Power series solutions near regular points (Sect. 5.2).

- We study: $P(x) y'' + Q(x) y' + R(x) y = 0$.
- Review of power series.
- Regular point equations.
- Solutions using power series.
- Examples of the power series method.
Review of power series.

Definition

The *power series* of a function \( y : \mathbb{R} \to \mathbb{R} \) centered at \( x_0 \in \mathbb{R} \) is

\[
y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.
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Example

$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$
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\begin{itemize}
  \item \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \). Here \( x_0 = 0 \) and \( |x| < 1 \).
  \item \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \)
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- The Taylor series of $y : \mathbb{R} \rightarrow \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is

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$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n = y(x_0) + y'(x_0)(x - x_0) + \cdots.$$
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Example
Find the Taylor series of $y(x) = \sin(x)$ centered at $x_0 = 0$.

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}
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Remark: The Taylor series of $y(x) = \cos(x)$ centered at $x_0 = 0$ is
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The function \( y(x) = \frac{1}{1 - x} \) is defined for \( x \in \mathbb{R} - \{1\} \).
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The power series

$$y(x) = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

converges only for $|x| < 1$. 

Review of power series.

Definition

The power series \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely iff the series \( \sum_{n=0}^{\infty} |a_n| |x - x_0|^n \) converges.
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The series \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges,
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The series \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges, but it does not converge absolutely,
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The series \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges, but it does not converge absolutely, since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.
Review of power series.

Definition
The radius of convergence of a power series

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

is the number \( \rho \geq 0 \) that satisfies both

(a) the series converges absolutely for \( |x - x_0| < \rho \);
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(4) \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \) has radius of convergence \( \rho = \infty \).
Theorem (Ratio test)

Given the power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, introduce the number $L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$. Then, the following statements hold:

1. The power series converges in the domain $|x - x_0|L < 1$.
2. The power series diverges in the domain $|x - x_0|L > 1$.
3. The power series may or may not converge at $|x - x_0|L = 1$.

Therefore, if $L \neq 0$, then $\rho = \frac{1}{L}$ is the series radius of convergence; if $L = 0$, then the radius of convergence is $\rho = \infty$. 

Review of power series.
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Remarks: On summation indices:

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]
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where \( m = n - 1 \), that is, \( n = m + 1 \).
Power series solutions near regular points (Sect. 5.2).  

- We study: $P(x) y'' + Q(x) y' + R(x) y = 0$.  
- Review of power series.  
- **Regular point equations.**  
- Solutions using power series.  
- Examples of the power series method.
Regular point equations.

**Problem:** We look for solutions \( y \) of the variable coefficients equation

\[
P(x) y'' + Q(x) y' + R(x) y = 0.
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around \( x_0 \in \mathbb{R} \) where \( P(x_0) \neq 0 \) using a power series representation of the solution centered at \( x_0 \), that is,

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**Definition**

Given continuous functions \( P, Q, R : (x_1, x_2) \rightarrow \mathbb{R} \), a point \( x_0 \in (x_1, x_2) \) is called a **regular point** of the equation

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iff \( P(x_0) \neq 0 \). The point \( x_0 \) is called a **singular point** iff \( P(x_0) = 0 \).
Regular point equations.

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**Definition**

Given continuous functions $P, Q, R : (x_1, x_2) \rightarrow \mathbb{R}$, a point $x_0 \in (x_1, x_2)$ is called a *regular point* of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0.$$ 

iff $P(x_0) \neq 0$. The point $x_0$ is called a *singular point* iff $P(x_0) = 0$.

**Remark:** The equation order does not change near regular points.
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- **Solutions using power series.**
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Solutions using power series.

Summary for regular points:

(1) Propose a power series representation of the solution centered at $x_0$, given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n;$$

(2) Introduce Eq. (1) into the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$ 

(3) Find a recurrence relation among the coefficients $a_n$;

(4) Solve the recurrence relation in terms of free coefficients;

(5) If possible, add up the resulting power series for the solution $y$. 

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Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$
Examples of the power series method.

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Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c y = 0, \quad c \in \mathbb{R}.\]

Solution: Recall: The solution is \( y(x) = a_0 e^{-cx} \).
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: The solution is $y(x) = a_0 e^{-c x}$.

We now use the power series method.
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + cy = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: The solution is $y(x) = a_0 e^{-cx}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: The solution is $y(x) = a_0 e^{-c x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)}$$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c \, y = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: The solution is \( y(x) = a_0 \, e^{-c \cdot x} \).

We now use the power series method. We propose a power series centered at \( x_0 = 0 \):
\[
y(x) = \sum_{n=0}^{\infty} a_n \, x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} n \, a_n \, x^{(n-1)} = \sum_{n=1}^{\infty} n \, a_n \, x^{(n-1)}.
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: The solution is $y(x) = a_0 e^{-c x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} na_n x^{(n-1)} = \sum_{n=1}^{\infty} na_n x^{(n-1)}.$$ 

Change the summation index: $m = n - 1$, so $n = m + 1$. 
Examples of the power series method.

Example

Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[
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\]

Change the summation index: \( m = n - 1 \), so \( n = m + 1 \).

\[
y'(x) = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: The solution is $y(x) = a_0 e^{-c x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

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Change the summation index: $m = n - 1$, so $n = m + 1$.

$$y'(x) = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n.$$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + cy = 0, \quad c \in \mathbb{R}.$$  

Solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and $y'(x) = \sum_{n=0}^{\infty} (n + 1)a_{n+1} x^n$. 
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[
y' + c y = 0, \quad c \in \mathbb{R}.
\]

Solution: \( y(x) = \sum_{n=0}^{\infty} a_n x^n \), and \( y'(x) = \sum_{n=0}^{\infty} (n + 1)a_{n+1} x^n \).

Introduce \( y \) and \( y' \) into the differential equation,

\[
\sum_{n=0}^{\infty} (n + 1)a_{n+1} x^n + \sum_{n=0}^{\infty} c a_n x^n = 0
\]
Examples of the power series method.

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Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

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\]

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Introduce \( y \) and \( y' \) into the differential equation,

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\]

\[
\sum_{n=0}^{\infty} [(n + 1)a_{n+1} + c a_n] x^n = 0
\]
Examples of the power series method.

Example

Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[
y' + c \, y = 0, \quad c \in \mathbb{R}.
\]

Solution: \( y(x) = \sum_{n=0}^{\infty} a_n \, x^n \), and \( y'(x) = \sum_{n=0}^{\infty} (n + 1) a_{n+1} \, x^n \).

Introduce \( y \) and \( y' \) into the differential equation,

\[
\sum_{n=0}^{\infty} (n + 1) a_{n+1} \, x^n + \sum_{n=0}^{\infty} c \, a_n \, x^n = 0
\]

\[
\sum_{n=0}^{\infty} \left[ (n + 1) a_{n+1} + c \, a_n \right] \, x^n = 0
\]

The recurrence relation is \( (n + 1) a_{n+1} + c \, a_n = 0 \) for all \( n \geq 0 \).
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recurrence relation: $(n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0.$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c y = 0, \quad c \in \mathbb{R}.
\]

Solution: Recurrence relation: \( (n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0. \)

Equivalently: \( a_{n+1} = -\frac{c}{n+1} a_n. \)
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + cy = 0, \quad c \in \mathbb{R}.
\]

Solution: Recurrence relation: \((n + 1)a_{n+1} + ca_n = 0, \quad n \geq 0.\)

Equivalently: \(a_{n+1} = -\frac{c}{n+1} a_n.\) That is,
\[
n = 0, \quad a_1 = -ca_0
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recurrence relation: $(n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1} a_n$. That is,

$$n = 0, \quad a_1 = -c a_0 \quad \Rightarrow \quad a_1 = -c a_0,$$
Examples of the power series method.

**Example**
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
\[ y' + c y = 0, \quad c \in \mathbb{R}. \]

**Solution:** Recurrence relation: \((n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0.\)

Equivalently: \(a_{n+1} = -\frac{c}{n+1} a_n.\) That is,

- \(n = 0, \quad a_1 = -c a_0 \quad \Rightarrow \quad a_1 = -c a_0,\)

- \(n = 1, \quad 2a_2 = -c a_1\)
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c y = 0, \quad c \in \mathbb{R}.
\]

Solution: Recurrence relation: \((n + 1) a_{n+1} + c a_n = 0, \quad n \geq 0.\)
Equivalently: \( a_{n+1} = -\frac{c}{n+1} a_n \). That is,
\[
n = 0, \quad a_1 = -c a_0 \quad \Rightarrow \quad a_1 = -c a_0,
\]
\[
n = 1, \quad 2a_2 = -c a_1 \quad \Rightarrow \quad a_2 = \frac{c^2}{2!} a_0,
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recurrence relation: $(n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1} a_n$. That is,

- $n = 0, \quad a_1 = -c a_0 \quad \Rightarrow \quad a_1 = -c a_0$,
- $n = 1, \quad 2a_2 = -c a_1 \quad \Rightarrow \quad a_2 = \frac{c^2}{2!} a_0$,
- $n = 2, \quad 3a_3 = -c a_2$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c y = 0, \quad c \in \mathbb{R}.
\]

Solution: Recurrence relation: \((n + 1)a(n+1) + c a_n = 0, \quad n \geq 0\).

Equivalently: \(a_{n+1} = -\frac{c}{n+1} a_n\). That is,
\[

to\quad a_1 = -c a_0,
\]
\[

to\quad a_2 = \frac{c^2}{2!} a_0,
\]
\[

to\quad a_3 = -\frac{c^3}{3!} a_0,
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recurrence relation: $(n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1} a_n$. That is,

- $n = 0, \quad a_1 = -c a_0 \quad \Rightarrow \quad a_1 = -c a_0$,
- $n = 1, \quad 2a_2 = -c a_1 \quad \Rightarrow \quad a_2 = \frac{c^2}{2!} a_0$,
- $n = 2, \quad 3a_3 = -c a_2 \quad \Rightarrow \quad a_3 = -\frac{c^3}{3!} a_0$,
- $n = 3, \quad 4a_4 = -c a_3$.
Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$  

Solution: Recurrence relation: $(n + 1)a_{n+1} + c a_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1} a_n$. That is,

- $n = 0$, $a_1 = -c a_0 \Rightarrow a_1 = -c a_0$,

- $n = 1$, $2a_2 = -c a_1 \Rightarrow a_2 = \frac{c^2}{2!} a_0$,

- $n = 2$, $3a_3 = -c a_2 \Rightarrow a_3 = -\frac{c^3}{3!} a_0$,

- $n = 3$, $4a_4 = -c a_3 \Rightarrow a_4 = \frac{c^4}{4!} a_0$. 
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[
y' + cy = 0, \quad c \in \mathbb{R}.
\]

Solution: Solved recurrence relation: \( a_n = \frac{(-c)^n}{n!} a_0 \).
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.\]

Solution: Solved recurrence relation: $a_n = \frac{(-c)^n}{n!} a_0$.

The solution $y$ of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} a_0 x^n$$
Examples of the power series method.

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Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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The solution $y$ of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} a_0 x^n \implies y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c x)^n}{n!}.$$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + c y = 0, \quad c \in \mathbb{R}.
\]

Solution: Solved recurrence relation: \( a_n = \frac{(-c)^n}{n!} a_0. \)

The solution \( y \) of the differential equation is given by
\[
y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} a_0 x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c x)^n}{n!}.
\]

If we recall the power series \( e^{ax} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} \),
Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$

Solution: Solved recurrence relation: $a_n = \frac{(-c)^n}{n!} a_0$.

The solution $y$ of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} a_0 x^n \Rightarrow y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c x)^n}{n!}.$$

If we recall the power series $e^{ax} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!}$,

then, we conclude that the solution is $y(x) = a_0 e^{-c x}$. △
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: Recall: The characteristic polynomial is $r^2 + 1 = 0,$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y'' + y = 0.
\]

Solution: Recall: The characteristic polynomial is \( r^2 + 1 = 0 \), hence the general solution is \( y(x) = a_0 \cos(x) + a_1 \sin(x) \).
Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$  

Solution: Recall: The characteristic polynomial is $r^2 + 1 = 0$, hence the general solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

We re-obtain this solution using the power series method:
Examples of the power series method.

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Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

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Solution: Recall: The characteristic polynomial is \( r^2 + 1 = 0 \), hence the general solution is \( y(x) = a_0 \cos x + a_1 \sin x \).

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y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} na_n x^{(n-1)}
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Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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$$y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m,$$
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where $m = n - 1$, so $n = m + 1$;
Examples of the power series method.

Example

Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[
y'' + y = 0.
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\[
y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m,
\]

where \( m = n - 1 \), so \( n = m + 1 \);

\[
y'' = \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2}
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: The characteristic polynomial is $r^2 + 1 = 0$, hence the general solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

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where $m = n - 1$, so $n = m + 1$;

$$y'' = \sum_{n=2}^{\infty} n(n - 1)a_n x^{(n-2)} = \sum_{m=0}^{\infty} (m + 2)(m + 1)a_{m+2} x^m.$$
Examples of the power series method.

**Example**

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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**Solution:** Recall: The characteristic polynomial is $r^2 + 1 = 0$, hence the general solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

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where $m = n - 1$, so $n = m + 1$;

$$y'' = \sum_{n=2}^{\infty} n(n - 1)a_n x^{(n-2)} = \sum_{m=0}^{\infty} (m + 2)(m + 1) a_{m+2} x^m.$$

where $m = n - 2$, so $n = m + 2$. 
Examples of the power series method.

Example

Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

\[ y'' + y = 0. \]

Solution: Introduce \( y \) and \( y'' \) into the differential equation,

\[
\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Introduce $y$ and $y''$ into the differential equation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1)a_{n+2} + a_n \right] x^n = 0.$$
Examples of the power series method.

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Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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$$\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} + a_n] x^n = 0.$$ 

The recurrence relation is $(n + 2)(n + 1)a_{n+2} + a_n = 0, \ n \geq 0.$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
$$y'' + y = 0.$$

Solution: Introduce $y$ and $y''$ into the differential equation,
$$
\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0
$$

$$
\sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} + a_n] x^n = 0.
$$

The recurrence relation is $(n + 2)(n + 1)a_{n+2} + a_n = 0, \quad n \geq 0$.

Equivalently: $(n + 2)(n + 1) a_{n+2} = -a_n$. 
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: $(n + 2)(n + 1) a_{(n+2)} = -a_n, \ n \geq 0.$
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
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Solution: Recall: \((n + 2)(n + 1) a_{n+2} = -a_n, \ n \geq 0.\)

For \( n \) even: \( n = 0, \ (2)(1)a_2 = -a_0 \)
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Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

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For \( n \) even: \( n = 0, \ (2)(1)a_2 = -a_0 \ \Rightarrow \ a_2 = -\frac{1}{2!} a_0, \)
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: $(n + 2)(n + 1)a_{n+2} = -a_n$, $n \geq 0$.

For $n$ even: $n = 0$, $(2)(1)a_2 = -a_0 \Rightarrow a_2 = -\frac{1}{2!}a_0,

n = 2, \quad (4)(3)a_4 = -a_2$
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Example
Find a power series solution \(y(x)\) around the point \(x_0 = 0\) of the equation
\[y'' + y = 0.\]

Solution: Recall: \((n + 2)(n + 1)a_{n+2} = -a_n, \ n \geq 0.\)

For \(n\) even: \(n = 0, \ (2)(1)a_2 = -a_0 \ \Rightarrow \ a_2 = -\frac{1}{2!} a_0,\)
\[n = 2, \ (4)(3)a_4 = -a_2 \ \Rightarrow \ a_4 = \frac{1}{4!} a_0,\]
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y'' + y = 0.
\]

Solution: Recall: \((n + 2)(n + 1) a_{n+2} = -a_n, \ n \geq 0.\)

For \( n \) even: \( n = 0, \ \ (2)(1)a_2 = -a_0 \ \Rightarrow \ a_2 = -\frac{1}{2!} a_0, \)
\[
n = 2, \ \ (4)(3)a_4 = -a_2 \ \Rightarrow \ a_4 = \frac{1}{4!} a_0,
\]
\[
n = 4, \ \ (6)(5)a_6 = -a_4
\]
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y'' + y = 0.
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Solution: Recall: \((n + 2)(n + 1) a_{n+2} = -a_n, \quad n \geq 0.\)

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n = 2, \quad (4)(3)a_4 = -a_2 \quad \Rightarrow \quad a_4 = \frac{1}{4!} a_0,
\]
\[
n = 4, \quad (6)(5)a_6 = -a_4 \quad \Rightarrow \quad a_6 = -\frac{1}{6!} a_0.
\]
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
\[ y'' + y = 0. \]

Solution: Recall: $(n + 2)(n + 1)a_{n+2} = -a_n$, $n \geq 0$.

For $n$ even:

- $n = 0$, $(2)(1)a_2 = -a_0$ \implies $a_2 = -\frac{1}{2!}a_0$,

- $n = 2$, $(4)(3)a_4 = -a_2$ \implies $a_4 = \frac{1}{4!}a_0$,

- $n = 4$, $(6)(5)a_6 = -a_4$ \implies $a_6 = -\frac{1}{6!}a_0$.

We obtain: $a_{2k} = \frac{(-1)^k}{(2k)!}a_0$, for $k \geq 0$. 
Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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For $n$ odd: $n = 1$, \hspace{1cm} (3)(2)a_3 = -a_1
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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For $n$ odd: $n = 1$, $(3)(2)a_3 = -a_1 \implies a_3 = -\frac{1}{3!} a_1$, 

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For \( n \) odd: \( n = 1 \), \( (3)(2)a_3 = -a_1 \) \( \Rightarrow \) \( a_3 = -\frac{1}{3!} a_1 \),

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Examples of the power series method.

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Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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$$n = 3, \quad (5)(4)a_5 = -a_3 \quad \Rightarrow \quad a_5 = \frac{1}{5!}a_1,$$
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For $n$ odd: $n = 1$, \( (3)(2)a_3 = -a_1 \quad \Rightarrow \quad a_3 = -\frac{1}{3!} a_1, \)

$n = 3$, \( (5)(4)a_5 = -a_3 \quad \Rightarrow \quad a_5 = \frac{1}{5!} a_1, \)

$n = 5$, \( (7)(6)a_7 = -a_5 \)
Examples of the power series method.

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$$n = 3, \quad (5)(4)a_5 = -a_3 \Rightarrow a_5 = \frac{1}{5!} a_1,$$

$$n = 5, \quad (7)(6)a_7 = -a_5 \Rightarrow a_7 = -\frac{1}{7!} a_1.$$

We obtain $a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1$ for $k \geq 0$. 
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation

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Solution: Recall: \( a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \) and \( a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1. \)
Examples of the power series method.

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Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

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$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \quad \text{and} \quad a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1.$$ 

Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}.$$
Examples of the power series method.

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Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
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One can check that these are precisely the power series representations of the cosine and sine functions,
Examples of the power series method.

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One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$
Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

\[ y'' - xy = 0. \]
Examples of the power series method.

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Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

**Solution:** We propose: $y = \sum_{n=0}^{\infty} a_n(x - 2)^n$. 
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It is convenient to rewrite the function $xy$ as follows,

$$xy = \sum_{n=0}^{\infty} a_n x(x - 2)^n$$
Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$  

Solution: We propose: $y = \sum_{n=0}^{\infty} a_n(x - 2)^n$.

It is convenient to rewrite the function $xy$ as follows,

$$xy = \sum_{n=0}^{\infty} a_n x(x - 2)^n = \sum_{n=0}^{\infty} a_n [(x - 2) + 2](x - 2)^n,$$
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Examples of the power series method.

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It is convenient to rewrite the function $xy$ as follows,

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We relabel the first sum: $\sum_{n=0}^{\infty} a_n (x - 2)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x - 2)^n$. 

Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

Solution: We relabel the $y''$, 

\[\begin{align*}
\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^{n+2} &= 0 \\
\sum_{n=1}^{\infty} a_n(x-2)^{n-1} &= 0
\end{align*}\]
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0.$$ 

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Introduce \( y'' \) and \( xy \) in the differential equation
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

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Introduce \( y'' \) and \( xy \) in the differential equation

\[ \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}(x - 2)^n - \sum_{n=0}^{\infty} 2a_n(x - 2)^n - \sum_{n=1}^{\infty} a_{n-1}(x - 2)^n = 0 \]
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0.$$ 

Solution: We relabel the $y''$, 

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n.$$ 

Introduce $y''$ and $xy$ in the differential equation

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{n-1}(x-2)^n = 0$$

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} \right] (x-2)^n = 0.$$
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Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

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$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}(x-2)^n.$$ 

Introduce $y''$ and $xy$ in the differential equation

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{(n-1)}(x-2)^n = 0$$ 

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)}\right] (x-2)^n = 0.$$ 

The recurrence relation for the coefficients $a_n$ is: 

$$a_2 - a_0 = 0, \quad (n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \quad n \geq 1.$$
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

\[ y'' - x \, y = 0. \]

Solution: The recurrence relation is:

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We solve this recurrence relation for the first four coefficients,
Examples of the power series method.

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\[ n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0, \]

\[ n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \]
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0.$$  

Solution: The recurrence relation is:

$$a_2 - a_0 = 0, \quad (n + 2)(n + 1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1.$$  

We solve this recurrence relation for the first four coefficients,

$$\begin{align*}
    n = 0 & \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0, \\
    n = 1 & \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},
\end{align*}$$
Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation
\[
y'' - xy = 0.
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Solution: The recurrence relation is:
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\[
n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0,
\]

\[
n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},
\]

\[
n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0
\]
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

\[ y'' - x y = 0. \]

Solution: The recurrence relation is:

\[ a_2 - a_0 = 0, \quad (n + 2)(n + 1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1. \]

We solve this recurrence relation for the first four coefficients,

\[ n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0, \]

\[ n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3}, \]

\[ n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}. \]
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Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

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$$n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0,$$

$$n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},$$

$$n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}.$$  

$$y \simeq a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x - 2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x - 2)^4.$$
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation 

$$y'' - x y = 0.$$ 

Solution: The first terms in the power series expression for $y$ are

$$y \approx a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x - 2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x - 2)^4.$$
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$ 

Solution: The first terms in the power series expression for $y$ are

$y \approx a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x - 2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x - 2)^4.$

$$y = a_0 \left[1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3 + \frac{1}{6}(x - 2)^4 + \cdots\right]$$

$$+ a_1 \left[(x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4 + \cdots\right]$$
Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0.$$ 

Solution: The first terms in the power series expression for $y$ are

$$y \approx a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x - 2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x - 2)^4.$$ 

$$y = a_0 \left[1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3 + \frac{1}{6}(x - 2)^4 + \cdots\right]$$

$$+ a_1 \left[(x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4 + \cdots\right]$$

So the first three terms on each fundamental solution are given by

$$y_1 \approx 1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3, \quad y_2 \approx (x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4.$$
The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[ (x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0. \]
- Solutions to the Euler equation near \( x_0 \).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Overview: Equations with singular points.

Recall: The point $x_0 \in \mathbb{R}$ is a **singular point** of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

iff holds $P(x_0) = 0$.
Overview: Equations with singular points.

Recall: The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

iff holds $P(x_0) = 0$.

Remarks:

- We are interested in finding solutions to the equation above arbitrary close to a singular point $x_0$. 

- The order of the differential equation changes in a neighborhood of a singular point.

- In the limit $x \to x_0$ the following could happen:
  
  (a) The two linearly independent solutions remain bounded.
  
  (b) Only one solution remains bounded.
  
  (c) None solution remains bounded.
Overview: Equations with singular points.

Recall: The point \( x_0 \in \mathbb{R} \) is a singular point of the equation

\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]

iff holds \( P(x_0) = 0 \).

Remarks:

- We are interested in finding solutions to the equation above arbitrary close to a singular point \( x_0 \).
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▶ If the singular point of a differential equation is not so singular, in a sense to be made precise later on, then it is known how to find solutions to such equation.
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▶ Singular points where the singular behavior of the solution is somehow mild, in a sense to be made precise later, will be called regular-singular points.

▶ The main example of a equation with a regular-singular point is the Euler differential equation.
The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \(x_0\).
- The roots of the indicial polynomial.
  - Different real roots.
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The Euler equation

Definition
Given real constants \( p_0, q_0 \), the Euler differential equation for the unknown \( y \) with singular point at \( x_0 \in \mathbb{R} \) is given by

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Given real constants $p_0, q_0$, the Euler differential equation for the unknown $y$ with singular point at $x_0 \in R$ is given by

$$(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$$ 

Remarks:

- The Euler equation has variable coefficients.
- Functions $y(x) = e^{rx}$ are not solutions of the Euler equation.
- The point $x_0 \in R$ is a singular point of the equation.
- The particular case $x_0 = 0$ is given by $x^2 y'' + p_0 x y' + q_0 y = 0$. 
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- The roots of the indicial polynomial.
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Solutions to the Euler equation near $x_0$.

**Summary of the main idea:**

- The main idea to find solution to the constant coefficients equation $y'' + a_1 y' + a_0 y = 0$ was to look for functions of the form $y(x) = e^{rx}$.
Solutions to the Euler equation near $x_0$.

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▶ The main idea to find solution to the constant coefficients equation $y'' + a_1 y' + a_0 y = 0$ was to look for functions of the form $y(x) = e^{rx}$. The exponential cancels out from the equation and we obtain an equation only for $r$ without $x$. 

\[
(e^{rx})'' + a_1 (e^{rx})' + a_0 (e^{rx}) = 0 \Leftrightarrow (r^2 + a_1 r + a_0) e^{rx} = 0.
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  $$ (x^2 r^2 + p_0 x r + q_0) e^{rx} = 0 \iff x^2 r^2 + p_0 x r + q_0 = 0, $$

  but the later equation still involves the variable $x$. 
Solutions to the Euler equation near $x_0$.

**Summary of the main idea:** Look for solutions like $y(x) = x^r$. 

Introduce $y = x^r$ into Euler's equation $x^2y'' + p_0xy' + q_0y = 0$, for $x \neq 0$ we obtain 

$$r(r-1) + p_0r + q_0 = 0.$$ 

The last equation involves only $r$, not $x$. 

This equation is called the **indicial equation** and is also called the **Euler characteristic equation**.
Solutions to the Euler equation near $x_0$.

**Summary of the main idea:** Look for solutions like $y(x) = x^r$.
These function have the following property:

$$y'(x) = r x^{r-1}$$
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y'(x) = rx^{r-1} \quad \Rightarrow \quad x\,y'(x) = rx^r;
\]
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y''(x) = r(r-1)x^{r-2} \quad \Rightarrow \quad x^2\,y''(x) = r(r-1)x^r.
\]

Introduce $y = x^r$ into Euler’s equation $x^2\,y'' + p_0\,x\,y' + q_0\,y = 0$, where $p_0$ and $q_0$ are constants.
Solutions to the Euler equation near $x_0$.

**Summary of the main idea:** Look for solutions like $y(x) = x^r$.

These functions have the following property:

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Solutions to the Euler equation near $x_0$.

Theorem (Euler equation)

Given $p_0$, $q_0$, $x_0 \in \mathbb{R}$, consider the Euler equation

$$(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0. \quad (3)$$

Let $r_+$, $r_-$ be solutions of $r(r - 1) + p_0 r + q_0 = 0$.

(a) If $r_+ \neq r_-$, then a real-valued general solution of Eq. (3) is

$$y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R}.$$ 

(b) If $r_+ = r_-$, then a real-valued general solution of Eq. (3) is

$$y(x) = \left[ c_0 + c_1 \ln(|x - x_0|) \right] |x - x_0|^{r_+}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R}.$$ 

Given $x_0 \neq x_1$, $y_0$, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$
The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
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  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Different real roots.

Example
Find the general solution of the Euler equation

\[ x^2 y'' + 4x y' + 2y = 0. \]
Different real roots.

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Find the general solution of the Euler equation
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Solution: We look for solutions of the form \( y(x) = x^r \),
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\[ x^2 y'' + 4x y' + 2y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),

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Different real roots.

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Introduce \( y(x) = x^r \) into Euler equation,
Different real roots.

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The solutions of \( r^2 + 3r + 2 = 0 \) are given by
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The solutions of \( r^2 + 3r + 2 = 0 \) are given by

\[
r_{\pm} = \frac{1}{2} \left[ -3 \pm \sqrt{9 - 8} \right]
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Different real roots.

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\[ r_{\pm} = \frac{1}{2} \left[ -3 \pm \sqrt{9 - 8} \right] \Rightarrow r_+ = -1 \quad r_- = -2. \]
The general solution is \( y(x) = c_1 |x|^{-1} + c_2 |x|^{-2}. \) △
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Solutions to the Euler equation near \(x_0\).

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- Different real roots.
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**Example**

Find the general solution of \( x^2 y'' - 3x y' + 4y = 0 \).
Repeated roots.

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[r(r - 1) - 3r + 4] x^r = 0
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Repeated roots.

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r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}]
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[r(r - 1) - 3r + 4] x^r = 0 \iff r(r - 1) - 3r + 4 = 0.
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The solutions of \( r^2 - 4r + 4 = 0 \) are given by
\[
r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \quad \Rightarrow \quad r_{\pm} = r_{-} = 2.
\]
Repeated roots.

Example
Find the general solution of \( x^2 y'' - 3x y' + 4 y = 0 \).

Solution: We look for solutions of the form \( y(x) = x^r \),
\[
x y'(x) = rx^r, \quad x^2 y''(x) = r(r - 1) x^r.
\]
Introduce \( y(x) = x^r \) into Euler equation,
\[
[r(r - 1) - 3r + 4] x^r = 0 \quad \Leftrightarrow \quad r(r - 1) - 3r + 4 = 0.
\]
The solutions of \( r^2 - 4r + 4 = 0 \) are given by
\[
r_{\pm} = \frac{1}{2} \left[ 4 \pm \sqrt{16 - 16} \right] \quad \Rightarrow \quad r_+ = r_- = 2.
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Two linearly independent solutions are
\[
y_1(x) = x^2, \quad y_2 = x^2 \ln(|x|).
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Repeated roots.

**Example**
Find the general solution of \( x^2 y'' - 3x y' + 4 y = 0 \).

**Solution:** We look for solutions of the form \( y(x) = x^r \),

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x y'(x) = r x^r, \quad x^2 y''(x) = r(r - 1) x^r.
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Introduce \( y(x) = x^r \) into Euler equation,

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Two linearly independent solutions are

\[
y_1(x) = x^2, \quad y_2 = x^2 \ln(|x|).
\]

The general solution is \( y(x) = c_1 x^2 + c_2 x^2 \ln(|x|) \). \( \triangleq \)
The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \(x_0\).
- **The roots of the indicial polynomial.**
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Different complex roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13 y = 0. \]
Different complex roots.

Example

Find the general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),

\[ r^2 - 4r + 13 = 0. \]

The solutions of the indicial equation are

\[ r_{\pm} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i. \]

The general solution is

\[ y(x) = c_1 |x|^{2+3i} + c_2 |x|^{2-3i}. \]
Different complex roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ x y'(x) = rx^r, \]
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Find the general solution of the Euler equation
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\[ x y'(x) = r x^r, \quad x^2 y''(x) = r(r-1) x^r. \]

Introduce \( y(x) = x^r \) into Euler equation
\[ r(r-1) - 3r + 13 = 0. \]
Different complex roots.

Example
Find the general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

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Introduce \( y(x) = x^r \) into Euler equation

\[ [r(r - 1) - 3r + 13] x^r = 0 \]
Different complex roots.

Example

Find the general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),

\[
\begin{align*}
x y'(x) &= rx^r, \\
x^2 y''(x) &= r(r - 1)x^r.
\end{align*}
\]

Introduce \( y(x) = x^r \) into Euler equation

\[
[r(r - 1) - 3r + 13] x^r = 0 \iff r(r - 1) - 3r + 13 = 0.
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Different complex roots.

Example

Find the general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),

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Introduce \( y(x) = x^r \) into Euler equation

\[ [r(r - 1) - 3r + 13] x^r = 0 \quad \iff \quad r(r - 1) - 3r + 13 = 0. \]

The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are

\[ r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 52}] \]
Different complex roots.

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Find the general solution of the Euler equation
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Example

Find the general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

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Introduce \( y(x) = x^r \) into Euler equation

\[ \left[ r(r - 1) - 3r + 13 \right] x^r = 0 \quad \Leftrightarrow \quad r(r - 1) - 3r + 13 = 0. \]

The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are

\[ r_\pm = \frac{1}{2} \left[ 4 \pm \sqrt{16 - 52} \right] \Rightarrow r_\pm = \frac{1}{2} \left[ 4 \pm \sqrt{-36} \right] \Rightarrow \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases} \]
Different complex roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ x y'(x) = r x^r, \quad x^2 y''(x) = r(r-1) x^r. \]

Introduce \( y(x) = x^r \) into Euler equation
\[ \left[ r(r-1) - 3r + 13 \right] x^r = 0 \iff r(r-1) - 3r + 13 = 0. \]

The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are
\[ r_\pm = \frac{1}{2} [4 \pm \sqrt{16 - 52}] \Rightarrow r_\pm = \frac{1}{2} [4 \pm \sqrt{-36}] \Rightarrow \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases} \]

The general solution is \( y(x) = c_1 |x|^{2+3i} + c_2 |x|^{2-3i}. \)
Different complex roots.

Theorem (Real-valued fundamental solutions)

If $p_0, q_0 \in \mathbb{R}$ satisfy that $[(p_0 - 1)^2 - 4q_0] < 0$, then the indicial polynomial $p(r) = r(r - 1) + p_0r + q_0$ of the Euler equation

$$x^2 y'' + p_0x y' + q_0 y = 0 \tag{4}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{(p_0 - 1)}{2}, \quad \beta = \frac{1}{2} \sqrt{4q_0 - (p_0 - 1)^2}.$$

Furthermore, a fundamental set of solution to Eq. (4) is

$$\tilde{y}_1(x) = |x|^{(\alpha+i\beta)}, \quad \tilde{y}_2(x) = |x|^{(\alpha-i\beta)},$$

while another fundamental set of solutions to Eq. (4) is

$$y_1(x) = |x|^{\alpha} \cos(\beta \ln |x|), \quad y_2(x) = |x|^{\alpha} \sin(\beta \ln |x|).$$
Different complex roots.

**Proof:** Given \( \tilde{y}_1 = |x|^{\alpha + i\beta} \) and \( \tilde{y}_2 = |x|^{\alpha - i\beta} \),
Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$
Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$. 
Different complex roots.

**Proof:** Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)}.$$
Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^{\alpha} |x|^{i\beta}.$$
Different complex roots.

Proof: Given \( \tilde{y}_1 = |x|^{(\alpha+i\beta)} \) and \( \tilde{y}_2 = |x|^{(\alpha-i\beta)} \), introduce

\[
y_1 = \frac{1}{2} (\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i} (\tilde{y}_1 - \tilde{y}_2).
\]

Use another Euler equation to rewrite \( \tilde{y}_1 \) and \( \tilde{y}_2 \),

\[
\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^\alpha e^{\text{ln}(|x|^{i\beta})}
\]
Different complex roots.

**Proof:** Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha |x|^i = |x|^\alpha e^{\ln(|x|^i)} = |x|^\alpha e^{i\beta \ln(|x|)}.$$
Different complex roots.

Proof: Given \( \tilde{y}_1 = |x|^{(\alpha+i\beta)} \) and \( \tilde{y}_2 = |x|^{(\alpha-i\beta)} \), introduce

\[
y_1 = \frac{1}{2} (\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i} (\tilde{y}_1 - \tilde{y}_2).
\]

Use another Euler equation to rewrite \( \tilde{y}_1 \) and \( \tilde{y}_2 \),

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\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^\alpha e^{\ln(|x|^{i\beta})} = |x|^\alpha e^{i\beta \ln(|x|)}.
\]

\[
\tilde{y}_1 = |x|^\alpha \left[ \cos(\beta \ln |x|) + 1 \sin(\beta \ln |x|) \right],
\]
Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^{\alpha} e^{\ln(|x|^{i\beta})} = |x|^{\alpha} e^{i\beta \ln(|x|)}.$$

$$\tilde{y}_1 = |x|^{\alpha} \left[ \cos(\beta \ln |x|) + 1 \sin(\beta \ln |x|) \right],$$

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Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^\alpha e^{\ln(|x|^{i\beta})} = |x|^\alpha e^{i\beta \ln(|x|)}.$$

$$\tilde{y}_1 = |x|^\alpha \left[ \cos(\beta \ln |x|) + 1 \sin(\beta \ln |x|) \right],$$

$$\tilde{y}_2 = |x|^\alpha \left[ \cos(\beta \ln |x|) - 1 \sin(\beta \ln |x|) \right].$$

We conclude that

$$y_1(x) = |x|^\alpha \cos(\beta \ln |x|), \quad y_2(x) = |x|^\alpha \sin(\beta \ln |x|).$$
Different complex roots.

Example

Find a real-valued general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]
Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

**Solution:** The indicial equation is \( r(r - 1) - 3r + 13 = 0. \)
Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

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**Solution:** The indicial equation is \( r(r - 1) - 3r + 13 = 0. \)

The solutions of the indicial equations are
Different complex roots.

Example
Find a real-valued general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: The indicial equation is \( r(r - 1) - 3r + 13 = 0 \).
The solutions of the indicial equations are

\[ r^2 - 4r + 13 = 0 \]
Different complex roots.

Example
Find a real-valued general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: The indicial equation is \( r(r - 1) - 3r + 13 = 0. \)
The solutions of the indicial equations are

\[ r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i. \]
Different complex roots.

Example
Find a real-valued general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: The indicial equation is \( r(r - 1) - 3r + 13 = 0. \)
The solutions of the indicial equations are
\[ r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i. \]
A complex-valued general solution is
\[ y(x) = \tilde{c}_1 |x|^{(2+3i)} + \tilde{c}_2 |x|^{(2-3i)} \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \]
Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

\[ x^2 y'' - 3x y' + 13 y = 0. \]

**Solution:** The indicial equation is \( r(r - 1) - 3r + 13 = 0 \).

The solutions of the indicial equations are

\[ r^2 - 4r + 13 = 0 \implies r_+ = 2 + 3i, \quad r_- = 2 - 3i. \]

A complex-valued general solution is

\[ y(x) = \tilde{c}_1 |x|^{2+3i} + \tilde{c}_2 |x|^{2-3i}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \]

A real-valued general solution is

\[ y(x) = c_1 |x|^2 \cos(3 \ln |x|) + c_2 |x|^2 \sin(3 \ln |x|), \quad c_1, c_2 \in \mathbb{R}. \]
Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- Example: Method to find solutions.
Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
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- Method to find solutions.
- Example: Method to find solutions.

Recall:
The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

iff holds that $P(x_0) = 0$. 
Equations with regular-singular points.

Definition
A singular point \( x_0 \in \mathbb{R} \) of the equation

\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]

is called a \textit{regular-singular} point iff the following limits are finite,

\[
\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)}, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)},
\]

and both functions

\[
\frac{(x - x_0) Q(x)}{P(x)}, \quad \frac{(x - x_0)^2 R(x)}{P(x)}
\]

admit convergent Taylor series expansions around \( x_0 \).
Equations with regular-singular points.

Remark:
- If $x_0$ is a regular-singular point of

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

and $P(x) \simeq (x - x_0)^n$ near $x_0$, then near $x_0$ holds

$$Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.$$
Remark:

- If \( x_0 \) is a regular-singular point of

\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]

and \( P(x) \simeq (x - x_0)^n \) near \( x_0 \), then near \( x_0 \) holds

\[
Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.
\]

- The main example is an Euler equation, case \( n = 2 \),

\[
(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.
\]
Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.
Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

$$(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,$$

where $p_0, q_0, x_0$, are real constants.
Equations with regular-singular points.

**Example**

Show that the singular point of every Euler equation is a regular-singular point.

**Solution:** Consider the general Euler equation

\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,\]

where \(p_0, q_0, x_0\), are real constants. This is an equation \(Py'' + Qy' + Ry = 0\) with

\[P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.\]
Equations with regular-singular points.

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Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

$$(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,$$

where $p_0$, $q_0$, $x_0$, are real constants. This is an equation $Py'' + Qy' + R y = 0$ with

$$P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.$$ 

Therefore, we obtain,

$$\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)}$$
Equations with regular-singular points.

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Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,\]

where \(p_0, q_0, x_0\), are real constants. This is an equation

\[P y'' + Q y' + R y = 0\]

with

\[P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.\]

Therefore, we obtain,

\[\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)} = p_0,\]
Equations with regular-singular points.

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Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,\]

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\[Py'' + Qy' + Ry = 0\]

with

\[P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.\]

Therefore, we obtain,

\[\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0, \quad \lim_{x \to x_0} \frac{(x - x_0)^2R(x)}{P(x)} = q_0.\]
Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,\]

where \(p_0, q_0, x_0,\) are real constants. This is an equation

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with

\[P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.\]

Therefore, we obtain,

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Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,\]

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Therefore, we obtain,

\[\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)} = p_0, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0.\]

We conclude that \(x_0\) is a regular-singular point.  \(\triangle\)
Equations with regular-singular points.

**Remark:** Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at $x_0$ is close to an Euler equation.
Equations with regular-singular points.

**Remark:** Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at $x_0$ is close to an Euler equation.

**Proof:**
For $x \neq x_0$ divide the equation by $P(x)$,
Equations with regular-singular points.

**Remark:** Every equation \( Py'' + Qy' + Ry = 0 \) with a regular-singular point at \( x_0 \) is close to an Euler equation.

**Proof:**
For \( x \neq x_0 \) divide the equation by \( P(x) \),
\[
y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0,
\]
Equations with regular-singular points.

**Remark:** Every equation \( Py'' + Qy' + Ry = 0 \) with a regular-singular point at \( x_0 \) is close to an Euler equation.

**Proof:**
For \( x \neq x_0 \) divide the equation by \( P(x) \),

\[
y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0,
\]

and multiply it by \((x - x_0)^2\),
Equations with regular-singular points.

Remark: Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at $x_0$ is close to an Euler equation.

Proof:
For $x \neq x_0$ divide the equation by $P(x)$,

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0,$$

and multiply it by $(x - x_0)^2$,

$$(x - x_0)^2 y'' + (x - x_0) \left[ \frac{(x - x_0)Q(x)}{P(x)} \right] y' + \left[ \frac{(x - x_0)^2 R(x)}{P(x)} \right] y = 0.$$
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The factors between $[ ]$ approach constants, say $p_0$, $q_0$, as $x \to x_0$. 
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Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- **Examples: Equations with regular-singular points.**
- Method to find solutions.
- Example: Method to find solutions.
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

$$(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,$$

where $\alpha$ is a real constant.

Solution:
Find the singular points of this equation,

$$0 = P(x) = (1 - x^2)(1 + x),$$

$\Rightarrow$

$$\{x_0 = 1, x_1 = -1\}.$$

Case $x_0 = 1$:
We then have

$$(x - 1)Q(x)P(x) = (x - 1)(-2x)(1 - x)(1 + x) = 2x^2 + x,$$

$$(x - 1)^2R(x)P(x) = (x - 1)^2[\alpha(\alpha + 1)](1 - x)(1 + x) = (x - 1)[\alpha(\alpha + 1)]1 + x;$$

both functions above have Taylor series around $x_0 = 1$. 

Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation
\[
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where \(\alpha\) is a real constant.

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Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

$$(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,$$

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$$0 = P(x)$$
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Find the singular points of this equation,

\[0 = P(x) = (1 - x^2)\]
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Find the singular points of this equation,

\[0 = P(x) = (1 - x^2) = (1 - x)(1 + x)\]
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\[0 = P(x) = (1 - x^2) = (1 - x)(1 + x) \quad \Rightarrow \quad \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases} \]
Examples: Equations with regular-singular points.

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Examples: Equations with regular-singular points.

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Case \(x_0 = 1\): We then have
\[
\frac{(x - 1) Q(x)}{P(x)}
\]
Examples: Equations with regular-singular points.

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Case \(x_0 = 1\): We then have

\[
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\[\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \frac{2x}{1 + x},\]

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Examples: Equations with regular-singular points.

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Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)}
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We conclude that \(x_0 = 1\) is a regular-singular point.
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution:
Case \(x_1 = -1:\)
Examples: Equations with regular-singular points.

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\[
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Examples: Equations with regular-singular points.

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\frac{(x + 1) Q(x)}{P(x)} = \frac{(x + 1)(-2x)}{(1 - x)(1 + x)} = -\frac{2x}{1 - x},
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Examples: Equations with regular-singular points.

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Examples: Equations with regular-singular points.

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\]

Both functions above have Taylor series \(x_1 = -1\).
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha (\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Recall:

\[
\frac{(x + 1) Q(x)}{P(x)} = -\frac{2x}{1 - x}, \quad \frac{(x + 1)^2 R(x)}{P(x)} = \frac{(x + 1) [\alpha (\alpha + 1)]}{1 - x}.
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\]

Furthermore, the following limits are finite,

\[
\lim_{x \to -1} \frac{(x + 1) Q(x)}{P(x)}
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Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

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\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Recall:

\[
\begin{align*}
\frac{(x + 1) Q(x)}{P(x)} &= -\frac{2x}{1-x}, \\
\frac{(x + 1)^2 R(x)}{P(x)} &= \frac{(x + 1)[\alpha(\alpha + 1)]}{1-x}.
\end{align*}
\]

Furthermore, the following limits are finite,

\[
\lim_{x \to -1} \frac{(x + 1) Q(x)}{P(x)} = 1, \quad \lim_{x \to -1} \frac{(x + 1)^2 R(x)}{P(x)} = 0.
\]

Therefore, the point \(x_1 = -1\) is a regular-singular point. \(\triangleright\)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

$$(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.$$
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Find the singular points:
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation
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Case \(x_0 = -2\):
\[
\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)}
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Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

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Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[
\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)}
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\]
So \(x_0 = -2\) is not a regular-singular point.

Case \(x_1 = 1\):
\[
\left(\frac{(x - 1)}{P(x)}\right) = \frac{3}{(x + 2)^2}, \quad \left(\frac{(x - 1)}{P(x)}\right) = \frac{2}{(x + 2)^2};
\]
Both functions have Taylor series around \(x_1 = 1\).
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.$$  

Solution: Find the singular points: $x_0 = -2$ and $x_1 = 1$.

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Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{(x + 2)} = \pm \infty.\]

So \(x_0 = -2\) is not a regular-singular point.
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).
Case \(x_0 = -2\):
\[
\lim_{{x \to -2}} \frac{(x + 2)Q(x)}{P(x)} = \lim_{{x \to -2}} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{{x \to -2}} \frac{3}{(x + 2)} = \pm \infty.
\]
So \(x_0 = -2\) is not a regular-singular point. Case \(x_1 = 1\):
\[
\frac{(x - 1)Q(x)}{P(x)}
\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2 (x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[
\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{(x + 2)} = \pm \infty.
\]

So \(x_0 = -2\) is not a regular-singular point. Case \(x_1 = 1\):

\[
\frac{(x - 1)Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)}
\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[
\lim_{x\to-2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x\to-2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x\to-2} \frac{3}{(x + 2)} = \pm\infty.
\]

So \(x_0 = -2\) is not a regular-singular point.

Case \(x_1 = 1\):

\[
\frac{(x - 1)Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = \frac{-3(x - 1)}{(x + 2)^2},
\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[
\lim_{x \to -2} \frac{(x + 2) Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{(x + 2)} = \pm \infty.
\]

So \(x_0 = -2\) is not a regular-singular point. Case \(x_1 = 1\):

\[
\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = -\frac{3(x - 1)}{(x + 2)^2},
\]

\[
\frac{(x - 1)^2 R(x)}{P(x)}
\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

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\frac{(x - 1)Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = -\frac{3(x - 1)}{(x + 2)^2},
\]

\[
\frac{(x - 1)^2R(x)}{P(x)} = \frac{2(x - 1)^2}{(x + 2)^2(x - 1)}.
\]
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

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So \(x_0 = -2\) is not a regular-singular point. Case \(x_1 = 1\):

\[
\frac{(x - 1)Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = -\frac{3(x - 1)}{(x + 2)^2},
\]
\[
\frac{(x - 1)^2R(x)}{P(x)} = \frac{2(x - 1)^2}{(x + 2)^2(x - 1)} = \frac{2(x - 1)}{(x + 2)^2};
\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.$$ 

Solution: Find the singular points: $x_0 = -2$ and $x_1 = 1$.

Case $x_0 = -2$:

$$\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{(x + 2)} = \pm\infty.$$ 

So $x_0 = -2$ is not a regular-singular point. Case $x_1 = 1$:

$$\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = -\frac{3(x - 1)}{(x + 2)^2},$$ 

$$\frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)^2}{(x + 2)^2(x - 1)} = \frac{2(x - 1)}{(x + 2)^2};$$

Both functions have Taylor series around $x_1 = 1$. 
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.$$
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution: Recall:

\[\frac{(x - 1) Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2},\]

\[\frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.\]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution: Recall:

\[\frac{(x - 1)Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.\]

Furthermore, the following limits are finite,
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation
\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

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\]

Furthermore, the following limits are finite,
\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)}
\]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ (x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0. \]

Solution: Recall:

\[
\frac{(x - 1)Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.
\]

Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1)Q(x)}{P(x)} = 0;
\]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Recall:

\[
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\]

Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 0; \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Recall:

\[
\frac{(x - 1) Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.
\]

Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 0; \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]

Therefore, the point \(x = 1\) is a regular-singular point.
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(x + 2)^2 (x - 1) y'' + 3(x - 1) y' + 2 y = 0.\]

Solution: Recall:

\[
\frac{(x - 1) Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.
\]

Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 0; \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]

Therefore, the point \(x_1 = -1\) is a regular-singular point.  \(\ trianguleleft\)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3xy = 0. \]

Solution: The singular point is \( x_0 = 0 \).
Examples: Equations with regular-singular points.

**Example**
Find the regular-singular points of the differential equation
\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

**Solution:** The singular point is \( x_0 = 0 \). We compute the limit
\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} \]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x \left[ -x \ln(|x|) \right]}{x} \]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

$$x y'' - x \ln(|x|) y' + 3x y = 0.$$  

Solution: The singular point is $x_0 = 0$. We compute the limit

$$\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x \left[ -x \ln(|x|) \right]}{x} = \lim_{x \to 0} - \frac{\ln(|x|)}{\frac{1}{x}}.$$
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3xy = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[
\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x \left[ -x \ln(|x|) \right]}{x} = \lim_{x \to 0} - \frac{\ln(|x|)}{\frac{1}{x}}.
\]

Use L’Hôpital’s rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} \)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3xy = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[
\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x[-x \ln(|x|)]}{x} = \lim_{x \to 0} -\frac{\ln(|x|)}{\frac{1}{x}}.
\]

Use L'Hôpital’s rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x} - \frac{1}{x^2} \).
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

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\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x[-x \ln(|x|)]}{x} = \lim_{x \to 0} - \frac{\ln(|x|)}{\frac{1}{x}}.
\]

Use L’Hôpital’s rule:

\[
\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} - \frac{\frac{1}{x}}{x^2} = \lim_{x \to 0} x.
\]
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

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\]

Use L'Hôpital’s rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x^2} = \lim_{x \to 0} x = 0. \)
Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

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Use L’Hôpital’s rule:

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} - \frac{1}{x} = \lim_{x \to 0} x = 0. \]

The other limit is:

\[ \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} \]
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x \left[ -x \ln(|x|) \right]}{x} = \lim_{x \to 0} - \frac{\ln(|x|)}{1/x}. \]

Use L’Hôpital’s rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{-1/x}{-1/x^2} = \lim_{x \to 0} x = 0. \)

The other limit is: \( \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2(3x)}{x} \).
Example: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

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Use L'Hôpital's rule: \( \lim_{x \to 0} \frac{x Q(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x} = \lim_{x \to 0} x = 0. \)

The other limit is: \( \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2 (3x)}{x} = \lim_{x \to 0} 3x^2 \).
Example

Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[
\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x[-x \ln(|x|)]}{x} = \lim_{x \to 0} -\frac{\ln(|x|)}{x}.
\]

Use L’Hôpital’s rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x^2} = \lim_{x \to 0} x = 0. \)

The other limit is: \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2(3x)}{x} = \lim_{x \to 0} 3x^2 = 0. \)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \) and \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0. \)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \) and \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0 \).

However, at the point \( x_0 = 0 \) the function \( xQ/P \) does not have a power series expansion around zero,
Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[ xy'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \) and \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0. \)

However, at the point \( x_0 = 0 \) the function \( xQ/P \) does not have a power series expansion around zero, since

\[ \frac{xQ(x)}{P(x)} = -x \ln(|x|), \]
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \) and \( \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = 0. \)

However, at the point \( x_0 = 0 \) the function \( xQ/P \) does not have a power series expansion around zero, since

\[ \frac{xQ(x)}{P(x)} = -x \ln(|x|), \]

and the log function does not have a Taylor series at \( x_0 = 0. \)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \(\lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0\) and \(\lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0\).

However, at the point \(x_0 = 0\) the function \(xQ/P\) does not have a power series expansion around zero, since

\[ \frac{xQ(x)}{P(x)} = -x \ln(|x|), \]

and the log function does not have a Taylor series at \(x_0 = 0\).

We conclude that \(x_0 = 0\) is not a regular-singular point. \(\triangle\)
Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- **Method to find solutions.**
- Example: Method to find solutions.
Method to find solutions.

Recall: If \( x_0 \) is a regular-singular point of

\[
P(x) y'' + Q(x) y' + R(x) y = 0,
\]

with limits

\[
\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0 \quad \text{and} \quad \lim_{x \to x_0} \frac{(x - x_0)^2R(x)}{P(x)} = q_0,
\]

then the coefficients of the differential equation above near \( x_0 \) are close to the coefficients of the Euler equation

\[
(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.
\]
Method to find solutions.

Recall: If $x_0$ is a regular-singular point of

\[ P(x) y'' + Q(x) y' + R(x) y = 0, \]

with limits \( \lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0 \) and \( \lim_{x \to x_0} \frac{(x - x_0)^2R(x)}{P(x)} = q_0, \)

then the coefficients of the differential equation above near $x_0$ are close to the coefficients of the Euler equation

\[ (x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0. \]

Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.
Method to find solutions.

Recall: If $x_0$ is a regular-singular point of

$$P(x) y'' + Q(x) y' + R(x) y = 0,$$

with limits $\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0$ and $\lim_{x \to x_0} \frac{(x - x_0)^2R(x)}{P(x)} = q_0$, then the coefficients of the differential equation above near $x_0$ are close to the coefficients of the Euler equation

$$(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.$$

Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

Recall: One solution of an Euler equation is $y(x) = (x - x_0)^r$. 
Method to find solutions.

Summary: Solutions for equations with regular-singular points:

1. Look for a solution $y(x)$ of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$;
2. Introduce this power series expansion into the differential equation and find both a the exponent $r$ and a recurrence relation for the coefficients $a_n$;
3. First find the solutions for the constant $r$. Then, introduce this result for $r$ into the recurrence relation for the coefficients $a_n$. Only then, solve this latter recurrence relation for the coefficients $a_n$. 
Method to find solutions.

**Summary:** Solutions for equations with regular-singular points:

1. Look for a solution $y$ of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

2. Introduce this power series expansion into the differential equation and find both $r$ and a recurrence relation for the coefficients $a_n$;

3. First find the solutions for the constant $r$. Then, introduce this result for $r$ into the recurrence relation for the coefficients $a_n$. Only then, solve this latter recurrence relation for the coefficients $a_n$. 
Method to find solutions.

**Summary:** Solutions for equations with regular-singular points:

(1) Look for a solution $y$ of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

(2) Introduce this power series expansion into the differential equation and find both $r$, the exponent, and a recurrence relation for the coefficients $a_n$.
Method to find solutions.

Summary: Solutions for equations with regular-singular points:

1. Look for a solution $y$ of the form

   $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

2. Introduce this power series expansion into the differential equation and find both a the exponent $r$ and a recurrence relation for the coefficients $a_n$;

3. First find the solutions for the constant $r$. Then, introduce this result for $r$ into the recurrence relation for the coefficients $a_n$. Only then, solve this latter recurrence relation for the coefficients $a_n$. 
Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- **Example:** Method to find solutions.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}$. 
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}$.

The first and second derivatives are given by
\[
y' = \sum_{n=0}^{\infty} (n + r) a_n x^{(n+r-1)},
\]
\[
y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r-2)}.
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In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n x^{(n-1)} = \sum_{n=1}^{\infty} n a_n x^{(n-1)},$$
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In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n x^{(n-1)} = \sum_{n=1}^{\infty} n a_n x^{(n-1)},$$

but for $r \neq 0$ this relation is not true.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: We now compute the term $(x + 3)y$, 

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$$= \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)}.$$
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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$$= - \sum_{n=0}^{\infty} (n + r) a_n x^{(n+r+1)} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{(n+r)},$$
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$$-x(x+3) y' = -\sum_{n=1}^{\infty} (n + r - 1) a_{n-1} x^{(n+r)} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{(n+r)}.$$
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: We compute the term $x^2 y''$,

$$x^2 y'' = x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r-2)}$$
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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$$x^2 y'' = x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r-2)}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r)}.$$
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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$$x^2 y'' = x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r-2)}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)}.$$

The guiding principle to rewrite each term is to have the power

function $x^{(n+r)}$ labeled in the same way on every term.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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Example: Method to find solutions.

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Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of
\[
x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
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Solution: The differential equation is given by
\[
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)} - \sum_{n=1}^{\infty} (n + r - 1)a_{(n-1)} x^{(n+r)}
\]
\[
- \sum_{n=0}^{\infty} 3(n + r)a_n x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)} = 0.
\]
Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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Solution: The differential equation is given by
\[
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)} - \sum_{n=1}^{\infty} (n + r - 1)a_{(n-1)} x^{(n+r)} \\
- \sum_{n=0}^{\infty} 3(n + r)a_n x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)} = 0.
\]

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geq 1$, 
Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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$$\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)} - \sum_{n=1}^{\infty} (n + r - 1)a_{(n-1)} x^{(n+r)}$$

$$- \sum_{n=0}^{\infty} 3(n + r)a_n x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)} = 0.$$ 

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geq 1$, that is,

$$0 = [r(r - 1) - 3r + 3]a_0 x^r +$$

$$\sum_{n=1}^{\infty} \left[ (n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n \right] x^{(n+r)}$$
Example: Method to find solutions.

Example

Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: Therefore, $[r(r - 1) - 3r + 3] = 0$ and

$$[(n+r)(n+r-1)a_n-(n+r-1)a_{n-1}-3(n+r)a_n+a_{n-1}+3a_n] = 0.$$
Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: Therefore, \[ r(r - 1) - 3r + 3 \] = 0 and
\[ \left[ (n+r)(n+r-1)a_n - (n+r-1)a_{n-1} - 3(n+r)a_n + a_{n-1} + 3a_n \right] = 0. \]
The last expression can be rewritten as follows,
\[ \left[ (n + r)(n + r - 1) - 3(n + r) + 3 \right] a_n - (n + r - 1 - 1) a_{n-1} = 0, \]
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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The last expression can be rewritten as follows,

$$\left[\left[(n+r)(n+r-1) - 3(n+r) + 3\right]a_n - (n+r-1-1)a_{n-1}\right] = 0,$$

$$\left[\left[(n+r)(n+r-1) - 3(n+r-1)\right]a_n - (n+r-2)a_{n-1}\right] = 0.$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: Hence, the recurrence relation is given by the equations

$$r(r - 1) - 3r + 3 = 0,$$

$$(n + r - 1)(n + r - 3)a_n - (n + r - 2)a_{(n-1)} = 0.$$
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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First: solve the first equation for $r_{\pm}$. 
Example: Method to find solutions.

Example

Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of

\[
x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
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\[
r(r - 1) - 3r + 3 = 0,
\]

\[
(n + r - 1)(n + r - 3)a_n - (n + r - 2)a_{(n-1)} = 0.
\]

First: solve the first equation for \( r_{\pm} \).

Second: Introduce the first solution \( r_+ \) into the second equation above and solve for the \( a_n \);
Example: Method to find solutions.

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Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of
\[
x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
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First: solve the first equation for \( r_\pm \).

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Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: Hence, the recurrence relation is given by the equations

$$r(r - 1) - 3r + 3 = 0,$$

$$(n + r - 1)(n + r - 3)a_n - (n + r - 2)a_{n-1} = 0.$$ 

First: solve the first equation for $r_{\pm}$.

Second: Introduce the first solution $r_+$ into the second equation above and solve for the $a_n$; the result is a solution $y_+$ of the original differential equation;

Third: Introduce the second solution $r_-$ into the second equation above and solve for the $a_n$;
Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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Example
Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of
\[
x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
\]

Solution: We first solve \( r(r - 1) - 3r + 3 = 0 \).
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: We first solve $r(r - 1) - 3r + 3 = 0$.
\[ r^2 - 4r + 3 = 0 \]
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Find the solution $y$ near the regular-singular point $x_0 = 0$ of

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Solution: We first solve $r(r - 1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 12}]$$
Example: Method to find solutions.

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$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 12}] \quad \Rightarrow \quad \{ r_+ = 3, \quad r_- = 1. \}$$
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$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 12}] \quad \Rightarrow \quad \left\{ \begin{array}{l} r_+ = 3, \\ r_- = 1. \end{array} \right.$$ 

Introduce $r_+ = 3$ into the equation for $a_n$:

$$(n + 2)n a_n - (n + 1)a_{n-1} = 0.$$
Example: Method to find solutions.

Example

Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: We first solve $r(r - 1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \implies r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 12}] \implies \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introduce $r_+ = 3$ into the equation for $a_n$:

$$(n + 2)n a_n - (n + 1)a_{n-1} = 0.$$ 

One can check that the solution $y_+$ is

$$y_+ = a_0 x^3 \left[1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right].$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: Introduce $r_+ = 1$ into the equation for $a_n$:

$$n(n - 2)a_n - (n - 1)a_{n-1} = 0.$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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Solution: Introduce $r_- = 1$ into the equation for $a_n$:
$$n(n - 2)a_n - (n - 1)a_{n-1} = 0.$$  

One can also check that the solution $y_-$ is
$$y_- = a_2 x \left[ x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \cdots \right].$$
Example: Method to find solutions.

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Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of
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x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
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\[
y_- = a_2 x \left[ x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \cdots \right].
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Notice:
\[
y_- = a_2 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right]
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y_- = a_2 x \left[ x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \cdots \right].
\]

Notice:

\[
y_- = a_2 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right] \implies y_- = \frac{a_2}{a_1} y_+.
\]
Example: Method to find solutions.

Example
Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of

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Solution: The solutions \( y_+ \) and \( y_- \) are not linearly independent.
Example: Method to find solutions.

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This Example shows that the method does not provide all solutions of a differential equation near a regular-singular point,
Example: Method to find solutions.

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Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial $r_+, r_-$ differ by
an integer, then the second solution $y_-$, the solution corresponding
to the smaller root, is not given by the method above.
Example: Method to find solutions.

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Find the solution $y$ near the regular-singular point $x_0 = 0$ of
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If the roots of the Euler characteristic polynomial $r_+, r_-$ differ by an integer, then the second solution $y_-$, the solution corresponding to the smaller root, is not given by the method above. This solution involves logarithmic terms.
Example: Method to find solutions.

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If the roots of the Euler characteristic polynomial $r_+, r_-$ differ by an integer, then the second solution $y_-$, the solution corresponding to the smaller root, is not given by the method above. This solution involves logarithmic terms.
We do not study this type of solutions in these notes.