Boundary Value Problems (Sect. 10.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.
Two-point Boundary Value Problem.

Definition

A two-point \textit{BVP} is the following: Given functions \( p, q, g \), and constants 
\[ x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2, \]
find a function \( y \) solution of the differential equation
\[ y'' + p(x) y' + q(x) y = g(x), \]
together with the extra, \textit{boundary conditions},
\[ b_1 \, y(x_1) + b_2 \, y'(x_1) = y_1, \]
\[ \tilde{b}_1 \, y(x_2) + \tilde{b}_2 \, y'(x_2) = y_2. \]
Two-point Boundary Value Problem.

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A *two-point BVP* is the following: Given functions $p$, $q$, $g$, and constants $x_1 < x_2$, $y_1, y_2$, $b_1, b_2$, $\tilde{b}_1, \tilde{b}_2$, find a function $y$ solution of the differential equation
\[ y'' + p(x) y' + q(x) y = g(x), \]
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**Remarks:**
- Both $y$ and $y'$ might appear in the boundary condition, evaluated at the same point.
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\]
 together with the extra, *boundary conditions*,
\[
    b_1 y(x_1) + b_2 y'(x_1) = y_1, \\
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\]

Remarks:
- Both \( y \) and \( y' \) might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,
\[
y'' + a_1 y' + a_0 y = g(x).
\]
Two-point Boundary Value Problem.

Example
Examples of BVP.
Two-point Boundary Value Problem.

Example
Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$
Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

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(2) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$
Two-point Boundary Value Problem.

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Examples of BVP. Assume $x_1 \neq x_2$.

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2. Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$

3. Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2.$$
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- **Example from physics.**
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Example from physics.

**Problem:** The equilibrium (time independent) temperature of a bar of length $L$ with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures $T_0$, $T_L$ is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$
Example from physics.

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- Two-point BVP.
- Example from physics.
- **Comparison: IVP vs BVP.**
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Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation
\[y'' + a_1 y' + a_0 y = g(t),\]

together with the initial conditions
\[y(t_0) = y_1, \quad y'(t_0) = y_2.\]
Comparison: IVP vs BVP.

Review: IVP:
Find the function values \( y(t) \) solutions of the differential equation
\[
y'' + a_1 y' + a_0 y = g(t),
\]
together with the initial conditions
\[
y(t_0) = y_1, \quad y'(t_0) = y_2.
\]

Remark: In physics:
\( y(t) \): Position at time \( t \).
Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1 y' + a_0 y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Remark: In physics:
- $y(t)$: Position at time $t$.
- Initial conditions: Position and velocity at the initial time $t_0$. 
Comparison: IVP vs BVP.

Review: BVP:
Find the function values $y(x)$ solutions of the differential equation

$$y'' + a_1 y' + a_0 y = g(x),$$

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Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position $x$. 
Comparison: IVP vs BVP.

Review: BVP:
Find the function values \( y(x) \) solutions of the differential equation

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\]

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\[
y(x_1) = y_1, \quad y(x_2) = y_2.
\]

Remark: In physics:
- \( y(x) \): A physical quantity (temperature) at a position \( x \).
- **Boundary conditions**: Conditions at the boundary of the object under study, where \( x_1 \neq x_2 \).
Boundary Value Problems (Sect. 10.1).

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Existence, uniqueness of solutions to BVP.

**Review:** The initial value problem.

**Theorem (IVP)**

Consider the homogeneous initial value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \]

and let \( r_{\pm} \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.
Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)

Consider the homogeneous initial value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \]

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If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.

Summary: The IVP above always has a unique solution, no matter what \( y_0 \) and \( y_1 \) we choose.
Existence, uniqueness of solutions to BVP.

Theorem (BVP)

Consider the homogeneous boundary value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(0) = y_0, \quad y(L) = y_1, \]

and let \( r_{\pm} \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

(A) If \( r_+ \neq r_- \), real, then for every choice of \( L \neq 0 \) and \( y_0, y_1 \), there exists a unique solution \( y \) to the BVP above.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \beta \neq 0 \), and \( \alpha, \beta \in \mathbb{R} \), then the solutions to the BVP above belong to one of these possibilities:

1. There exists a unique solution.
2. There exists no solution.
3. There exist infinitely many solutions.
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case \( r_+ \neq r_- \).
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case \( r_+ \neq r_- \). The general solution is

\[
y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t},
\]

where \( c_1, c_2 \in \mathbb{R} \). The initial conditions determine \( c_1 \) and \( c_2 \) as follows:

\[
y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0},
\]

\[
y_1 = y'(t_0) = c_1 (r_- - e^{r_- t_0}) + c_2 (r_+ + e^{r_+ t_0}).
\]

Using matrix notation,

\[
\begin{bmatrix}
    e^{r_- t_0} & e^{r_+ t_0} \\
    r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2
\end{bmatrix}
= 
\begin{bmatrix}
    y_0 \\
    y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff

\[
\text{det}(Z) \neq 0,
\]

where

\[
Z = 
\begin{bmatrix}
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Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}.$$
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$$y_0 = y(t_0)$$
Existence, uniqueness of solutions to BVP.

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y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0}
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\[
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Existence, uniqueness of solutions to BVP.

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$$y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}.$$ 

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$$y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0}$$

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Existence, uniqueness of solutions to BVP.

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$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$ 

The linear system above has a unique solution $c_1$ and $c_2$ for every constants $y_0$ and $y_1$ iff

$$\det(Z) \neq 0,$$ 

where $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

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Existence, uniqueness of solutions to BVP.

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y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}.
\]
The initial conditions determine \( c_1 \) and \( c_2 \) as follows:
\[
y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0}
\]
\[
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Using matrix notation,
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y_0 \\
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\]
The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \text{det}(Z) \neq 0 \),
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

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Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\
 r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\
 c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\
 y_1 \end{bmatrix}. $$

The linear system above has a unique solution $c_1$ and $c_2$ for every constants $y_0$ and $y_1$ iff the det($Z$) $\neq 0$, where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\
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Existence, uniqueness of solutions to BVP.

**Proof of IVP:** We study the case \( r_+ \neq r_- \). The general solution is

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y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}.
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  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix}
\Rightarrow 
Z \begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= 
\begin{bmatrix}
  y_0 \\
  y_1
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\]
Existence, uniqueness of solutions to BVP.

Proof of IVP:
Recall: \( Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).
Existence, uniqueness of solutions to BVP.

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Recall: \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[
\text{det}(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0}
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP:
Recall: \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[
\det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.
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Existence, uniqueness of solutions to BVP.

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Recall: \[ Z = \begin{bmatrix} e^{r_-t_0} & e^{r_+t_0} \\ r_- e^{r_-t_0} & r_+ e^{r_+t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

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Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible
Proof of IVP:

Recall:  \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[ \det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \neq 0 \iff r_+ \neq r_. \]

Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so

\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]
Existence, uniqueness of solutions to BVP.

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Recall: \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

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Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so
\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]

We conclude that for every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the IVP above has a unique solution. \( \square \)
Existence, uniqueness of solutions to BVP.

**Proof of BVP:** The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 e^{r^- x} + c_2 e^{r^+ x}, \quad c_1, c_2 \in \mathbb{R}.$$
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}.$$ 

The boundary conditions determine $c_1$ and $c_2$
Existence, uniqueness of solutions to BVP.

**Proof of BVP:** The general solution is

\[ y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

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\[ y_0 = y(0) = c_1 + c_2. \]

\[ y_1 = y(L) \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine $c_1$ and $c_2$ as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L}$$
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]

\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L} 
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1 
\end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is
\[ y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:
\[ y_0 = y(0) = c_1 + c_2. \]
\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,
\[
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\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]
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Using matrix notation,

\[
\begin{bmatrix}
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\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \),
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]
\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} =
\begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where

\[
Z =
\begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix}
\]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r^- x} + c_2 e^{r^+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]

\[ y_1 = y(L) = c_1 e^{r^- L} + c_2 e^{r^+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
e^{r^- L} & e^{r^+ L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} =
\begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where

\[
Z = \begin{bmatrix}
1 & 1 \\
e^{r^- L} & e^{r^+ L}
\end{bmatrix} \Rightarrow Z \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \)
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows

\[ \det(Z) = e^{r_+ L} - e^{r_- L} \]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows
\[
\text{det}(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows
\[
\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued,

(1) If \( \beta L \neq n\pi \), then BVP has a unique solution.
(2) If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions.
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0. \)
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_-L} & e^{r_+L} \end{bmatrix} \Rightarrow \begin{bmatrix} Z \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r_+L} - e^{r_-L} \neq 0 \iff e^{r_+L} \neq e^{r_-L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{-L} & e^{L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[
\det(Z) = e^{L} - e^{-L} \neq 0 \iff e^{L} \neq e^{-L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \),
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_-L} & e^{r_+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \)

A simple calculation shows
\[
\det(Z) = e^{r_+L} - e^{r_-L} \neq 0 \iff e^{r_+L} \neq e^{r_-L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0. \)

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[
\det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L})
\]
Existence, uniqueness of solutions to BVP.

**Proof of IVP:** Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows

\[
\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
\]

**(A)** If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

**(B)** If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then

\[
\det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L).
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{-rL} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows

\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then

\[ \det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then

\[ \det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,

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Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall:  \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[ \det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,

1. If \( \beta L \neq n\pi \), then BVP has a unique solution.
2. If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions. \( \square \)
Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.
\]
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1
\]
Existence, uniqueness of solutions to BVP.

Example

Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$ 

Solution: The characteristic polynomial is 

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$ 

The general solution is 

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$
Existence, uniqueness of solutions to BVP.

**Example**

Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

**Solution:** The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$
Existence, uniqueness of solutions to BVP.

Example

Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

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$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1.$$
Existence, uniqueness of solutions to BVP.

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The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.
\]

The general solution is

\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]

The boundary conditions are

\[
1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}
\]

We conclude: \( y(x) = \cos(x) + c_2 \sin(x) \), with \( c_2 \in \mathbb{R} \).
Existence, uniqueness of solutions to BVP.

Example

Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$ 

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free}.$$ 

We conclude: $y(x) = \cos(x) + c_2 \sin(x)$, with $c_2 \in \mathbb{R}$. 

The BVP has infinitely many solutions. \hspace{1cm} \triangleleft
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
\[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0. \]

Solution: The characteristic polynomial is
\[ p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i. \]

The general solution is
\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
\[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0. \]

Solution: The characteristic polynomial is
\[ p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i. \]

The general solution is
\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]

The boundary conditions are
\[ 1 = y(0) = c_1, \]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \implies r_\pm = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$ 

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$

The BVP has no solution.  \[\triangle\]
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP
\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.
\]

Solution:
The characteristic polynomial is
\[
p(r) = r^2 + 1 \Rightarrow r = \pm i.
\]
The general solution is
\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]
The boundary conditions are
\[
1 = y(0) = c_1,
\]
\[
1 = y(\pi/2) = c_2 \Rightarrow c_1 = c_2 = 1.
\]
We conclude:
\[
y(x) = \cos(x) + \sin(x).
\]
The BVP has a unique solution.
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

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$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2$$
Existence, uniqueness of solutions to BVP.

**Example**
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

**Solution:**
The characteristic polynomial is

$$p(r) = r^2 + 1 \implies r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \implies c_1 = c_2 = 1.$$
Example

Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.
\]

The general solution is

\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]

The boundary conditions are

\[
1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.
\]

We conclude: \( y(x) = \cos(x) + \sin(x) \).
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
\[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1. \]

Solution: The characteristic polynomial is
\[ p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i. \]

The general solution is
\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]

The boundary conditions are
\[ 1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1. \]

We conclude: $y(x) = \cos(x) + \sin(x)$.

The BVP has a unique solution.
Boundary Value Problems (Sect. 10.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.
Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number $\lambda$ and a non-zero function $y$ solutions to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$
Particular case of BVP: Eigenvalue-eigenfunction problem.

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Remarks: We will show that:

(1) If $\lambda \leq 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues $\lambda_n$ and eigenfunctions $y_n$, with $n$ any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi}{L}x\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0$, $y'(L) = 0$; or for $y'(0) = 0$, $y'(L) = 0$. 
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Since $y = 0$, there are NO non-zero solutions for $\lambda = 0.$
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We need to solve the linear system

$$
\begin{bmatrix}
1 & 1 \\
e^{\mu L} & e^{-\mu L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L > 0$, matrix $Z$ is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$. Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$. 
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Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

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$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$  \quad \triangleq
Overview of Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.
Origins of the Fourier Series.

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Fourier found particular solutions to the heat equation.

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Fourier also realized that $U_N(t,x) = \sum_{n=1}^{N} a_n \sin(n\pi x L) e^{-k(n\pi L)^2 t}$, $a_n \in \mathbb{R}$ is also solution of the heat equation with initial condition $F_N(x) = \sum_{n=1}^{N} a_n \sin(n\pi x L)$.

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Given an initial data function $F$, satisfying $F(0) = F(L) = 0$, but otherwise arbitrary, Fourier proved that one can construct an expansion $F_N$ as follows,

$$F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n \pi x}{L}\right),$$

for $N$ any positive integer, where the $a_n$ are given by

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To find all solutions to the heat equation problem above one must prove one more thing: That $F_N$ approximates $F$ for large enough $N$. That is, $\lim_{N \to \infty} F_N = F$. Fourier didn't show this.
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Origins of the Fourier Series.

Remarks:

- However, Fourier went farther than Bernoulli. Fourier found a formula for the coefficients $a_n$ in terms of the function $F$.

- Given an initial data function $F$, satisfying $F(0) = F(L) = 0$, but otherwise arbitrary, Fourier proved that one can construct an expansion $F_N$ as follows,

$$F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right),$$

for $N$ any positive integer, where the $a_n$ are given by

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Remarks:

- Based on Bernoulli and Fourier works, people have been able to prove that.

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F(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right],
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satisfies

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for every \(x \in \mathbb{R}\).
Origins of the Fourier Series.

Remarks:
- Based on Bernoulli and Fourier works, people have been able to prove that. Every continuous, $\tau$-periodic function can be expressed as an infinite linear combination of sine and cosine functions.
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▶ Based on Bernoulli and Fourier works, people have been able to prove that. Every continuous, $\tau$-periodic function can be expressed as an infinite linear combination of sine and cosine functions.

▶ More precisely: Every continuous, $\tau$-periodic function $F$, there exist constants $a_0$, $a_n$, $b_n$, for $n = 1, 2, \cdots$ such that

$$F_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right],$$

satisfies $\lim_{N \to \infty} F_N(x) = F(x)$ for every $x \in \mathbb{R}$. 
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- Based on Bernoulli and Fourier works, people have been able to prove that. Every continuous, \( \tau \)-periodic function can be expressed as an infinite linear combination of sine and cosine functions.

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Notation: \( F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{2n\pi x}{\tau} \right) + b_n \sin \left( \frac{2n\pi x}{\tau} \right) \right] \).
The main problem in our class:
Given a continuous, $\tau$-periodic function $f$, find the formulas for $a_n$ and $b_n$ such that

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Remarks: We need to review two main concepts:

- The notion of periodic functions.
- The notion of orthogonal functions, in particular the orthogonality of Sines and Cosines.
Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- **Periodic functions.**
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.
Periodic functions.

Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$
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A \textit{period} \( T \) of a periodic function \( f \) is the smallest value of \( \tau \) such that \( f(x + \tau) = f(x) \) holds.
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Definition
A *period* $T$ of a periodic function $f$ is the smallest value of $\tau$ such that $f(x + \tau) = f(x)$ holds.

Notation:
A periodic function with period $T$ is also called $T$-periodic.
Periodic functions.

Example

The following functions are periodic, with period $T$,

\[
\begin{align*}
    f(x) &= \sin(x), & T &= 2\pi. \\
    f(x) &= \cos(x), & T &= 2\pi. \\
    f(x) &= \tan(x), & T &= \pi. \\
    f(x) &= \sin(ax), & T &= \frac{2\pi}{a}.
\end{align*}
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$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right)$$
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    &= \sin(ax + 2\pi) \\
    &= \sin(ax) \\
    &= f(x).
\end{align*}
\]
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Example
Show that the function below is periodic, and find its period,

\[ f(x) = e^x, \quad x \in [0, 2), \quad f(x - 2) = f(x). \]
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Show that the function below is periodic, and find its period,

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Solution: We just graph the function,

So the function is periodic with period \( T = 2 \).
Periodic functions.

Theorem

A linear combination of $T$-periodic functions is also $T$-periodic.

Example

$f(x) = 2 \sin(3x) + 7 \cos(3x)$ is periodic with period $T = \frac{2\pi}{3}$.

Remark:

The functions below are periodic with period $T = \tau n$,

$f(x) = \cos(2\pi nx \tau)$,

$g(x) = \sin(2\pi nx \tau)$,

Since $f$ and $g$ are invariant under translations by $\tau/n$, they are also invariant under translations by $\tau$. 
Periodic functions.

**Theorem**

*A linear combination of $T$-periodic functions is also $T$-periodic.*

**Proof:** If $f(x + T) = f(x)$ and $g(x + T) = g(x)$, then

$$af(x + T) + bg(x + T) = af(x) + bg(x),$$

so $(af + bg)$ is also $T$-periodic. \hfill \square
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Remark: The functions below are periodic with period $T = \frac{\tau}{n}$,

$$f(x) = \cos\left(\frac{2\pi nx}{\tau}\right), \quad g(x) = \sin\left(\frac{2\pi nx}{\tau}\right),$$
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**Theorem**
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Since $f$ and $g$ are invariant under translations by $\tau/n$, they are also invariant under translations by $\tau$.\(\triangleleft\)
Periodic functions.

Corollary

Any function \( f \) given by

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f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right]
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Remark: We will show that the converse statement is true.
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Theorem

A function $f$ is $\tau$-periodic iff holds

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right].$$
Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- Main result on Fourier Series.
Orthogonality of Sines and Cosines.

Remark:
From now on we work on the following domain: \([-L, L]\).
Orthogonality of Sines and Cosines.

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Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all \( n, m \in \mathbb{N} \),

\[
\int_{-L}^{L} \cos\left( \frac{n\pi x}{L} \right) \cos\left( \frac{m\pi x}{L} \right) \, dx = \left\{ \begin{array}{ll}
0 & n \neq m, \\
L & n = m \neq 0, \\
2L & n = m = 0,
\end{array} \right.
\]

\[
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Remark:

The operation \( f \cdot g = \int_{-L}^{L} f(x) g(x) \, dx \) is an inner product in the vector space of functions. Like the dot product in \( \mathbb{R}^2 \).

Two functions \( f, g \), are orthogonal iff \( f \cdot g = 0 \).
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- The operation $f \cdot g = \int_{-L}^{L} f(x) g(x) \, dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$. 
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- The operation $f \cdot g = \int_{-L}^{L} f(x) g(x) \, dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions $f, g$, are orthogonal iff $f \cdot g = 0$. 
Orthogonality of Sines and Cosines.

Recall: \[ \cos(\theta) \cos(\phi) = \frac{1}{2} \left[ \cos(\theta + \phi) + \cos(\theta - \phi) \right]; \]
\[ \sin(\theta) \sin(\phi) = \frac{1}{2} \left[ \cos(\theta - \phi) - \cos(\theta + \phi) \right]; \]
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Proof: First formula:
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Proof: First formula: If \( n = m = 0 \), it is simple to see that

\[ \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \int_{-L}^{L} \, dx = 2L. \]
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\]
In the case where one of \( n \) or \( m \) is non-zero, use the relation
\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n + m)\pi x}{L} \right) \, dx \\
+ \frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n - m)\pi x}{L} \right) \, dx.
\]
Orthogonality of Sines and Cosines.

Proof: Since one of $n$ or $m$ is non-zero,
Orthogonality of Sines and Cosines.

Proof: Since one of \( n \) or \( m \) is non-zero, holds

\[
\frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n + m)\pi x}{L} \right) \, dx = \frac{L}{2(n + m)\pi} \sin \left( \frac{(n + m)\pi x}{L} \right) \bigg|_{-L}^{L} = 0.
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Proof: Since one of $n$ or $m$ is non-zero, holds

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We obtain that

$$
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\]

If we further restrict \( n \neq m \), then

\[
\frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n - m) \pi x}{L} \right) \, dx = \frac{L}{2(n - m) \pi} \sin \left[ \frac{(n - m) \pi x}{L} \right] \bigg|_{-L}^{L} = 0.
\]
Orthogonality of Sines and Cosines.

Proof: Since one of \( n \) or \( m \) is non-zero, holds

\[
\frac{1}{2} \int_{-L}^{L} \cos \left[ \frac{(n + m)\pi x}{L} \right] \, dx = \frac{L}{2(n + m)\pi} \sin \left[ \frac{(n + m)\pi x}{L} \right] \bigg|_{-L}^{L} = 0.
\]

We obtain that

\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \cos \left[ \frac{(n - m)\pi x}{L} \right] \, dx.
\]

If we further restrict \( n \neq m \), then

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\]

If \( n = m \neq 0 \), we have that

\[
\frac{1}{2} \int_{-L}^{L} \cos \left[ \frac{(n - m)\pi x}{L} \right] \, dx = \frac{1}{2} \int_{-L}^{L} \, dx = L.
\]
Orthogonality of Sines and Cosines.

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\]

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. \( \square \)
Overview of Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- **Main result on Fourier Series.**
Main result on Fourier Series.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then $f$ can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left( \frac{n\pi x}{L} \right) + b_n \sin\left( \frac{n\pi x}{L} \right) \right] \quad (1)$$

with the constants $a_n$ and $b_n$ given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left( \frac{n\pi x}{L} \right) \, dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left( \frac{n\pi x}{L} \right) \, dx, \quad n \geq 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$-periodic extension of $f$ from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$. 
Examples of the Fourier Theorem (Sect. 10.3).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.
The Fourier Theorem: Continuous case.

**Theorem (Fourier Series)**

*If the function $f : [-L, L] \subset \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $f$ can be expressed as an infinite series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$  \hspace{1cm} (2)

*with the constants $a_n$ and $b_n$ given by*

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.$$

*Furthermore, the Fourier series in Eq. (2) provides a $2L$-periodic extension of function $f$ from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.***
The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

\[ f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \]

- Express \( f_N \) as a convolution of Sine, Cosine, functions and the original function \( f \).

- Use the convolution properties to show that \( \lim_{N \to \infty} f_N(x) = f(x), x \in [-L, L] \).
The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

\[ f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \]

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\[ \lim_{N \to \infty} f_N(x) = f(x), \quad x \in [-L, L]. \]
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- The Fourier Theorem: Continuous case.
- **Example: Using the Fourier Theorem.**
- The Fourier Theorem: Piecewise continuous case.
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Example: Using the Fourier Theorem.

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: In this case \( L = 1 \).

The Fourier series expansion is

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right], \]

where the \( a_n \), \( b_n \) are given in the Theorem.

We start with \( a_0 \),

\[ a_0 = \frac{1}{2} \left[ \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx \right]. \]

We obtain:

\[ a_0 = 1. \]
Example: Using the Fourier Theorem.

Example

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\[ a_0 = \left( x + \frac{x^2}{2} \right) \bigg|_{-1}^{0} + \left( x - \frac{x^2}{2} \right) \bigg|_{0}^{1} \]
Example: Using the Fourier Theorem.

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We obtain: \( a_0 = 1 \).
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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1 + x & x \in [-1, 0), \\
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Solution: Recall: \( a_0 = 1 \). Similarly, the rest of the \( a_n \) are given by,

\[ a_n = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx \]
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Find the Fourier series expansion of the function

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\[ a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n\pi x) \, dx. \]
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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\[ a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n\pi x) \, dx. \]

Recall the integrals \( \int \cos(n\pi x) \, dx = \frac{1}{n\pi} \sin(n\pi x) \),
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
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Recall the integrals \( \int \cos(n\pi x) \, dx = \frac{1}{n\pi} \sin(n\pi x) \), and

\[ \int x \cos(n\pi x) \, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x). \]
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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\end{cases} \]

Solution: It is not difficult to see that

\[
a_n = \frac{1}{n\pi} \sin(n\pi x) \bigg|_{-1}^{0} + \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{-1}^{0} \\
+ \frac{1}{n\pi} \sin(n\pi x) \bigg|_{0}^{1} - \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{0}^{1}
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Example: Using the Fourier Theorem.

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\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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\]

\[
= \left[ \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi) \right] - \left[ \frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2} \right].
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\[
+ \frac{1}{n\pi} \sin(n\pi x) \bigg|_0^1 - \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_0^1
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\]

We then conclude that \( a_n = \frac{2}{n^2\pi^2} \left[ 1 - \cos(n\pi) \right]. \)
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2\pi^2} \left[ 1 - \cos(n\pi) \right] \).
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Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2 \pi^2} [1 - \cos(n\pi)] \).

Finally, we must find the coefficients \( b_n \).
Example: Using the Fourier Theorem.

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A similar calculation shows that \( b_n = 0 \).
Example: Using the Fourier Theorem.

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Finally, we must find the coefficients \( b_n \).

A similar calculation shows that \( b_n = 0 \).

Then, the Fourier series of \( f \) is given by

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[ 1 - \cos(n\pi) \right] \cos(n\pi x). \]
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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We can obtain a simpler expression for the Fourier coefficients \( a_n \).
Example: Using the Fourier Theorem.

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We can obtain a simpler expression for the Fourier coefficients \( a_n \).

Recall the relations \( \cos(n \pi) = (-1)^n \).
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[ 1 + (-1)^{n+1} \right] \cos(n\pi x). \]
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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1 + x & x \in [-1, 0), \\
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Solution: Recall: 

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 + (-1)^{n+1}] \cos(n\pi x). \]
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If \( n = 2k \), so \( n \) is even, so \( n + 1 = 2k + 1 \) is odd,
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\[ a_{2k-1} = \frac{2}{(2k - 1)^2 \pi^2} (1 + 1) \Rightarrow a_{2k-1} = \frac{4}{(2k - 1)^2 \pi^2}. \]
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We conclude:

\[ f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k - 1)^2 \pi^2} \cos((2k - 1) \pi x). \]
Examples of the Fourier Theorem (Sect. 10.3).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
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The Fourier Theorem: Piecewise continuous case.

Recall:

Definition
A function \( f : [a, b] \rightarrow \mathbb{R} \) is called \textit{piecewise continuous} iff holds,

(a) \([a, b]\) can be partitioned in a finite number of sub-intervals such that \( f \) is continuous on the interior of these sub-intervals.

(b) \( f \) has finite limits at the endpoints of all sub-intervals.
The Fourier Theorem: Piecewise continuous case.

**Theorem (Fourier Series)**

If \( f : [-L, L] \subset \mathbb{R} \rightarrow \mathbb{R} \) is piecewise continuous, then the function

\[
f_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]
\]

where \( a_n \) and \( b_n \) given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.
\]

satisfies that:

(a) \( f_f(x) = f(x) \) for all \( x \) where \( f \) is continuous;

(b) \( f_f(x_0) = \frac{1}{2} \left[ \lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right] \) for all \( x_0 \) where \( f \) is discontinuous.
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Find the Fourier series of \( f(x) = \begin{cases} 
-1 & x \in [-1, 0), \\
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a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx,
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Solution: Recall: \( b_{2k} = 0 \), and \( b_{2k} = \frac{4}{(2k - 1)\pi} \).

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad L = 1,
\]

\[
a_n = \int_{-1}^{0} (-1) \cos(n\pi x) \, dx + \int_{0}^{1} (1) \cos(n\pi x) \, dx,
\]

\[
a_n = \frac{(-1)}{n\pi} \left[ \sin(n\pi x) \right]_{-1}^{0} + \frac{1}{n\pi} \left[ \sin(n\pi x) \right]_{0}^{1},
\]

\[
a_n = \frac{(-1)}{n\pi} [0 - \sin(-n\pi)] + \frac{1}{n\pi} [\sin(n\pi) - 0]
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Example: Using the Fourier Theorem.

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Therefore, we conclude that

\[
f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)} \sin((2k - 1)\pi x).
\]