Review for Exam 2.

▶ 5 or 6 problems.
▶ No multiple choice questions.
▶ No notes, no books, no calculators.
▶ Problems similar to homeworks.
▶ Exam covers:
  ▶ Regular-singular points (5.5).
  ▶ Euler differential equation (5.4).
  ▶ Power series solutions (5.2).
  ▶ Variation of parameters (3.6).
  ▶ Undetermined coefficients (3.5)
  ▶ Constant coefficients, homogeneous, (3.1)-(3.4).
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  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).
Regular-singular points (5.5).

Summary:

- Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)} \).
Regular-singular points (5.5).

Summary:

▷ Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \).

▷ Recall: Since \( r \neq 0 \), holds

\[
y' = \sum_{n=0}^{\infty} (n+r) a_n (x - x_0)^{n+r-1} \neq \sum_{n=1}^{\infty} (n+r) a_n (x - x_0)^{n+r-1},
\]

(a) If \( (r+ - r- -) \) is not an integer, then each \( r+ \) and \( r- \) define linearly independent solutions.

(b) If \( (r+ - r- -) \) is an integer, then both \( r+ \) and \( r- \) define proportional solutions.
Regular-singular points (5.5).

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  \]

- Find the indicial equation for \( r \), the recurrence relation for \( a_n \).
Regular-singular points (5.5).

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- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.

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- Find the indicial equation for $r$, the recurrence relation for $a_n$.

- Introduce the larger root $r_+$ of the indicial polynomial into the recurrence relation and solve for $a_n$. 

- If $(r_+ - r_-)$ is not an integer, then each $r_+$ and $r_-$ define linearly independent solutions.

- If $(r_+ - r_-)$ is an integer, then both $r_+$ and $r_-$ define proportional solutions.
Regular-singular points (5.5).

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▶ Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.

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▶ Find the indicial equation for $r$, the recurrence relation for $a_n$.

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Regular-singular points (5.5).

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- Find the indicial equation for $r$, the recurrence relation for $a_n$.

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(a) If $(r_+ - r_-)$ is not an integer, then each $r_+$ and $r_-$ define linearly independent solutions.

(b) If $(r_+ - r_-)$ is an integer, then both $r_+$ and $r_-$ define proportional solutions.
Example
Consider the equation $x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, 

\[
\begin{align*}
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Regular-singular points (5.5).

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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 
\[
\begin{align*}
y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2},
\end{align*}
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Example
Consider the equation \( x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( y = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \quad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r-2)}, \)

\[ x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r)} \]
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r-2)}$,

$$x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r)}$$

We also need to compute

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$$
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Solution: $\left(x^2 + \frac{1}{4}\right)y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$. 


Regular-singular points (5.5).

Example
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Re-label $m = n + 2$ in the first term and then switch back to $n$, ...
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Consider the equation $x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left( x^2 + \frac{1}{4} \right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$.

Re-label $m = n + 2$ in the first term and then switch back to $n$,

$\left( x^2 + \frac{1}{4} \right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$,
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \((x^2 + \frac{1}{4}) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}\).

Re-label $m = n + 2$ in the first term and then switch back to $n$, \((x^2 + \frac{1}{4}) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}\),

The equation is \(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0\).
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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:
$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$
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Solution:
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\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.
\]

\[
\left[ r(r-1) + \frac{1}{4} \right] a_0 x^r + \left[ (r+1)r + \frac{1}{4} \right] a_1 x^{(r+1)} + \sum_{n=2}^{\infty} \left[ (n+r)(n+r-1)a_n + a_{(n-2)} + \frac{1}{4} a_n \right] x^{(n+r)} = 0.
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Regular-singular points (5.5).

Example
Consider the equation \(x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0\). Use a power series centered at the regular-singular point \(x_0 = 0\) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:
\[
\begin{align*}
[r(r - 1) + \frac{1}{4}] a_0 &= 0, \\
(r + 1)r + \frac{1}{4} a_1 &= 0, \\
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The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r_\pm = \frac{1}{2}$. 


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Solution: \[
\left[ r(r - 1) + \frac{1}{4} \right] a_0 = 0, \quad \left[ (r + 1)r + \frac{1}{4} \right] a_1 = 0, \quad \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n + a_{n-2} = 0.
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The indicial equation \( r^2 - r + \frac{1}{4} = 0 \) implies \( r_\pm = \frac{1}{2} \).

The indicial equation \( r^2 + r + \frac{1}{4} = 0 \) implies \( r_\pm = -\frac{1}{2} \).
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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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Choose \( r = \frac{1}{2} \).
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Choose \( r = \frac{1}{2} \). That implies \( a_0 \) arbitrary.
Regular-singular points (5.5).

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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 

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\begin{align*}
(r(r-1) + \frac{1}{4}) a_0 &= 0, \\
(r+1)r + \frac{1}{4} a_1 &= 0, \\
(n+r)(n+r-1) + \frac{1}{4} a_n + a_{n-2} &= 0.
\end{align*}
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The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r_{\pm} = \frac{1}{2}$.

The indicial equation $r^2 + r + \frac{1}{4} = 0$ implies $r_{\pm} = -\frac{1}{2}$.

Choose $r = \frac{1}{2}$. That implies $a_0$ arbitrary and $a_1 = 0$. 
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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \frac{1}{2}, \quad a_1 = 0, \quad \left[(n + r)(n + r - 1) + \frac{1}{4}\right]a_n = -a_{(n-2)}$. 
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Consider the equation \(x^2 y'' + \left(x^2 + \frac{1}{4}\right)y = 0\). Use a power series centered at the regular-singular point \(x_0 = 0\) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \(r = \frac{1}{2}, \ a_1 = 0, \ \left[(n + r)(n + r - 1) + \frac{1}{4}\right]a_n = -a_{(n-2)}.\)

\[\left[(n+\frac{1}{2})(n-\frac{1}{2}) + \frac{1}{4}\right]a_n = -a_{(n-2)}\]
Regular-singular points (5.5).

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Consider the equation \( x^2 y'' + \left(x^2 + \frac{1}{4}\right)y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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\[
\left[(n+\frac{1}{2})(n-\frac{1}{2})+\frac{1}{4}\right]a_n = -a_{(n-2)} \Rightarrow \left[n^2 - \frac{1}{4} + \frac{1}{4}\right]a_n = -a_{(n-2)}
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Example

Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)}. \)

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\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \Rightarrow \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\( n^2 a_n = -a_{(n-2)} \)
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left(x^2 + \frac{1}{4}\right)y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ [\left(n + r\right)(n + r - 1) + \frac{1}{4}] a_n = -a_{(n-2)}. \)

\[
\left[\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) + \frac{1}{4}\right] a_n = -a_{(n-2)} \Rightarrow \ [n^2 - \frac{1}{4} + \frac{1}{4}] a_n = -a_{(n-2)}
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\[
n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2}
\]
Regular-singular points (5.5).

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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \quad a_1 = 0 \), \( \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)}. \)

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\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \Rightarrow \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\[
n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2} \Rightarrow \begin{cases} a_2 = -\frac{a_0}{4}, \\ a_4 = -\frac{a_2}{16} \end{cases}
\]
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2} \), \( a_1 = 0 \), \( \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)} \).

\[
\left[ \left( n+\frac{1}{2} \right) \left( n-\frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \quad \Rightarrow \quad \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\[ n^2 a_n = -a_{(n-2)} \quad \Rightarrow \quad a_n = -\frac{a_{(n-2)}}{n^2} \quad \Rightarrow \quad \left\{ \begin{array}{l}
a_2 = -\frac{a_0}{4}, \\
a_4 = -\frac{a_2}{16} = \frac{a_0}{64}.
\end{array} \right. \]
Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \frac{1}{2}, \ a_1 = 0, \ a_2 = -\frac{a_0}{4}, \ \text{and} \ a_4 = \frac{a_0}{64}$. 
Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ a_2 = -\frac{a_0}{4}, \) and \( a_4 = \frac{a_0}{64} \). Then,

\[
y(x) = x^r \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).
\]
Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ a_2 = -\frac{a_0}{4}, \) and \( a_4 = \frac{a_0}{64} \). Then,

\[
y(x) = x^r \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).
\]

Recall: \( a_1 = 0 \) and the recurrence relation imply \( a_n = 0 \) for \( n \) odd.
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \frac{1}{2}$, $a_1 = 0$, $a_2 = -\frac{a_0}{4}$, and $a_4 = \frac{a_0}{64}$. Then,

$$y(x) = x^r \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for $n$ odd. Therefore,

$$y(x) = a_0 x^{1/2} \left(1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 + \cdots \right).$$

\(\triangle\)
Review for Exam 2.

- 5 problems.
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- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
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  - Constant coefficients, homogeneous, (3.1)-(3.4).
Euler differential equation (5.4).

Summary:

\[
(x - x_0)^2 y'' + (x - x_0) p_0 y' + q_0 y = 0.
\]
Euler differential equation (5.4).

Summary:

- \((x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0\).
- Find \(r_{\pm}\) solutions of \(r(r - 1) + p_0 r + q_0 = 0\).
Euler differential equation (5.4).

Summary:

- $(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0$.
- Find $r_{\pm}$ solutions of $r(r - 1) + p_0 r + q_0 = 0$.
- If $r_+ \neq r_-$ and both are real, then fundamental solutions are
  
  $$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}.$$
Euler differential equation (5.4).

Summary:

- \((x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0\).
- Find \(r_\pm\) solutions of \(r(r-1) + p_0 r + q_0 = 0\).
- If \(r_+ \neq r_-\) and both are real, then fundamental solutions are
  \[ y_+ = |x - x_0|^r, \quad y_- = |x - x_0|^{-r}. \]
- If \(r_\pm = \alpha \pm i\beta\), then real-valued fundamental solutions are
  \[ y_+ = |x - x_0|^\alpha \cos(\beta \ln |x - x_0|), \quad y_- = |x - x_0|^\alpha \sin(\beta \ln |x - x_0|). \]
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\[r(r - 1) + 5r + 8 = 0,\]

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Example
Using a power series centered at $x_0 = 0$ find the three first terms of the general solution of $(4 - x^2) y'' + 2y = 0$. 

Solution:
We look for solutions $y = \sum_{n=0}^{\infty} a_n x^n$. Therefore, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$. The differential equation is then given by

$$(4 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

which simplifies to

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\[ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \]
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Use the variation of parameters to find the general solution of

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Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2 \),
Variation of parameters (3.6).

Example
Use the variation of parameters to find the general solution of
\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

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Now we find the functions \( u_1 \) and \( u_2 \),
\[ u'_1 = -\frac{y_2 g}{W} \]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

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Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2 \),

\[
\quad u'_1 = -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}}.
\]
Variation of parameters (3.6).

Example
Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x} \), \( y_2 = x e^{-2x} \), \( g = x^{-2} e^{-2x} \), \( W = e^{-4x} \).

Now we find the functions \( u_1 \) and \( u_2 \),

\[
\begin{align*}
u_1' &= -\frac{y_2 g}{W} = -\frac{xe^{-2x}}{e^{-4x}} = -\frac{1}{x}, \\
u_2' &= y_1 g = xe^{-2x}.
\end{align*}
\]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x \ e^{-2x}, \ g = x^{-2} \ e^{-2x}, \ \mathcal{W} = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)

\[
\begin{align*}
  u'_1 &= -\frac{y_2 g}{W} = \frac{-x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.
\end{align*}
\]
Variation of parameters (3.6).

Example
Use the variation of parameters to find the general solution of
\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)

\[ u'_1 = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \Rightarrow u_1 = - \ln |x|. \]

\[ u'_2 = \frac{y_1 g}{W} \]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2 \),

\[
  \begin{align*}
  u'_1 &= -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln |x|. \\
  u'_2 &= \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}}
  \end{align*}
\]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of
\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x} \), \( y_2 = xe^{-2x} \), \( g = x^{-2} e^{-2x} \), \( W = e^{-4x} \).

Now we find the functions \( u_1 \) and \( u_2 \),

\[
    u'_1 = - \frac{y_2 g}{W} = - \frac{xe^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \quad \Rightarrow \quad u_1 = - \ln |x|.
\]

\[
    u'_2 = \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2}
\]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

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\[
\begin{align*}
 u'_1 &= -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln |x|. \\
 u'_2 &= \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}.
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Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2} e^{-2x}.$$ 

Solution: $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$, $g = x^{-2} e^{-2x}$, $W = e^{-4x}$.

Now we find the functions $u_1$ and $u_2$,

$$u_1' = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \Rightarrow u_1 = - \ln |x|.$$ 

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \Rightarrow u_2 = - \frac{1}{x}.$$ 

$$y_p = u_1 y_1 + u_2 y_2$$
Variation of parameters (3.6).

Example
Use the variation of parameters to find the general solution of
\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)

\[
\begin{align*}
    u'_1 &= -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|. \\
    u'_2 &= \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = \frac{1}{x}. 
\end{align*}
\]

\[ y_p = u_1 y_1 + u_2 y_2 = -\ln|x| e^{-2x} - \frac{1}{x} x e^{-2x} \]
Variation of parameters (3.6).

Example
Use the variation of parameters to find the general solution of
\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = xe^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)
\[ u_1' = -\frac{y_2g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln |x|. \]
\[ u_2' = \frac{y_1g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}. \]

\[ y_p = u_1y_1 + u_2y_2 = -\ln |x| e^{-2x} - \frac{1}{x} xe^{-2x} = -(1 + \ln |x|) e^{-2x}. \]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)

\[
\begin{align*}
u_1' &= - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \quad \Rightarrow \quad u_1 = - \ln |x|. \\
u_2' &= \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = - \frac{1}{x}.
\end{align*}
\]

\[ y_p = u_1 y_1 + u_2 y_2 = - \ln |x| e^{-2x} - \frac{1}{x} xe^{-2x} = -(1 + \ln |x|) e^{-2x}. \]

Since \( \tilde{y}_p = - \ln |x| e^{-2x} \) is solution,
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2} e^{-2x}.$$ 

Solution: $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$, $g = x^{-2} e^{-2x}$, $W = e^{-4x}$.

Now we find the functions $u_1$ and $u_2$,

$$u_1' = -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x} \Rightarrow u_1 = -\ln|x|.$$ 

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \Rightarrow u_2 = -\frac{1}{x}.$$ 

$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x| e^{-2x} - \frac{1}{x} x e^{-2x} = -(1 + \ln|x|) e^{-2x}.$$ 

Since $\tilde{y}_p = -\ln|x| e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|) e^{-2x}$. ◢
Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - **Undetermined coefficients (3.5)**
  - Constant coefficients, homogeneous, (3.1)-(3.4).
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]
Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3\sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3\sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,

\[ r^2 + 4 = 0 \]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,

\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]

\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

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Start with the first source, \( f_1(x) = 3 \sin(2x) \).
Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]
\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]

Start with the first source, \( f_1(x) = 3 \sin(2x) \).
The function \( \tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x) \) is the
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3\sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,

\[ r^2 + 4 = 0 \implies r_{\pm} = \pm 2i. \]

\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]

Start with the first source, \( f_1(x) = 3\sin(2x) \).

The function \( \tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess,
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3 \sin(2x) + e^{3x}$$

Solution: Find the solutions of the homogeneous problem,

$$r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i.$$

$$y_1 = \cos(2x), \quad y_2 = \sin(2x).$$

Start with the first source, $$f_1(x) = 3 \sin(2x).$$

The function $$\tilde{y}_{p1} = k_1 \sin(2x) + k_2 \cos(2x)$$ is the wrong guess, since it is solution of the homogeneous equation.
Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]
\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]

Start with the first source, \( f_1(x) = 3 \sin(2x) \).
The function \( \tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess, since it is solution of the homogeneous equation. We guess:
\[ y_p = x[k_1 \sin(2x) + k_2 \cos(2x)]. \]
Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,

\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]

\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]

Start with the first source, \( f_1(x) = 3 \sin(2x). \)

The function \( \tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess, since it is solution of the homogeneous equation. We guess:

\[ y_p = x\left[ k_1 \sin(2x) + k_2 \cos(2x) \right]. \]

\[ y_p' = \left[ k_1 \sin(2x) + k_2 \cos(2x) \right] + 2x\left[ k_1 \cos(2x) - k_2 \sin(2x) \right]. \]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3\sin(2x) + e^{3x} \]

Solution: Find the solutions of the homogeneous problem,
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_\pm = \pm 2i. \]
\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]

Start with the first source, \( f_1(x) = 3\sin(2x) \).
The function \( \tilde{y}_p_1 = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess, since it is solution of the homogeneous equation. We guess:
\[ y_p = x\left[k_1 \sin(2x) + k_2 \cos(2x)\right]. \]
\[ y'_p = \left[k_1 \sin(2x) + k_2 \cos(2x)\right] + 2x\left[k_1 \cos(2x) - k_2 \sin(2x)\right]. \]
\[ y''_p = 4\left[k_1 \cos(2x) - k_2 \sin(2x)\right] + 4x\left[-k_1 \sin(2x) - k_2 \cos(2x)\right].\]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3 \sin(2x) + e^{3x}.$$ 

Solution: Recall: \( y_1 = \sin(2x) \), and \( y_2 = \cos(2x) \).
Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x), \) and \( y_2 = \cos(2x). \)

\[
4\left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x\left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] + 4x\left[ k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x),
\]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x), \) and \( y_2 = \cos(2x). \)

\[
4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x \left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] + \\
4x \left[ k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x),
\]

Therefore, \( 4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x). \)
Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x) \), and \( y_2 = \cos(2x) \).

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4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x \left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] + \\
4x \left[ k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x),
\]

Therefore, \[ 4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x). \]

Evaluating at \( x = 0 \) and \( x = \pi/4 \).
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x) \), and \( y_2 = \cos(2x) \).

\[
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\]

Therefore, \( 4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x) \).

Evaluating at \( x = 0 \) and \( x = \pi/4 \) we get

\[
4k_1 = 0, \quad -4k_2 = 3
\]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3\sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x), \) and \( y_2 = \cos(2x). \)

\[ 4\left[k_1 \cos(2x) - k_2 \sin(2x)\right] + 4x\left[-k_1 \sin(2x) - k_2 \cos(2x)\right] + 4x\left[k_1 \sin(2x) + k_2 \cos(2x)\right] = 3 \sin(2x), \]

Therefore, \( 4\left[k_1 \cos(2x) - k_2 \sin(2x)\right] = 3 \sin(2x). \)

Evaluating at \( x = 0 \) and \( x = \pi/4 \) we get

\[ 4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}. \]
Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_1 = \sin(2x) \), and \( y_2 = \cos(2x) \).

\[
4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x \left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] +
4x \left[ k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x),
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Therefore, \( 4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x) \).

Evaluating at \( x = 0 \) and \( x = \pi/4 \) we get

\[
4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.
\]

Therefore, \( y_{p_1} = -\frac{3}{4} x \cos(2x) \).
Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3\sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p_1} = -\frac{3}{4}x \cos(2x). \)
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p_1} = -\frac{3}{4} x \cos(2x). \)

We now compute \( y_{p_2} \) for \( f_2(x) = e^{3x} \).
Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \[ y_{p_1} = -\frac{3}{4} x \cos(2x). \]

We now compute \( y_{p_2} \) for \( f_2(x) = e^{3x} \).

We guess: \( y_{p_2} = ke^{3x}. \) Then, \( y''_{p_2} = 9e^{3x}, \)
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3\sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p1} = -\frac{3}{4} x \cos(2x). \)

We now compute \( y_{p2} \) for \( f_2(x) = e^{3x}. \)

We guess: \( y_{p2} = k e^{3x}. \) Then, \( y''_{p2} = 9 e^{3x}, \)

\[ (9 + 4)ke^{3x} = e^{3x} \]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p1} = -\frac{3}{4}x \cos(2x). \)

We now compute \( y_{p2} \) for \( f_2(x) = e^{3x}. \)

We guess: \( y_{p2} = k e^{3x}. \) Then, \( y_{p2}'' = 9 e^{3x}, \)

\[ (9 + 4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13}. \]
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p_1} = -\frac{3}{4} x \cos(2x). \)

We now compute \( y_{p_2} \) for \( f_2(x) = e^{3x}. \)

We guess: \( y_{p_2} = k e^{3x}. \) Then, \( y_{p_2}'' = 9 e^{3x}, \)

\[(9 + 4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13} e^{3x}.\]
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x) + e^{3x}. \]

Solution: Recall: \( y_{p_1} = -\frac{3}{4} x \cos(2x). \)

We now compute \( y_{p_2} \) for \( f_2(x) = e^{3x}. \)

We guess: \( y_{p_2} = k e^{3x}. \) Then, \( y_{p_2}'' = 9 e^{3x}, \)

\[ (9 + 4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13} e^{3x}. \]

Therefore, the general solution is

\[ y(x) = c_1 \sin(2x) + \left( c_2 - \frac{3}{4} x \right) \cos(2x) + \frac{1}{13} e^{3x}. \]
The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
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The definition of the Laplace Transform.

Definition

The function $F : D_F \rightarrow \mathbb{R}$ is the **Laplace transform** of a function $f : [0, \infty) \rightarrow \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt,$$

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Review: Improper integrals.

Recall: Improper integral are defined as a limit.

\[ \int_{t_0}^{\infty} g(t) \, dt = \lim_{N \to \infty} \int_{t_0}^{N} g(t) \, dt. \]

Example
Compute the improper integral \( \int_{0}^{\infty} e^{-at} \, dt \), with \( a > 0 \).

Solution:
\[ \int_{0}^{\infty} e^{-at} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt. \]

Since \( \lim_{N \to \infty} e^{-aN} = 0 \) for \( a > 0 \), we conclude \( \int_{0}^{\infty} e^{-at} \, dt = \frac{1}{a} \).
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- The integral converges iff the limit exists.

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\[ \begin{align*}
\int_{0}^{\infty} e^{-at} \, dt &= \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt \\
&= \lim_{N \to \infty} \left[ -\frac{1}{a} e^{-at} \right]_{0}^{N} \\
&= \lim_{N \to \infty} \left( -\frac{1}{a} e^{-aN} + \frac{1}{a} \right) \\
&= \frac{1}{a}
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Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[1]$. 

Solution:

We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt$$

But

$$\int_0^\infty e^{-at} dt = \frac{1}{a} \quad \text{for} \quad a > 0,$$

and divergence for $a \leq 0$. Therefore

$$\mathcal{L}[1] = \frac{1}{s}, \quad \text{for} \quad s > 0,$$

and $\mathcal{L}[1]$ does not exist for $s \leq 0$. In other words,

$$F(s) = \mathcal{L}[1] \quad \text{is the function} \quad F: \mathbb{D}_F \rightarrow \mathbb{R} \quad \text{given by} \quad f(t) = 1, \quad F(s) = \frac{1}{s}, \quad \mathbb{D}_F = (0, \infty).$$
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Therefore $\mathcal{L}[1] = \frac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leq 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

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Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.
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Integrating by parts twice it is not difficult to obtain:

$$\int_{0}^{N} e^{-st} \sin(at) \, dt =$$

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This identity implies

$$\left( 1 + \frac{a^2}{s^2} \right) \int_0^N e^{-st} \sin(at) \, dt = -\frac{1}{s} \left[ e^{-st} \sin(at) \right]_0^N - \frac{a}{s^2} \left[ e^{-st} \cos(at) \right]_0^N.$$
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Hence, it is not difficult to see that

$$(\frac{s^2 + a^2}{s^2}) \int_0^\infty e^{-st} \sin(at) \, dt = \frac{a}{s^2}, \quad s > 0,$$
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A table of Laplace Transforms.

<table>
<thead>
<tr>
<th>$f(t)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$f(t) = 1$</td>
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</tr>
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Properties of the Laplace Transform.

Theorem (Sufficient conditions)

*If the function $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and there exist positive constants $k$ and $a$ such that

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Theorem (Linear combination)
If the $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined and $a$, $b$ are constants, then

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$$\mathcal{L}[f'] = s \mathcal{L}[f] - f(0).$$  \hfill (1)

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Proof of Eq (2):

Use Eq. (1) twice:

$$\mathcal{L}[f''] = \mathcal{L}[ (f')' ] = s \mathcal{L}[f'] - f'(0) = s \left( s \mathcal{L}[f] - f(0) \right) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0).$$
Properties of the Laplace Transform.

**Theorem (Derivatives)**

*If the \( \mathcal{L}[f] \) and \( \mathcal{L}[f'] \) are well-defined, then holds,

\[
\mathcal{L}[f'] = s \mathcal{L}[f] - f(0). \tag{1}
\]

*Furthermore, if \( \mathcal{L}[f''] \) is well-defined, then it also holds

\[
\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0). \tag{2}
\]

**Proof of Eq (2):** Use Eq. (1) twice:
Properties of the Laplace Transform.

Theorem (Derivatives)

If the \( \mathcal{L}[f] \) and \( \mathcal{L}[f'] \) are well-defined, then holds,

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Proof of Eq (1): Recall the definition of the Laplace Transform,

\[ \mathcal{L}[f'] = \int_0^\infty e^{-st} f'(t) \, dt \]
Properties of the Laplace Transform.

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Integrating by parts,

\[ \lim_{n \to \infty} \int_0^n e^{-st} f'(t) \, dt = \lim_{n \to \infty} \left[ \left. e^{-st} f(t) \right|_0^n - \int_0^n (-s) e^{-st} f(t) \, dt \right] \]
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\[ \mathcal{L}[f'] = \lim_{n \to \infty} \left[ e^{-sn} f(n) - f(0) \right] + s \int_0^\infty e^{-st} f(t) \, dt \]
Properties of the Laplace Transform.

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$$\mathcal{L}[f'] = \lim_{n \to \infty} \left[ e^{-sn} f(n) - f(0) \right] + s \int_0^\infty e^{-st} f(t) \, dt = -f(0) + s \mathcal{L}[f],$$
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where we used that \( \lim_{n \to \infty} e^{-sn} f(n) = 0 \) for \( s \) big enough, and we also used that \( \mathcal{L}[f] \) is well-defined.

We then conclude that \( \mathcal{L}[f'] = s \mathcal{L}[f] - f(0) \).
The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
Laplace Transform and differential equations.

**Remark:** Laplace Transforms can be used to find solutions to differential equations with constant coefficients.
Laplace Transform and differential equations.

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Laplace Transform and differential equations.

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\[ \mathcal{L} \left[ \text{Differential Eq.} \right] \]

for \( y(t) \).
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\[ \mathcal{L} \left[ \text{Differential Eq. for } y(t) \right] \rightarrow \text{Algebraic Eq. for } \mathcal{L}[y(t)]. \]
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Idea of the method:

\[ \mathcal{L} \begin{bmatrix} \text{Differential Eq.} \\ \text{for } y(t) \end{bmatrix} \quad \overset{(1)}{\rightarrow} \quad \text{Algebraic Eq. for } \mathcal{L}[y(t)] \]

\[ \overset{(2)}{\rightarrow} \quad \text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)]. \]
Laplace Transform and differential equations.

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\[
\mathcal{L} \left[ \text{Differential Eq. for } y(t) \right] \quad \text{(1)} \quad \text{Algebraic Eq. for } \mathcal{L}[y(t)].
\]

\[
\text{(2)} \quad \text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)].
\]

\[
\text{(3)} \quad \text{Transform back to obtain } y(t). \\
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Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y' + 2y = 0, \quad y(0) = 3.
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Solution: We know the solution: \( y(t) = 3e^{-2t} \).
Laplace Transform and differential equations.

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Find an algebraic equation for $\mathcal{L}[y]$. 
Laplace Transform and differential equations.

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Find an algebraic equation for \( \mathcal{L}[y] \). Recall linearity:

\[
\mathcal{L}[y'] + 2 \mathcal{L}[y] = 0.
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Laplace Transform and differential equations.

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Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity:

$$ \mathcal{L}[y'] + 2 \mathcal{L}[y] = 0. $$

Also recall the property: $\mathcal{L}[y'] = s \mathcal{L}[y] - y(0)$,
Laplace Transform and differential equations.

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Also recall the property: $\mathcal{L}[y'] = s \mathcal{L}[y] - y(0)$, that is,

$$\left[ s \mathcal{L}[y] - y(0) \right] + 2 \mathcal{L}[y] = 0 \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0).$$
Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
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y' + 2y = 0, \quad y(0) = 3.
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Solution: Recall: \((s + 2) \mathcal{L}[y] = y(0)\).
Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$ y' + 2y = 0, \quad y(0) = 3. $$

Solution: Recall: $(s + 2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$. 

\[ \mathcal{L} \left[ y(t) \right] = \mathcal{L} \left[ 3e^{-2t} \right] \]

Hence, 

\[ \mathcal{L} \left[ y(t) \right] = 3e^{-2t}. \]
Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: Recall: $(s + 2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s + 2},$$
Laplace Transform and differential equations.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP
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y' + 2y = 0, \quad y(0) = 3.
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Solution: Recall: \((s + 2)\mathcal{L}[y] = y(0).\)

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\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3,
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Laplace Transform and differential equations.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP

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y' + 2y = 0, \quad y(0) = 3.
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Solution: Recall: \((s + 2)\mathcal{L}[y] = y(0)\).

(2): Solve the algebraic equation for \( \mathcal{L}[y] \).

\[
\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.
\]
Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: Recall: $(s + 2)L[y] = y(0)$.

(2): Solve the algebraic equation for $L[y]$.

$$L[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad L[y] = \frac{3}{s + 2}.$$ 

(3): Transform back to $y(t)$. 

$$y(t) = 3e^{-2t}.$$
Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
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y' + 2y = 0, \quad y(0) = 3.
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Solution: Recall: \((s + 2)\mathcal{L}[y] = y(0)\).

(2): Solve the algebraic equation for \( \mathcal{L}[y] \).
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\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.
\]

(3): Transform back to \( y(t) \). From the table:
\[
\mathcal{L}[e^{at}] = \frac{1}{s - a}
\]
Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[ y' + 2y = 0, \quad y(0) = 3. \]

Solution: Recall: \( (s + 2)L[y] = y(0). \)

(2): Solve the algebraic equation for \( L[y] \).

\[ L[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad L[y] = \frac{3}{s + 2}. \]

(3): Transform back to \( y(t) \). From the table:

\[ L[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{3}{s + 2} = 3L[e^{-2t}] \]
Laplace Transform and differential equations.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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$$L[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{3}{s + 2} = 3L[e^{-2t}] \quad \Rightarrow \quad \frac{3}{s + 2} = L[3e^{-2t}].$$
Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: Recall: $(s + 2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$. 

$$\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.$$ 

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{3}{s + 2} = 3\mathcal{L}[e^{-2t}] \quad \Rightarrow \quad \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}].$$ 

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}]$.
Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: Recall: $(s + 2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.$$ 

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \Rightarrow \frac{3}{s + 2} = 3\mathcal{L}[e^{-2t}] \Rightarrow \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}].$$

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \Rightarrow y(t) = 3e^{-2t}$.  \triangle
The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - Non-homogeneous IVP.
Solving differential equations using $\mathcal{L}[\cdot]$.

**Remark:** The method works with:

1. Constant coefficient equations.
2. Homogeneous and non-homogeneous equations.
3. First, second, higher order equations.

**Idea of the method:**

1. Transform the differential equation into an algebraic equation for $\mathcal{L}[y(t)]$.
2. Solve the algebraic equation for $\mathcal{L}[y(t)]$.
3. Transform back to obtain $y(t)$ (using the table).
Solving differential equations using $\mathcal{L}[\ ]$.

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Solving differential equations using $\mathcal{L}[]$.

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Solving differential equations using \( \mathcal{L} [ ] \).

Remark: The method works with:

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**Idea of the method:**

$$
\mathcal{L} \left[ \begin{array}{c}
\text{Differential Eq.} \\
\text{for } y(t).
\end{array} \right]
$$
Solving differential equations using $\mathcal{L}[\ ]$.

Remark: The method works with:
- Constant coefficient equations.
- Homogeneous and non-homogeneous equations.
- First, second, higher order equations.

Idea of the method:

\[
\mathcal{L} \left[ \begin{array}{c} \text{Differential Eq. for } y(t). \\ \end{array} \right] \rightarrow (1) \rightarrow \text{Algebraic Eq. for } \mathcal{L}[y(t)].
\]
Solving differential equations using $\mathcal{L}[\cdot]$.

Remark: The method works with:
- Constant coefficient equations.
- Homogeneous and non-homogeneous equations.
- First, second, higher order equations.

Idea of the method:

\[
\mathcal{L} \left[ \begin{array}{c} \text{Differential Eq.} \\
\text{for } y(t). \end{array} \right] \quad \rightarrow \quad (1) \quad \text{Algebraic Eq.} \quad \rightarrow \quad (2)
\]

Solve the Algebraic Eq. for $\mathcal{L}[y(t)]$. 

\[
\rightarrow \quad (2) \quad \text{Algebraic Eq.} \quad \rightarrow \quad (3)
\]

Solve for $\mathcal{L}[y(t)]$. 

Solving differential equations using $\mathcal{L}[\ ]$.

**Remark:** The method works with:

- Constant coefficient equations.
- Homogeneous and non-homogeneous equations.
- First, second, higher order equations.

**Idea of the method:**

1. $\mathcal{L}$ Differential Eq. for $y(t)$.
2. Solve the Algebraic Eq. for $\mathcal{L}[y(t)]$.
3. Transform back to obtain $y(t)$. (Using the table.)
Solving differential equations using $\mathcal{L} [ ]$.

Idea of the method:

\[ \mathcal{L} \left[ \text{Differential Eq. for } y(t) \right] \quad \rightarrow \quad (1) \quad \text{Algebraic Eq. for } \mathcal{L}[y(t)]. \]

\[ \rightarrow \quad (2) \quad \text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)]. \]

\[ \rightarrow \quad (3) \quad \text{Transform back to obtain } y(t). \quad \text{(Using the table.)} \]
Solving differential equations using $\mathcal{L}[\ ]$.

Idea of the method:

$$\mathcal{L} \left[ \begin{array}{c}
\text{Differential Eq.} \\
\text{for } y(t).
\end{array} \right] \rightarrow (1) \rightarrow \text{Algebraic Eq.} \rightarrow (2) \rightarrow \text{Algebraic Eq.} \rightarrow (3) \rightarrow \text{Transform back to obtain } y(t).$$

(Using the table.)

Recall:

(a) $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)];$
Solving differential equations using $\mathcal{L}[\cdot]$.

Idea of the method:

$$
\mathcal{L} \left[ \begin{array}{c}
\text{Differential Eq.} \\
\text{for } y(t).
\end{array} \right] \quad (1) \quad \rightarrow \quad \text{Algebraic Eq.} \quad (2) \quad \text{for } \mathcal{L}[y(t)].
$$

\[ (2) \quad \rightarrow \quad \text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)]. \]

\[ (3) \quad \rightarrow \quad \text{Transform back to obtain } y(t). \quad (\text{Using the table.}) \]

Recall:

(a) $\mathcal{L}[a f(t) + b g(t)] = a \mathcal{L}[f(t)] + b \mathcal{L}[g(t)]$;

(b) $\mathcal{L}[y^{(n)}] = s^n \mathcal{L}[y] - s^{(n-1)} y(0) - s^{(n-2)} y'(0) - \cdots - y^{(n-1)}(0)$. 
The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\cdot]$.
  - **Homogeneous IVP.**
  - First, second, higher order equations.
  - Non-homogeneous IVP.
Homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
\]
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$  

Solution: Compute the $L[\ ]$ of the differential equation,

$$L[y'' - y' - 2y] = L[0] \quad \Rightarrow \quad L[y'' - y' - 2y] = 0.$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

\[ y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

\[ \mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'' - y' - 2y] = 0. \]

The $\mathcal{L}[\ ]$ is a linear function,
Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
\]

Solution: Compute the \( \mathcal{L}[\ ] \) of the differential equation,
\[
\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'' - y' - 2y] = 0.
\]
The \( \mathcal{L}[\ ] \) is a linear function, so
\[
\mathcal{L}[y''] - \mathcal{L}[y'] - 2 \mathcal{L}[y] = 0.
\]
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'' - y' - 2y] = 0.$$ 

The $\mathcal{L}[\ ]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0.$$ 

Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0)\right] - \left[s \mathcal{L}[y] - y(0)\right] - 2 \mathcal{L}[y] = 0,$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
 y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
\]

Solution: Compute the \( \mathcal{L}[\ ] \) of the differential equation,
\[
 \mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \implies \mathcal{L}[y'' - y' - 2y] = 0.
\]
The \( \mathcal{L}[\ ] \) is a linear function, so
\[
 \mathcal{L}[y''] - \mathcal{L}[y'] - 2 \mathcal{L}[y] = 0.
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Derivatives are transformed into power functions,
\[
 \left[s^2 \mathcal{L}[y] - s y(0) - y'(0)\right] - \left[s \mathcal{L}[y] - y(0)\right] - 2 \mathcal{L}[y] = 0,
\]
We the obtain \((s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0).\)
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP
\[ y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

Solution: Recall: $(s^2 - s - 2) \mathcal{L}[y] = (s - 1)y(0) + y'(0)$. 
Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[ y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

Solution: Recall: \( (s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0). \)

Differential equation for \( y \) \( \xrightarrow{\mathcal{L}[\quad]} \) Algebraic equation for \( \mathcal{L}[y] \).
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

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Differential equation for $y$ $\xrightarrow{\mathcal{L}[\cdot]}$ Algebraic equation for $\mathcal{L}[y]$. 

Introduce the initial condition,
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$  

Solution: Recall:  

$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0).$$  

Differential equation for $y$  

$$\mathcal{L}[y] \quad \xrightarrow{\text{L}} \quad \text{Algebraic equation for } \mathcal{L}[y].$$  

Introduce the initial condition,  

$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1).$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$  

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Differential equation for $y$ \quad $\mathcal{L}[\int]$ \quad Algebraic equation for $\mathcal{L}[y]$.

Introduce the initial condition,

$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1).$$

We can solve for the unknown $\mathcal{L}[y]$ as follows,
Homogeneous IVP.

Example
Use the Laplace transform to find the solution \(y(t)\) to the IVP
\[y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.\]

Solution: Recall:
\[(s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0).\]

Differential equation for \(y\) \(\xrightarrow{\mathcal{L}[\ ]}\) Algebraic equation for \(\mathcal{L}[y]\).

Introduce the initial condition,
\[(s^2 - s - 2) \mathcal{L}[y] = (s - 1).\]

We can solve for the unknown \(\mathcal{L}[y]\) as follows,
\[\mathcal{L}[y] = \frac{(s - 1)}{(s^2 - s - 2)}.\]
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$  

Solution: Recall: $\mathcal{L}[y] = \frac{(s - 1)}{(s^2 - s - 2)}$.  

The partial fraction method: Find the zeros of the denominator, $s^2 - s - 2 = 0 \Rightarrow s = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$.  

Therefore, we rewrite:

$$\mathcal{L}[y] = \frac{(s - 1)}{(s - 2)(s + 1)}.$$  

Find constants $a$ and $b$ such that

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.$$  


Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

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The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $\mathcal{L}[y] = \frac{(s - 1)}{(s^2 - s - 2)}$.

The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 8} \right]$$
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

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The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2} \left[1 \pm \sqrt{1 + 8}\right] \quad \Rightarrow \quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

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Therefore, we rewrite: $\mathcal{L}[y] = \frac{(s - 1)}{(s - 2)(s + 1)}$. 
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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The partial fraction method: Find the zeros of the denominator,

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 s_- = -1, \end{cases}$$

Therefore, we rewrite: $\mathcal{L}[y] = \frac{(s - 1)}{(s - 2)(s + 1)}$.

Find constants $a$ and $b$ such that

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.$$
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

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A simple calculation shows

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.$$
Homogeneous IVP.

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A simple calculation shows

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1} = \frac{a(s + 1) + b(s - 2)}{(s - 2)(s + 1)}.$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

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y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
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Solution: Recall:

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\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.
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A simple calculation shows

\[
\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1} = \frac{a(s + 1) + b(s - 2)}{(s - 2)(s + 1)}
\]

\[
(s - 1) = s(a + b) + (a - 2b)
\]
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

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A simple calculation shows

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1} = \frac{a(s + 1) + b(s - 2)}{(s - 2)(s + 1)}$$ 

$$(s - 1) = s(a + b) + (a - 2b) \quad \Rightarrow \quad \begin{cases} 
  a + b = 1, \\
  a - 2b = -1
\end{cases}$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
\]

Solution: Recall:
\[
\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.
\]
A simple calculation shows
\[
\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1} = \frac{a(s + 1) + b(s - 2)}{(s - 2)(s + 1)}
\]
\[
(s - 1) = s(a + b) + (a - 2b) \quad \Rightarrow \quad \left\{\begin{array}{l}
a + b = 1, \\
a - 2b = -1
\end{array}\right.
\]
Hence, \( a = \frac{1}{3} \) and \( b = \frac{2}{3} \).
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Recall:

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1}.$$ 

A simple calculation shows

$$\frac{(s - 1)}{(s - 2)(s + 1)} = \frac{a}{s - 2} + \frac{b}{s + 1} = \frac{a(s + 1) + b(s - 2)}{(s - 2)(s + 1)}.$$ 

$$(s - 1) = s(a + b) + (a - 2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

Hence, $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Then, $\mathcal{L}[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{2}{3} \frac{1}{s + 1}$. 
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{2}{3} \frac{1}{s + 1}$. 
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{2}{3} \frac{1}{s + 1}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a}$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{s-2} + \frac{2}{3} \frac{1}{s+1}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \left( \frac{1}{s - 2} + \frac{2}{3} \frac{1}{s + 1} \right)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}], \quad \frac{1}{s + 1} = \mathcal{L}[e^{-t}].$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \left( \frac{1}{s - 2} \right) + \frac{2}{3} \left( \frac{1}{s + 1} \right)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}], \quad \frac{1}{s + 1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3} \mathcal{L}[e^{2t}] + \frac{2}{3} \mathcal{L}[e^{-t}]$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}], \quad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3} \mathcal{L}[e^{2t}] + \frac{2}{3} \mathcal{L}[e^{-t}] = \mathcal{L}\left[\frac{1}{3}(e^{2t} + 2e^{-t})\right]$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{2}{3} \frac{1}{s + 1}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}], \quad \frac{1}{s + 1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3} \mathcal{L}[e^{2t}] + \frac{2}{3} \mathcal{L}[e^{-t}] = \mathcal{L}\left[\frac{1}{3}(e^{2t} + 2e^{-t})\right].$$

We conclude that: $y(t) = \frac{1}{3}(e^{2t} + 2e^{-t})$. △
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,
Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.
\]
Solution: Compute the \( L[ \ ] \) of the differential equation,
\[
L[y'' - 4y' + 4y] = L[0] = 0.
\]
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$ 

The $\mathcal{L}[\ ]$ is a linear function,

$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = 0.$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Compute the $\mathcal{L}[]$ of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$ 

The $\mathcal{L}[]$ is a linear function,

$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = 0.$$ 

Derivatives are transformed into power functions,

$$\left[ s^2 \mathcal{L}[y] - sy(0) - y'(0) \right] - 4 \left[ s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = 0,$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP
$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,
$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$ 

The $\mathcal{L}[\ ]$ is a linear function,
$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = 0.$$ 

Derivatives are transformed into power functions,
$$[s^2 \mathcal{L}[y] - s y(0) - y'(0)] - 4 [s \mathcal{L}[y] - y(0)] + 4 \mathcal{L}[y] = 0,$$

Therefore,
$$(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0).$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0)$. 
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Introduce the initial conditions,
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Recall: $$(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0).$$

Introduce the initial conditions, $$(s^2 - 4s + 4) \mathcal{L}[y] = s - 3.$$
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP
\[ y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1. \]

Solution: Recall: 
\[ (s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0). \]
Introduce the initial conditions, 
\[ (s^2 - 4s + 4) \mathcal{L}[y] = s - 3. \]

Solve for $\mathcal{L}[y]$ as follows:
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4)y(0) + y'(0)$.  
Introduce the initial conditions, $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3$.

Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)}$. 
Homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.
\]

Solution: Recall: \((s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0)\).
Introduce the initial conditions, \((s^2 - 4s + 4) \mathcal{L}[y] = s - 3\).
Solve for \( \mathcal{L}[y] \) as follows: \( \mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} \).

The partial fraction method:
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP
\[ y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1. \]

Solution: Recall: \((s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0)\).
Introduce the initial conditions, \((s^2 - 4s + 4) \mathcal{L}[y] = s - 3\).
Solve for \(\mathcal{L}[y]\) as follows: \(\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)}\).

The partial fraction method: Find the roots of the denominator, \(s^2 - 4s + 4 = 0\).
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4)y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3$.

Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)}$.

The partial fraction method: Find the roots of the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}]$$
Homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \quad \Rightarrow \quad s_+ = s_- = 2.$$ 

We obtain: \(\mathcal{L}[y] = \frac{(s - 3)}{(s - 2)^2}\).
Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Homogeneous IVP.

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If $s = 2$, 

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Homogeneous IVP.

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$$\frac{(s - 3)}{(s - 2)^2} = \frac{a}{(s - 2)} + \frac{b}{(s - 2)^2} \quad \Rightarrow \quad s - 3 = a(s - 2) + b$$

If $s = 2$, then $b = -1$. 
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$ 

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$$\frac{(s - 3)}{(s - 2)^2} = \frac{a}{s - 2} + \frac{b}{(s - 2)^2} \quad \Rightarrow \quad s - 3 = a(s - 2) + b$$

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Homogeneous IVP.

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If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}.$$
Homogeneous IVP.

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From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a}$$
Homogeneous IVP.

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$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}],$$
Homogeneous IVP.

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$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s - a)(n+1)}$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$ 

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$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{1}{s - 2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s - a)(n + 1)} \quad \Rightarrow \quad \frac{1}{(s - 2)^2} = \mathcal{L}[te^{2t}].$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.
\]

Solution: Recall: \( \mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} \) and

\[
\frac{1}{s - 2} = \mathcal{L}[e^{2t}], \quad \frac{1}{(s - 2)^2} = \mathcal{L}[te^{2t}].
\]
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$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}]$$
Homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

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\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}[e^{2t} - te^{2t}].
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\[
\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}[e^{2t} - te^{2t}].
\]

We conclude that \( y(t) = e^{2t} - te^{2t} \). \( \triangleq \)
The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[ ]$.
  - Homogeneous IVP.
  - **First, second, higher order equations.**
  - Non-homogeneous IVP.
First, second, higher order equations.

Example
Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)
\[
y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.
\]
First, second, higher order equations.

Example
Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)
\( y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0. \)

Solution: Compute the \( \mathcal{L}[ \ ] \) of the equation,

\[
\mathcal{L}[y^{(4)}] - 4 \mathcal{L}[y] = 0.
\]
First, second, higher order equations.

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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)
\[ y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0. \]

Solution: Compute the \( L[\ ] \) of the equation,
\[ L[y^{(4)}] - 4 L[y] = 0. \]

\[ [s^4 L[y] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] - 4 L[y] = 0. \]
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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0 \),
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\mathcal{L}[y^{(4)}] - 4 \mathcal{L}[y] = 0.
\]

\[
[s^4 \mathcal{L}[y] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] - 4 \mathcal{L}[y] = 0.
\]

\[
[s^4 \mathcal{L}[y] - s^3 + 2s] - 4 \mathcal{L}[y] = 0
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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0 \),
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\]

\[
\left[ s^4 L[y] - s^3 + 2s \right] - 4 L[y] = 0 \quad \Rightarrow \quad (s^4 - 4) L[y] = s^3 - 2s,
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First, second, higher order equations.

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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)
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\left[ s^4 \mathcal{L}[y] - s^3 + 2s \right] - 4 \mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4) \mathcal{L}[y] = s^3 - 2s,
\]
We obtain, \( \mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}. \)
First, second, higher order equations.

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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)
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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0 \), \( y(0) = 1 \), \( y'(0) = 1 \), \( y''(0) = -2 \), \( y'''(0) = 0 \).

Solution: Recall:
\[
\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}.
\]

\[
\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)}
\]
Example

Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0, \)

\[
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\[
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First, second, higher order equations.

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The last expression is in the table of Laplace Transforms,
Example

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The last expression is in the table of Laplace Transforms,

\[
\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)}
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Example
Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0 \),
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The last expression is in the table of Laplace Transforms,
\[
\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)} = \mathcal{L}[\cos(\sqrt{2} t)].
\]
First, second, higher order equations.

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Use the Laplace Transform to find the solution of \( y^{(4)} - 4y = 0 \),
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y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.
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Solution: Recall: \( \mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)} \).

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The last expression is in the table of Laplace Transforms,
\[
\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)} = \mathcal{L}[\cos(\sqrt{2} \, t)].
\]

We conclude that \( y(t) = \cos(\sqrt{2} \, t) \).
The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L} [\cdot]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - **Non-homogeneous IVP.**
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$
Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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Solution: Compute the Laplace transform of the equation,
Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Compute the Laplace transform of the equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3 \sin(2t)].$$
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

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The right-hand side above can be expressed as follows,
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

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Non-homogeneous IVP.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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\[
\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3 \sin(2t)].
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The right-hand side above can be expressed as follows,
\[
\mathcal{L}[3 \sin(2t)] = 3 \mathcal{L}[\sin(2t)] = 3 \frac{2}{s^2 + 2^2}.
\]
Non-homogeneous IVP.

**Example**
Use the Laplace transform to find the solution $y(t)$ to the IVP
\[ y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1. \]

**Solution:** Compute the Laplace transform of the equation,
\[
\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3 \sin(2t)].
\]
The right-hand side above can be expressed as follows,
\[
\mathcal{L}[3 \sin(2t)] = 3 \mathcal{L}[\sin(2t)] = 3 \frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.
\]
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP
\[ y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1. \]

Solution: Compute the Laplace transform of the equation,
\[ \mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3 \sin(2t)]. \]

The right-hand side above can be expressed as follows,
\[ \mathcal{L}[3 \sin(2t)] = 3 \mathcal{L}[\sin(2t)] = 3 \frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}. \]

Introduce this source term in the differential equation,
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
\]

Solution: Compute the Laplace transform of the equation,
\[
L[y'' - 4y' + 4y] = L[3 \sin(2t)].
\]
The right-hand side above can be expressed as follows,
\[
L[3 \sin(2t)] = 3L[\sin(2t)] = 3 \frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.
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Introduce this source term in the differential equation,
\[
L[y''] - 4L[y'] + 4L[y] = \frac{6}{s^2 + 4}.
\]
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Recall: $\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
\]

Solution: Recall:

\[
\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.
\]

Derivatives are transformed into power functions,

\[
\left[ s^2 \mathcal{L}[y] - sy(0) - y'(0) \right] - 4 \left[ s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.
\]
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Recall:

$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$ 

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$$\left[ s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - 4 \left[ s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$ 

Rewrite the above equation,

$$\left( s^2 - 4s + 4 \right) \mathcal{L}[y] = (s - 4) y(0) + y'(0) + \frac{6}{s^2 + 4}.$$
Non-homogeneous IVP.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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Solution: Recall: 

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\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.
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\]

Rewrite the above equation,

\[
(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0) + \frac{6}{s^2 + 4}.
\]

Introduce the initial conditions,

\[
(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.
\]
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

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Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Recall: 

$$(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.$$ 

Therefore, 

$$\mathcal{L}[y] = \frac{s - 3}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4 + 4)(s^2 + 4)}.$$ 


Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Recall: $$(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.$$ 

Therefore, $$\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4 + 4)(s^2 + 4)}.$$ 

From an Example above: $s^2 - 4s + 4 = (s - 2)^2,$
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

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y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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Solution: Recall: \((s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}\).

Therefore, \( \mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4 + 4)(s^2 + 4)} \).

From an Example above: \( s^2 - 4s + 4 = (s - 2)^2 \),

\[
\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}.
\]
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

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From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}.$$

From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}.$$
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \frac{6}{(s - 2)^2(s^2 + 4)}$. 
Non-homogeneous IVP.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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Solution: Recall: \( \mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \frac{6}{(s - 2)^2(s^2 + 4)} \).

Use Partial fractions to simplify the last term above.
Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

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Use Partial fractions to simplify the last term above.

Find constants $a, b, c, d$, such that

$$\frac{6}{(s - 2)^2(s^2 + 4)} = \frac{as + b}{s^2 + 4} + \frac{c}{s - 2} + \frac{d}{(s - 2)^2}.$$
Non-homogeneous IVP.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP

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y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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\]

\[
\frac{6}{(s - 2)^2(s^2 + 4)} = \frac{(as + b)(s - 2)^2 + c(s - 2)(s^2 + 4) + d(s^2 + 4)}{(s^2 + 4)(s - 2)^2}
\]
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$  

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \frac{6}{(s - 2)^2(s^2 + 4)}$.

Use Partial fractions to simplify the last term above.

Find constants $a$, $b$, $c$, $d$, such that

$$\frac{6}{(s - 2)^2(s^2 + 4)} = \frac{as + b}{s^2 + 4} + \frac{c}{s - 2} + \frac{d}{(s - 2)^2}.$$  

$$\frac{6}{(s - 2)^2(s^2 + 4)} = \frac{(as + b)(s - 2)^2 + c(s - 2)(s^2 + 4) + d(s^2 + 4)}{(s^2 + 4)(s - 2)^2}.$$  

$$6 = (as + b)(s - 2)^2 + c(s - 2)(s^2 + 4) + d(s^2 + 4).$$
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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Solution: \( 6 = (as + b)(s - 2)^2 + c(s - 2)(s^2 + 4) + d(s^2 + 4). \)
Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution:  

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$$6 = (as + b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

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$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).$$
Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

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y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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\[
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\]

\[
6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).
\]

\[
6 = (a + c)s^3 + (-4a + b - 2c + d)s^2 \\
+ (4a - 4b + 4c)s + (4b - 8c + 4d).
\]
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Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
\]

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\]

\[
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\]

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6 = (a + c)s^3 + (-4a + b - 2c + d)s^2
\]

\[
+ (4a - 4b + 4c)s + (4b - 8c + 4d).
\]

We obtain the system

\[
a + c = 0, \quad -4a + b - 2c + d = 0, \quad 4b - 8c + 4d = 6.
\]
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$ 

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \quad b = 0, \quad c = -\frac{3}{8}, \quad d = \frac{3}{4}.$$
Non-homogeneous IVP.

Example
Use the Laplace transform to find the solution \( y(t) \) to the IVP
\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
\]

Solution: The solution for this linear system is
\[
a = \frac{3}{8}, \quad b = 0, \quad c = -\frac{3}{8}, \quad d = \frac{3}{4}.
\]

\[
\frac{6}{(s - 2)^2 (s^2 + 4)} = \frac{3}{8} \frac{s}{s^2 + 4} - \frac{3}{8} \frac{1}{s - 2} + \frac{3}{4} \frac{1}{(s - 2)^2}.
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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Use the table of Laplace Transforms

$$\frac{6}{(s - 2)^2 (s^2 + 4)} = \frac{3}{8} \mathcal{L}[\cos(2t)] - \frac{3}{8} \mathcal{L}[e^{2t}] + \frac{3}{4} \mathcal{L}[te^{2t}].$$
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Use the Laplace transform to find the solution $y(t)$ to the IVP

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$$\frac{6}{(s - 2)^2 (s^2 + 4)} = \mathcal{L}\left[\frac{3}{8} \cos(2t) - \frac{3}{8} e^{2t} + \frac{3}{4} te^{2t}\right].$$
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\[
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$$\mathcal{L}[y(t)] = \mathcal{L}\left[(1 - t) e^{2t} + \frac{3}{8} (-1 + 2t) e^{2t} + \frac{3}{8} \cos(2t)\right].$$
Non-homogeneous IVP.

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Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[
y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.
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\]

\[
\mathcal{L}[y(t)] = \mathcal{L}\left[(1 - t) e^{2t} + \frac{3}{8} (-1 + 2t) e^{2t} + \frac{3}{8} \cos(2t)\right].
\]

We conclude that

\[
y(t) = (1 - t) e^{2t} + \frac{3}{8} (2t - 1) e^{2t} + \frac{3}{8} \cos(2t).
\]