathbf{x}^{(2)} &= [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}.
\end{align*}
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