Review 2 for Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Linear equations (2.1).
  - Separable equations (2.2).
  - Homogeneous equations (2.2).
  - Modeling using differential equations (2.3).
  - Non-linear equations (2.4).
  - Bernoulli equation (2.4).
  - Autonomous systems (2.5).
  - Exact equations (2.6).
  - Exact equations with integrating factors (2.6).
Example

Find the integrating factor that converts the equation below into an exact equation, where

\[ (x^3 e^y + \frac{x}{y}) y' + (2x^2 e^y + 1) = 0. \]
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Solution: We first verify if the equation is not exact.

\[
N = \left( x^3 e^y + \frac{x}{y} \right)
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\[ N = \left( x^3 e^y + \frac{x}{y} \right) \Rightarrow \partial_x N = 3x^2 e^y + \frac{1}{y}. \]
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So the equation is not exact. We now compute

\[
\frac{\partial_y M - \partial_x N}{N}
\]
Example

Find the integrating factor that converts the equation below into an exact equation, where

\[ (x^3e^y + \frac{x}{y}) \frac{dy}{dx} + (2x^2e^y + 1) = 0. \]

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\[ N = \left( x^3e^y + \frac{x}{y} \right) \] \[ \Rightarrow \] \[ \partial_x N = 3x^2e^y + \frac{1}{y}. \]

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So the equation is not exact. We now compute

\[ \frac{\partial_y M - \partial_x N}{N} = \frac{2x^2e^y - \left( 3x^2e^y + \frac{1}{y} \right)}{\left( x^3e^y + \frac{x}{y} \right)} \]
Example

Find the integrating factor that converts the equation below into an exact equation, where

\( (x^3e^y + \frac{x}{y}) y' + (2x^2e^y + 1) = 0. \)

Solution: We first verify if the equation is not exact.

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\begin{align*}
N &= (x^3e^y + \frac{x}{y}) \quad \Rightarrow \quad \partial_x N = 3x^2e^y + \frac{1}{y}.
\end{align*}
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Review 2 Exam 1.

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Solution: Recall: \( \frac{\partial_y M - \partial_x N}{N} = -\frac{1}{x} \). Therefore,

\[
\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x}
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So the equation \(\left(x^2e^y + \frac{1}{y}\right)y' + \left(2xe^y + \frac{1}{x}\right) = 0\) is exact.
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So the equation \( \left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2xe^y + \frac{1}{x} \right) = 0 \) is exact. Indeed,

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\tilde{N} = \left( x^2 e^y + \frac{1}{y} \right)
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\tilde{N} &= \left( x^2e^y + \frac{1}{y} \right) \quad \Rightarrow \quad \partial_x \tilde{N} = 2xe^y, \\
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\[ \Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}. \]
Example

Find every solution $y$ of the equation

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Solution: The equation is exact. We need to find the potential function \( \psi \).
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\partial_y \psi = N, \quad \partial_x \psi = M.
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Introduce the expression for \( \psi \) in the equation \( \partial_x \psi = M \), that is,

\[
2xe^y + g'(x) = \partial_x \psi
\]
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2xe^y + g'(x) = \partial_x \psi = M = 2x e^y + \frac{1}{x} \quad \Rightarrow \quad g'(x) = \frac{1}{x}.
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Example

Find every solution $y$ of the equation

$$(x^2 e^y + \frac{1}{y}) y' + (2x e^y + \frac{1}{x}) = 0.$$  

Solution: Recall: $g'(x) = \frac{1}{x}$.  

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.
Example

Find every solution $y$ of the equation

$$\left(x^2 e^y + \frac{1}{y}\right) y' + \left(2x e^y + \frac{1}{x}\right) = 0.$$ 

Solution: Recall: $g'(x) = \frac{1}{x}$. Therefore $g(x) = \ln(x)$. 

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$$\left(x^2 e^y + \frac{1}{y}\right) y' + \left(2x e^y + \frac{1}{x}\right) = 0.$$ 

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The potential function is $\psi = x^2 e^y + \ln(y) + \ln(x)$. 

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Find every solution \( y \) of the equation

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Solution: Recall: \( g'(x) = \frac{1}{x} \). Therefore \( g(x) = \ln(x) \).

The potential function is \( \psi = x^2 e^y + \ln(y) + \ln(x) \).

The solution \( y \) satisfies \( x^2 e^{y(x)} + \ln(y(x)) + \ln(x) = c \). \( \triangleright \)
Example

Find every solution y of the equation

$$\left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2x e^y + \frac{1}{x} \right) = 0. $$

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Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.
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Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.

\[
2xe^y + x^2 e^y y' + \frac{1}{y} y' + \frac{1}{x} = 0.
\]
Example

Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]
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Solution: The equation is: Not linear.
It is a Bernoulli equation: \[ y' - 4x y = 4x y^n, \text{ with } n = 1/2. \]
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Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: The equation is: Not linear.
It is a Bernoulli equation: \( y' - 4x y = 4x y^n \), with \( n = 1/2 \).
It is separable: \( \frac{y'}{y + \sqrt{y}} = 4x \).
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Solution: The equation is: Not linear.
It is a Bernoulli equation: \( y' - 4x y = 4x y^n \), with \( n = 1/2 \).
It is separable: \( \frac{y'}{y + \sqrt{y}} = 4x \).
The equation is not homogeneous.
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The equation is not homogeneous. It is not exact.
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The equation is not homogeneous. It is not exact.

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Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

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The integral on the left-hand side requires an integration table.
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\[ y' - 4x \, y = 4x \, y^{1/2} \]
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Review 2 for Exam 1.

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The coefficient function is \( a(x) = -2x, \)
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\[ y' - 4x y = 4x y^{1/2} \implies \frac{y'}{y^{1/2}} - 4x y^{1/2} = 4x. \]

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\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: Recall: \( v' - 2xv = 2x \) and \( \mu(x) = e^{-x^2} \).
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We conclude that \( v = c e^{x^2} - 1. \)
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We finally find \( y = v^2 \), that is, \( y(x) = (3e^{x^2} - 1)^2. \)
Review 2 for Exam 1.

Example

Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$
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Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$ 

Solution: We first need to find the solution $y$. 

The equation is separable. 

$$\int y' \, dt = \int -\frac{2t}{y} \, dt + c \Rightarrow y^2 = -t^2 + c.$$ 

At $t=1$,

$$y^2(1) = -1 + c \Rightarrow c = 3.$$

Therefore, 

$$y(t) = \sqrt{2(3-t^2)}.$$ 

The domain of the solution $y$ is 

$$D = (-\sqrt{3}, \sqrt{3}).$$ 

The points $\pm \sqrt{3}$ do not belong to the domain of $y$, since $y'$ and the differential equation are not defined there.
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Find the domain of the function $y$ solution of the IVP

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$\triangleright$
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$$y' = -\frac{2t}{y}, \quad y(t_0) = y_0 > 0.$$ 

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$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2}.$$
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\[ \frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \Rightarrow c = \frac{y_0^2}{2} + t_0^2 \]
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Solution: The solution $y$ is given as above, $\frac{y^2}{2} = -t^2 + c$.

The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \quad \Rightarrow \quad c = \frac{y_0^2}{2} + t_0^2 \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$
Example
Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(t_0) = y_0 > 0.$$ 

Solution: The solution $y$ is given as above, $\frac{y^2}{2} = -t^2 + c$. The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \Rightarrow c = \frac{y_0^2}{2} + t_0^2 \Rightarrow \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$ 

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}$. 

D = $\left(-\sqrt{t_0^2 + y_0^2}, +\sqrt{t_0^2 + y_0^2}\right)$. 

◁
Example
Find the domain of the function $y$ solution of the IVP

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The solution to the IVP is

\[y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}.
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The domain of the solution depends on the initial condition $t_0, y_0$: 

\[D = (-\sqrt{-t_0^2 + y_0^2}, +\sqrt{-t_0^2 + y_0^2}).\]
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Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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Example

Find every solution \( y \) to the equation \( y' = -\frac{2x + 3y}{3x + 4y} \).

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Is it exact?
Review 2 for Exam 1.

Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by $\frac{1}{x}$.)

Is it exact? $(3x + 4y)y' + (2x + 3y) = 0$
Review 2 for Exam 1.

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Is it exact? \((3x + 4y)\ y' + (2x + 3y) = 0\) implies \(\partial_x N = 3 = \partial_y M\).
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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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Review 2 for Exam 1.

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We choose here the exact equation method.
Review 2 for Exam 1.

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We choose here the exact equation method. (Finding the potential function is sometimes simpler that solving homogeneous Eqs.)
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We need to find the potential function $\psi$:

\begin{align*}
\partial_y \psi &= N \\
\partial_x \psi &= M
\end{align*}

\begin{align*}
\partial_y \psi &= 3x + 4y \\
\partial_x \psi &= 2x + 3y
\end{align*}

We conclude:
$\psi(x, y) = 3xy + 2y^2 + x^2,$
and $\psi(x, y(x)) = c$.\]
Review 2 for Exam 1.

Example

Find every solution \( y \) to the equation \( y' = -\frac{2x + 3y}{3x + 4y} \).

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by \( \frac{1}{x} \).)

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$$\partial_y \psi = N \quad \Rightarrow \quad \psi = 3xy + 2y^2 + g(x).$$

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We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$. 

\[
\begin{align*}
    \psi(x, y) &= 3xy + 2y^2 + x^2, \\
    \partial_x \psi &= 3y + g'(x) = 2x + 3y, \\
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\end{align*}
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We conclude: \( \psi(x, y) = 3xy + 2y^2 + x^2 \), and \( \psi(x, y(x)) = c \).\( \triangleleft \)
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Review 2 for Exam 1.

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Find every solution \( y \) to the equation \( y' = -\frac{2x + 3y}{3x + 4y} \).

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation.
Review 2 for Exam 1.

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Review 2 for Exam 1.

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Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x + 3y)}{(3x + 4y)} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$
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\[
y' = -\frac{(2x + 3y)}{(3x + 4y)} \left( \frac{1}{x} \right) = -\frac{2 + 3\left( \frac{y}{x} \right)}{3 + 4\left( \frac{y}{x} \right)}.
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Review 2 for Exam 1.

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The change $v = y/x$ implies $y = xv$ and $y' = v + x v'$.
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The change $v = y/x$ implies $y = xv$ and $y' = v + x \cdot v'$. Hence

$$v + x \cdot v' = \frac{2 + 3v}{3 + 4v}.$$
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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

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The change $v = y/x$ implies $y = xv$ and $y' = v + xv'$. Hence

$$v + x v' = \frac{2 + 3v}{3 + 4v} \Rightarrow x v' = \frac{2 + 3v}{3 + 4v} - v.$$
Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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The change $v = y/x$ implies $y = xv$ and $y' = v + x v'$. Hence

$$v + x v' = \frac{2 + 3v}{3 + 4v} \Rightarrow x v' = \frac{2 + 3v}{3 + 4v} - v = \frac{2 + 3v - 3v + 4v^2}{3 + 4v}.$$
Review 2 for Exam 1.

Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x + 3y)}{(3x + 4y)} \frac{1}{\frac{1}{x}} = -\frac{2 + 3\left(\frac{y}{x}\right)}{3 + 4\left(\frac{y}{x}\right)}.$$

The change $v = \frac{y}{x}$ implies $y = xv$ and $y' = v + xv'$. Hence

$$v + xv' = \frac{2 + 3v}{3 + 4v} \quad \Rightarrow \quad xv' = \frac{2 + 3v}{3 + 4v} - v = \frac{2 + 3v - 3v + 4v^2}{3 + 4v}.$$

We conclude that $v$ satisfies $\frac{3 + 4v}{2 - 4v^2} \cdot v' = \frac{1}{x}$. 
Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: Recall: 

$$\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}.$$
Review 2 for Exam 1.

Example

Find every solution \( y \) to the equation \( y' = -\frac{2x + 3y}{3x + 4y} \).

Solution: Recall: \( \frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x} \).

This equation is complicated to integrate.

\[
\int \frac{3 v'}{2 - 4v^2} \, dx + \int \frac{4v v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c
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Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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\frac{3 + 4v}{2 - 4v^2} \cdot v' = \frac{1}{x}.
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This equation is complicated to integrate.

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\int \frac{3 \, v'}{2 - 4v^2} \, dx + \int \frac{4v \, v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c = \ln(x) + c.
\]

The usual substitution $u = v(x)$ implies $du = v' \, dx$, 

\[
\int \frac{3 \, v'}{2 - 4v^2} \, dx + \int \frac{4v \, v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c = \ln(x) + c.
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Example

Find every solution \( y \) to the equation \( y' = -\frac{2x + 3y}{3x + 4y} \).

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This equation is complicated to integrate.

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\int \frac{3 v'}{2 - 4v^2} \, dx + \int \frac{4v \, v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c = \ln(x) + c.
\]

The usual substitution \( u = v(x) \) implies \( du = \frac{dv}{v} \, dx \), so

\[
\int \frac{3 \, du}{2 - 4u^2} + \int \frac{4u \, du}{2 - 4u^2} = \ln(x) + c.
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Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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The first integral on the left-hand side requires integration tables.
Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: Recall: $\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}$.

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The usual substitution $u = v(x)$ implies $du = v' \, dx$, so

$$\int \frac{3 \, du}{2 - 4u^2} + \int \frac{4u \, du}{2 - 4u^2} = \ln(x) + c.$$ 

The first integral on the left-hand side requires integration tables. This is why the exact method is simpler to use in this case.
Example

Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$. 
Example

Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

**Solution:** We first plot the function $f(y) = -(y - 2)^2$. 
Example
Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$. Find the equilibrium solutions, $f(y) = 0$. 

Semistable equilibrium

$y' < 0$

$y'' = f'(y) f(y)$

$y'' > 0$

$2y$
Review 2 for Exam 1.

Example
Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$. Find the equilibrium solutions, $f(y) = 0$. Determine the increasing-decreasing intervals for $y$. 
Review 2 for Exam 1.

Example

Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$.
Find the equilibrium solutions, $f(y) = 0$.
Determine the increasing-decreasing intervals for $y$. (Sign of $y'$.)

Semistable

$y' < 0$
$y'' = f'(y) f(y)$
$y'' > 0$

Semi−stable equilibrium

$y_0$
$y_0$
t
$t$
$2$

Semistable equilibrium
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Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$. Find the equilibrium solutions, $f(y) = 0$. Determine the increasing-decreasing intervals for $y$. (Sign of $y'$.) Determine the curvature of $y$. 
Review 2 for Exam 1.

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Find the equilibrium solutions, $f(y) = 0$.
Determine the increasing-decreasing intervals for $y$. (Sign of $y'$.)
Determine the curvature of $y$. (Sign of $y''$.)

\[
\begin{aligned}
f(y) &= -(y - 2)^2 \\
y' &= f(y) \quad y'' = f'(y) f(y) \\
y' < 0 & \quad y' < 0 \\
y'' < 0 & \quad y'' > 0 \\
\end{aligned}
\]
Example

Sketch the graph of the function $y$ solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$. Find the equilibrium solutions, $f(y) = 0$. Determine the increasing-decreasing intervals for $y$. (Sign of $y'$.) Determine the curvature of $y$. (Sign of $y''$.)

\begin{align*}
f(y) &= -(y - 2)^2 \\
y' &= f(y) \\
y'' &= f'(y) f(y) \\
y' &< 0 \\
y'' &< 0 \\
y' &< 0 \\
y'' &> 0
\end{align*}
Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter,
find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.
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Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
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\[ V(t) = V_0, \]
\[ Q(t) = Q_0 e^{-rt/V_0}, \]
\[ q(t) = r - \frac{Q(t)}{V(t)} = r - \frac{Q_0 e^{-rt/V_0}}{V_0} = r - \frac{Q_0}{V_0} e^{-rt/V_0}. \]
\[ q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0}. \]

Since $q(t_1)$ is 1% of $Q_0/V_0$, we have
\[ e^{-rt_1/V_0} = \frac{1}{100}. \]

Therefore, 
\[ rt_1/V_0 = \ln(100), \]
\[ t_1 = \frac{V_0 r \ln(100)}{Q_0}. \]
Review 2 for Exam 1.

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We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \).
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We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \). Hence: \( t_1 = 100 \ln(100) \).
Variable coefficients second order linear ODE (Sect. 3.2).

**Summary:** The study the main properties of solutions to second order, linear, variable coefficients, ODE.

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel’s theorem on the Wronskian.
Review: Second order linear ODE.

Definition
Given functions $a_1, a_0, b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)y' + a_0(t)y = b(t)$$

is called a second order linear differential equation with variable coefficients.
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Theorem
If the functions $y_1$ and $y_2$ are solutions to the homogeneous linear equation

$$y'' + a_1(t) y' + a_0(t) y = 0,$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants $c_1, c_2 \in \mathbb{R}$. 
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
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Existence and uniqueness of solutions.

Theorem (Variable coefficients)

If the functions \( a, b : (t_1, t_2) \to \mathbb{R} \) are continuous, the constants \( t_0 \in (t_1, t_2) \) and \( y_0, y_1 \in \mathbb{R} \), then there exists a unique solution \( y : (t_1, t_2) \to \mathbb{R} \) to the initial value problem

\[
y'' + a_1(t) y' + a_0(t) y = b(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.
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Remarks:

▶ Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.
▶ Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.
Existence and uniqueness of solutions.

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Existence and uniqueness of solutions.

Example
Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t - 1)y'' - 3ty' + 4y = t(t - 1), \quad y(-2) = 2, \quad y'(-2) = 1.$$
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The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are \( I_1 = (-\infty, 1) \) and \( I_2 = (1, \infty) \). Since the initial condition belongs to \( I_1 \), the solution domain is

\[I_1 = (-\infty, 1).\]
Existence and uniqueness of solutions.

Remarks:

▶ Every solution of the first order linear equation

\[ y' + a(t) y = 0 \]

is given by

\[ y(t) = c e^{-A(t)} \]

with

\[ A(t) = \int a(s) \, ds. \]

▶ All solutions above are proportional to each other:

\[ y_1(t) = c_1 e^{-A(t)} \]
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\[ \Rightarrow y_1(t) = c_1 c_2 y_2(t) \]

Remark:

The above statement is not true for solutions of second order, linear, homogeneous equations,

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Before we prove this statement we need few definitions:

▶ Proportional functions (linearly dependent).

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Remark: The above statement is not true for solutions of second order, linear, homogeneous equations, \( y'' + a_1(t) y' + a_0(t) y = 0 \). Before we prove this statement we need few definitions:

▶ Proportional functions (linearly dependent).
Existence and uniqueness of solutions.

Remarks:

▶ Every solution of the first order linear equation

\[ y' + a(t) y = 0 \]

is given by \( y(t) = c \, e^{-A(t)} \), with \( A(t) = \int a(s) \, ds \).

▶ All solutions above are proportional to each other:

\[ y_1(t) = c_1 \, e^{-A(t)}, \quad y_2(t) = c_2 \, e^{-A(t)} \quad \Rightarrow \quad y_1(t) = \frac{c_1}{c_2} \, y_2(t) \]

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▶ Wronskian of two functions.
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- **Linearly dependent and independent functions.**
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel’s theorem on the Wronskian.
Definition
Two continuous functions $y_1, y_2 : (t_1, t_2) \subset \mathbb{R} \rightarrow \mathbb{R}$ are called *linearly dependent, (ld)*, on the interval $(t_1, t_2)$ iff there exists a constant $c$ such that for all $t \in I$ holds

$$y_1(t) = c y_2(t).$$

The two functions are called *linearly independent, (li)*, on the interval $(t_1, t_2)$ iff they are not linearly dependent.
Linearly dependent and independent functions.

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Remarks:

- $y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R}$ are ld $\iff$ there exist constants $c_1, c_2$, not both zero, such that $c_1 y_1(t) + c_2 y_2(t) = 0$ for all $t \in (t_1, t_2)$. 
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Remarks:
\begin{itemize}
  \item \( y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R} \) are ld \iff there exist constants \( c_1, c_2 \), not 
  both zero, such that \( c_1 \, y_1(t) + c_2 \, y_2(t) = 0 \) for all \( t \in (t_1, t_2) \).
  \item \( y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R} \) are li \iff the only constants \( c_1, c_2 \), solutions 
  of \( c_1 \, y_1(t) + c_2 \, y_2(t) = 0 \) for all \( t \in (t_1, t_2) \) are \( c_1 = c_2 = 0 \).
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  \item These definitions are not given in the textbook.
\end{itemize}
Example

(a) Show that \( y_1(t) = \sin(t) \), \( y_2(t) = 2 \sin(t) \) are ld.

(b) Show that \( y_1(t) = \sin(t) \), \( y_2(t) = t \sin(t) \) are li.
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Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain
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$$c_1 + \frac{\pi}{2} c_2 = 0, \quad c_1 + \frac{3\pi}{2} c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$  
We conclude: The functions $y_1$ and $y_2$ are li.
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
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The Wronskian of two functions.

**Remark:** The Wronskian is a function that determines whether two functions are ld or li.
The Wronskian of two functions.

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**Definition**
The *Wronskian* of functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ is the function

$$W_{y_1y_2}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$
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- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}$,
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**Remark:**

- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}$, then $W_{y_1y_2}(t) = \det(A(t))$.

- An alternative notation is: $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$. 
The Wronskian of two functions.

Example
Find the Wronskian of the functions:
(a) \( y_1(t) = \sin(t) \) and \( y_2(t) = 2\sin(t) \). (Id)
(b) \( y_1(t) = \sin(t) \) and \( y_2(t) = t\sin(t) \). (li)
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Solution:
Case (a): \( W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \)

\[ W_{y_1y_2}(t) = \sin(t) \left[ \sin(t) + t\cos(t) \right] - \cos(t) \cdot t\sin(t) \]

We obtain \( W_{y_1y_2}(t) = \sin^2(t) \). \( \sqsubset \)
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Therefore,
\[
W_{y_1y_2}(t) = \sin(t) \cdot 2 \cos(t) - \cos(t) \cdot 2 \sin(t) = 0.
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Case (b):
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W_{y_1y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}.
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Case (b): \( W_{y_1y_2} = \begin{vmatrix} \sin(t) & t \sin(t) \\ \cos(t) & \sin(t) + t \cos(t) \end{vmatrix} \). Therefore,

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We obtain \( W_{y_1y_2}(t) = \sin^2(t) \). \( \triangle \)
The Wronskian of two functions.

**Remark:** The Wronskian determines whether two functions are linearly dependent or independent.

Theorem (Wronskian and linearly dependence)

The continuously differentiable functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are linearly dependent iff $W_{y_1 y_2}(t) = 0$ for all $t \in (t_1, t_2)$.

**Remark:** Importance of the Wronskian:

▶ Sometimes it is not simple to decide whether two functions are proportional to each other.

▶ The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)
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The Wronskian of two functions.

Example
Show whether the following two functions form a l.d. or l.i. set:

\[ y_1(t) = \cos(2t) - 2\cos^2(t), \quad y_2(t) = \cos(2t) + 2\sin^2(t). \]
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\[ W_{y_1y_2}(t) = \left[ \cos(2t) - 2 \cos^2(t) \right] \left[ -2 \sin(2t) + 4 \sin(t) \cos(t) \right] \]

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\[ \sin(2t) = 2 \sin(t) \cos(t) \Rightarrow \left[ -2 \sin(2t) + 4 \sin(t) \cos(t) \right] = 0. \]
The Wronskian of two functions.

Example
Show whether the following two functions form a l.d. or l.i. set:

\[ y_1(t) = \cos(2t) - 2\cos^2(t), \quad y_2(t) = \cos(2t) + 2\sin^2(t). \]

Solution: Compute their Wronskian:

\[ W_{y_1y_2}(t) = y_1 y_2' - y_1' y_2. \]

\[
W_{y_1y_2}(t) = \left[ \cos(2t) - 2\cos^2(t) \right] \left[ -2\sin(2t) + 4\sin(t)\cos(t) \right]
- \left[ -2\sin(2t) + 4\sin(t)\cos(t) \right] \left[ \cos(2t) + 2\sin^2(t) \right].
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We conclude \( W_{y_1y_2}(t) = 0 \), so the functions \( y_1 \) and \( y_2 \) are l.d. \( \triangle \)
The Wronskian of two functions.

Theorem (Variable coefficients)

- If \( a_1, a_0, b : (t_1, t_2) \to \mathbb{R} \) are continuous, then there exist two linearly independent solutions \( y_1, y_2 : (t_1, t_2) \to \mathbb{R} \) to the equation

\[
y'' + a_1(t)y' + a_0(t)y = b(t). \tag{1}
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$$y'' + a_1(t)y' + a_0(t)y = b(t). \quad (1)$$

- Every other solution $y$ of Eq. (1) can be decomposed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

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The Wronskian of two functions.

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- Every other solution $y$ of Eq. (1) can be decomposed as
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  for appropriate constants $c_1, c_2$.

- For every constant $t_0 \in (t_1, t_2)$ and $y_0, y_1 \in \mathbb{R}$, there exists a unique solution $y : (t_1, t_2) \to \mathbb{R}$ to the initial value problem given by Eq. (1) with the initial conditions
  \[ y(t_0) = y_0, \quad y'(t_0) = y_1. \]
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- **General and fundamental solutions.**
- Abel’s theorem on the Wronskian.
General and fundamental solutions.

Remark: The Theorem above justifies the following definitions.
General and fundamental solutions.

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Definition
Two solutions \( y_1, y_2 \) of the homogeneous equation

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are called \textit{fundamental solutions} iff the functions \( y_1, y_2 \) are linearly independent, that is, iff \( W_{y_1y_2} \neq 0 \).
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**Definition**
Given any two fundamental solutions \( y_1, y_2 \), and arbitrary constants \( c_1, c_2 \), the function

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y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

is called the **general solution** of Eq. (1).
Example
Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2 y'' + 3t y' - y = 0.$$
General and fundamental solutions.

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$$y_1 = t^{1/2}, \quad y_1' = \frac{1}{2} t^{-1/2}, \quad y_1'' = -\frac{1}{4} t^{-3/2},$$

Now show that $y_2$ is a solution:

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\mathcal{W}_{y_1y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}
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Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
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- **Abel’s theorem on the Wronskian.**
Abel’s theorem on the Wronskian.

Theorem (Abel)

If \( a_1, a_0 : (t_1, t_2) \rightarrow \mathbb{R} \) are continuous functions and \( y_1, y_2 \) are continuously differentiable solutions of the equation

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then the Wronskian \( W_{y_1 y_2} \) is a solution of the equation

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Therefore, for any \( t_0 \in (t_1, t_2) \), the Wronskian \( W_{y_1 y_2} \) is given by

\[
W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0) e^{A(t)} \quad A(t) = \int_{t_0}^{t} a_1(s) \, ds.
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**Remarks:** If the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.
Abel’s theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

\[ t^2 y'' - t(t + 2) y' + (t + 2) y = 0, \quad t > 0. \]
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Solution: Write the equation as in Abel’s Theorem,
\[ y'' - \left( \frac{2}{t} + 1 \right) y' + \left( \frac{2}{t^2} + \frac{1}{t} \right) y = 0. \]
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so, the solution is \( W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0) \frac{t^2}{t_0^2} e^{(t-t_0)}. \)
Abel’s theorem on the Wronskian.

Example
Find the Wronskian of two solutions of the equation

\[ t^2 y'' - t(t + 2) y' + (t + 2)y = 0, \quad t > 0. \]

Solution: \( A(t) = -2 \ln \left( \frac{t}{t_0} \right) - (t - t_0) = \ln \left( \frac{t_0^2}{t^2} \right) - (t - t_0). \)

The integrating factor is \( \mu = \frac{t_0^2}{t^2} e^{-(t-t_0)}. \) Therefore,

\[
\left[ \mu(t) W_{y_1y_2}(t) \right]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1y_2}(t) - \mu(t_0) W_{y_1y_2}(t_0) = 0
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Denoting \( c = \left( W_{y_1y_2}(t_0) / t_0^2 \right) e^{-t_0}, \) then \( W_{y_1y_2}(t) = c \ t^2 e^t. \) ◀
Second order linear homogeneous ODE (Sect. 3.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0. \)
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
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Review: On solutions of \( y'' + a_1y' + a_0y = 0 \).

**Definition**

Any two solutions \( y_1, y_2 \) of the homogeneous equation

\[
y'' + a_1(t)y' + a_0(t)y = 0,
\]

are called *fundamental solutions* iff the functions \( y_1, y_2 \) are linearly independent, that is, iff \( W_{y_1y_2} \neq 0 \).

Remark: Fundamental solutions are not unique.

**Definition**

Given any two fundamental solutions \( y_1, y_2 \), and arbitrary constants \( c_1, c_2 \), the function

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y(t) = c_1y_1(t) + c_2y_2(t)
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Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Theorem (Constant coefficients)

Given real constants $a_1$, $a_0$, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1 y' + a_0 y = 0. \quad (2)$$

Let $r_+$, $r_-$ be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let $c_0$, $c_1$ be arbitrary constants. Then, any solution of Eq. (2) belongs to only one of the following cases:

(a) If $r_+ \neq r_-$, the general solution is $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$.

(b) If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_1 + c_2 t) e^{r_+ t}$.

Furthermore, given real constants $t_0$, $y_1$ and $y_2$, there is a unique solution to the initial value problem given by Eq. (2) and the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example
Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution:
Since solutions have the form $e^{rt}$, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$, that is, $r = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2}$.

So, $r_1 = 3$, $r_2 = -2$.

A fundamental solution set is formed by $y_1(t) = e^{3t}$, $y_2(t) = e^{-2t}$.

The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is, $y(t) = c_1 e^{3t} + c_2 e^{-2t}$, $c_1, c_2 \in \mathbb{R}$.

Remark: Since $c_1, c_2 \in \mathbb{R}$, then $y$ is real-valued.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example
Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution: Since solutions have the form $e^{rt}$, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$, 

\[ p(r) = r^2 - r - 6 = (r - 3)(r + 2) \]

Thus, $r = 3$ and $r = -2$. Therefore, the general solution is 

\[ y(t) = c_1 e^{3t} + c_2 e^{-2t} \]

where $c_1$ and $c_2$ are arbitrary constants.
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Example
Find the general solution of the equation \( y'' - y' - 6y = 0 \).

Solution: Since solutions have the form \( e^{rt} \), we need to find the roots of the characteristic polynomial \( p(r) = r^2 - r - 6 \), that is,

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r_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 + 24} \right)
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$$r_\pm = \frac{1}{2} (1 \pm \sqrt{1 + 24}) = \frac{1}{2} (1 \pm 5) \implies r_+ = 3, \quad r_- = -2.$$
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So, \( r_{\pm} \) are real-valued. A fundamental solution set is formed by

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y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}.
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Remark: Since \( c_1, c_2 \in \mathbb{R} \), then \( y \) is real-valued.
Second order linear homogeneous ODE.

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- **Characteristic polynomial with complex roots.**
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.
Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants \( a_1, a_0 \in \mathbb{R} \) satisfy that \( a_1^2 - 4a_0 < 0 \), then the characteristic polynomial \( p(r) = r^2 + a_1r + a_0 \) of the equation

\[
y'' + a_1 y' + a_0 y = 0
\]

has complex roots \( r_+ = \alpha + i\beta \) and \( r_- = \alpha - i\beta \), where

\[
\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}.
\]

Furthermore, a fundamental set of solutions to Eq. (3) is

\[
\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},
\]

while another fundamental set of solutions to Eq. (3) is

\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Two main sets of fundamental solutions.

Example
Find the general solution of the equation \( y'' - 2y' + 6y = 0 \).
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\[
y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.
\]
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- One way to find the real-valued general solution is to find real-valued fundamental solutions.
Second order linear homogeneous ODE.

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- **Characteristic polynomial with complex roots.**
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The Theorem above says that a real-valued fundamental set is

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\[ y(t) = [c_1 \cos(\sqrt{5} t) + c_2 \sin(\sqrt{5} t)] e^t, \quad c_1, c_2 \in \mathbb{C}. \]
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The real-valued general solution is simple to obtain:
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We just restricted the coefficients \( c_1, c_2 \) to be real-valued. △
A real-valued fundamental and general solutions.

Example
Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$. 
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Show that $y_1(t) = e^t \cos(\sqrt{5} \, t)$ and $y_2(t) = e^t \sin(\sqrt{5} \, t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: We start with the complex-valued fundamental solutions,

$\tilde{y}_1(t) = e^{(1+i\sqrt{5}) \, t}, \quad \tilde{y}_2(t) = e^{(1-i\sqrt{5}) \, t}.$
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Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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Any linear combination of these functions is solution of the differential equation. In particular,

$$
y_1(t) = \frac{1}{2} [\tilde{y}_1(t) + \tilde{y}_2(t)], \quad y_2(t) = \frac{1}{2i} [\tilde{y}_1(t) - \tilde{y}_2(t)].
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Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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Now, recalling \( e^{(1\pm i\sqrt{5})t} = e^t e^{\pm i\sqrt{5} t} \)
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Now, recalling \( e^{(1\pm i\sqrt{5})t} = e^{t} e^{\pm i\sqrt{5}t} \)
\[
y_1(t) = \frac{1}{2} \left[ e^{t} e^{i\sqrt{5}t} + e^{t} e^{-i\sqrt{5}t} \right], \quad y_2(t) = \frac{1}{2i} \left[ e^{t} e^{i\sqrt{5}t} - e^{t} e^{-i\sqrt{5}t} \right],
\]
A real-valued fundamental and general solutions.

Example
Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1 = \frac{e^t}{2} [e^{i\sqrt{5}t} + e^{-i\sqrt{5}t}]$, $y_2 = \frac{e^t}{2i} [e^{i\sqrt{5}t} - e^{-i\sqrt{5}t}]$. 
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The Euler formula and its complex-conjugate formula

$e^{i\sqrt{5}t} = \left[ \cos(\sqrt{5} t) + i \sin(\sqrt{5} t) \right]$. 
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Example

Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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So functions $y_1$ and $y_2$ can be written as

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Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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A real-valued fundamental and general solutions.

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- \( y \) is complex-valued for \( c_1, c_2 \in \mathbb{C} \).
A real-valued fundamental and general solutions.

Remark:

- The proof of the Theorem follow exactly the same ideas given in the example above.

The real-valued fundamental solutions are
\[ y_1(t) = e^{\alpha t} \cos(\beta t), \]
\[ y_2(t) = e^{\alpha t} \sin(\beta t). \]
A real-valued fundamental and general solutions.

Remark:
- The proof of the Theorem follow exactly the same ideas given in the example above.
- One has to replace the roots of the characteristic polynomial
  \[1 + i\sqrt{5} \rightarrow \alpha + i\beta, \quad 1 - i\sqrt{5} \rightarrow \alpha - i\beta.\]
A real-valued fundamental and general solutions.

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Example
Find real-valued fundamental solutions to the equation
\[ y'' + 2y' + 6y = 0. \]
A real-valued fundamental and general solutions.

Example
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Solution:
The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \)
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\[ y'' + 2y' + 6y = 0. \]

Solution:
The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \) are

\[
 r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 24} \right]
\]

These are complex-valued roots, with \( \alpha = -1 \), \( \beta = \sqrt{5} \).

Real-valued fundamental solutions are

\[ y_1(t) = e^{-t} \cos(\sqrt{5}t), \quad y_2(t) = e^{-t} \sin(\sqrt{5}t). \]
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A real-valued fundamental and general solutions.

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Find real-valued fundamental solutions to the equation

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Differential equations like the one in this example describe physical processes related to damped oscillations. For example, pendulums with friction.
Example

Find the real-valued general solution of \( y'' + 5y = 0 \).

Solution:
The characteristic polynomial is \( p(r) = r^2 + 5 \).

Its roots are \( r = \pm \sqrt{5}i \). This is the case \( \alpha = 0 \), and \( \beta = \sqrt{5} \).

Real-valued fundamental solutions are \( y_1(t) = \cos(\sqrt{5}t) \), \( y_2(t) = \sin(\sqrt{5}t) \).

The real-valued general solution is \( y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) \), \( c_1, c_2 \in \mathbb{R} \). \( \triangleright \)

Remark:
Equations like the one in this example describe oscillatory physical processes without dissipation.
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of $y'' + 5y = 0$.

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Real-valued fundamental solutions are

$$y_1(t) = \cos(\sqrt{5} t), \quad y_2(t) = \sin(\sqrt{5} t).$$
A real-valued fundamental and general solutions.

**Example**

Find the real-valued general solution of $y'' + 5y = 0$.

**Solution:** The characteristic polynomial is $p(r) = r^2 + 5$.

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Real-valued fundamental solutions are

$$y_1(t) = \cos(\sqrt{5} t), \quad y_2(t) = \sin(\sqrt{5} t).$$

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A real-valued fundamental and general solutions.

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Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.
Second order linear homogeneous ODE.

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.

- **Application:** The RLC circuit.
Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

The electric current flowing in such circuit satisfies:

$$L \frac{d^2 I(t)}{dt^2} + R \frac{dI(t)}{dt} + \frac{1}{C} \int_0^t I(s) \, ds = 0.$$ 

Derivate both sides above:

$$L \frac{d^2 I(t)}{dt^2} + R \frac{dI(t)}{dt} + \frac{1}{C} I(t) = 0.$$ 

Divide by $L$:

$$\frac{d^2 I(t)}{dt^2} + 2 \left( \frac{R}{L} \right) I'(t) + \frac{1}{LC} I(t) = 0.$$ 

Introduce $\alpha = \frac{R}{L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then

$$\frac{d^2 I(t)}{dt^2} + 2 \alpha I'(t) + \omega^2 I(t) = 0.$$
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$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0.$$
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Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}},$
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$$I'' + 2\alpha I' + \omega^2 I = 0.$$
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. 
Application: The RLC circuit.

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Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

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r_{\pm} = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right]
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Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

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Case (a) \( R = 0 \).
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2} \left[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\right] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$  

Case (a) $R = 0$. This implies $\alpha = 0$, 

Remark: When the circuit has no resistance, the current oscillates without dissipation.
Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2} \left[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\right] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. 

Remark: When the circuit has no resistance, the current oscillates without dissipation.
Application: The RLC circuit.

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Solution: Recall: \( r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}. \)

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R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.
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The resistance $R$ damps the current oscillations.