The integrating factor method (Sect. 2.1)

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- The integrating factor method.
  - Constant coefficients.
  - The Initial Value Problem.
  - Variable coefficients.

Read:

- The direction field. Example 2 in Section 1.1 in the Textbook.
- See direction field plotters in Internet. For example, see: http://math.rice.edu/~dfield/dfpp.html
  This link is given in our class webpage.
Overview of differential equations.

Definition
A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Remark:
- **Ordinary Differential Equations (ODE):** Derivatives with respect to only one variable appear in the equation. Example: Newton's second law of motion: \( ma = F \).
- **Partial differential Equations (PDE):** Partial derivatives of two or more variables appear in the equation. Example: The wave equation for sound propagation in air.
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Newton’s second law of motion is an ODE: The unknown is $x(t)$, the particle position as function of time $t$ and the equation is

$$\frac{d^2}{dt^2}x(t) = \frac{1}{m} F(t, x(t)),$$

with $m$ the particle mass and $F$ the force acting on the particle.
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with \( m \) the particle mass and \( F \) the force acting on the particle.

Example
The wave equation is a PDE: The unknown is \( u(t, x) \), a function that depends on two variables, and the equation is
\[
\frac{\partial^2}{\partial t^2} u(t, x) = v^2 \frac{\partial^2}{\partial x^2} u(t, x),
\]
with \( v \) the wave speed. Sound propagation in air is described by a wave equation, where \( u \) represents the air pressure.
Overview of differential equations.

Remark: Differential equations are a central part in a physical description of nature:
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- Classical Mechanics:
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- Electromagnetism:
  - Maxwell’s equations. (PDE)
- Quantum Mechanics:
  - Schrödinger’s equation. (PDE)
- General Relativity:
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- **Linear Ordinary Differential Equations.**
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  - Constant coefficients.
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Remark: Given a function $y : \mathbb{R} \to \mathbb{R}$, we use the notation

$$y'(t) = \frac{dy}{dt}(t).$$
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Definition

Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, a \textit{first order ODE} in the unknown function $y : \mathbb{R} \rightarrow \mathbb{R}$ is the equation

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The first order ODE above is called linear iff there exist functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $f(t, y) = -a(t)y + b(t)$. That is, $f$ is linear on its argument $y$, hence a first order linear ODE is given by

$$y'(t) = -a(t)y(t) + b(t).$$
Example
A first order linear ODE is given by

\[ y'(t) = -2y(t) + 3. \]
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In this case function \( a(t) = -2 \) and \( b(t) = 3 \). Since these function do not depend on \( t \), the equation above is called of constant coefficients.
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A first order linear ODE is given by

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Remark: Solutions to first order linear ODE can be obtained using the integrating factor method.
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Theorem (Constant coefficients)

*Given constants* \( a, b \in \mathbb{R} \) *with* \( a \neq 0 \), *the linear differential equation*

\[
y'(t) = -ay(t) + b
\]

*has infinitely many solutions, one for each value of* \( c \in \mathbb{R} \), *given by*

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y(t) = c e^{-at} + \frac{b}{a}.
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Remark: A proof is given in the Lecture Notes. Here we present the main idea of the proof, showing and exponential integrating factor.
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**Remark:** A proof is given in the Lecture Notes. Here we present the main idea of the proof, showing and exponential integrating factor. In the Lecture Notes it is shown that this is essentially the only integrating factor.
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Main ideas of the Proof: Write down the differential equation as

\[ y'(t) + a y(t) = b. \]
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Key idea: The left-hand side above is a total derivative if we multiply it by the exponential \( e^{at} \).
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\[ y'(t) + ay(t) = b. \]

Key idea: The left-hand side above is a total derivative if we multiply it by the exponential \( e^{at} \). Indeed,
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Example
Find all functions $y$ solution of the ODE $y' = 2y + 3$. 

Solution: 
The ODE is $y' = -ay + b$ with $a = -2$ and $b = 3$.

The functions $y(t) = ce^{-at} + b/a$, with $c \in \mathbb{R}$, are solutions.

We conclude that the ODE has infinitely many solutions, given by $y(t) = ce^{2t} - 3/2$, $c \in \mathbb{R}$.

Since we did one integration, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.

Verification: $ce^{2t} = y + (3/2)$, so $2ce^{2t} = y'$, therefore we conclude that $y$ satisfies the ODE $y' = 2y + 3$. 

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The Initial Value Problem (IVP) for a linear ODE is the following:
Given functions \( a, b : \mathbb{R} \to \mathbb{R} \) and constants \( t_0, y_0 \in \mathbb{R} \), find a solution \( y : \mathbb{R} \to \mathbb{R} \) of the problem
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Remark: The initial condition selects one solution of the ODE.
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Theorem (Constant coefficients)
Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = -ay + b, \quad y(t_0) = y_0$$

has the unique solution

$$y(t) = \left( y_0 - \frac{b}{a} \right) e^{-a(t-t_0)} + \frac{b}{a}.$$
The Initial Value Problem.

Example

Find the solution to the initial value problem

\[ y' = 2y + 3, \quad y(0) = 1. \]
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Example
Find the solution to the initial value problem

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Solution: Every solution of the ODE above is given by

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\[ y(t) = c e^{2t} - \frac{3}{2}, \quad c \in \mathbb{R}. \]

The initial condition \( y(0) = 1 \) selects only one solution:

\[ 1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}. \]

We conclude that \( y(t) = \frac{5}{2} e^{2t} - \frac{3}{2}. \)
The integrating factor method (Sect. 2.1).

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- **The integrating factor method.**
  - Constant coefficients.
  - The Initial Value Problem.
  - **Variable coefficients.**
The integrating factor method.

Theorem (Variable coefficients)

Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ and given constants $t_0, y_0 \in \mathbb{R}$, the IVP

$$y' = -a(t)y + b(t) \quad y(t_0) = y_0$$

has the unique solution

$$y(t) = \frac{1}{\mu(t)} \left[ y_0 + \int_{t_0}^{t} \mu(s)b(s)ds \right],$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \quad A(t) = \int_{t_0}^{t} a(s)ds.$$

Remark: See the proof in the Lecture Notes.
The integrating factor method.

Example

Find the solution $y$ to the IVP

$$t \ y' + 2y = 4t^2, \quad y(1) = 2.$$
The integrating factor method.

Example
Find the solution \( y \) to the IVP

\[
t y' + 2y = 4t^2, \quad y(1) = 2.
\]

Solution: We first express the ODE as in the Theorem above,

\[
y' = -\frac{2}{t} y + 4t.
\]
The integrating factor method.

Example
Find the solution $y$ to the IVP

$$t\ y' + 2y = 4t^2, \quad y(1) = 2.$$  

Solution: We first express the ODE as in the Theorem above,

$$y' = -\frac{2}{t}y + 4t.$$  

Therefore, $a(t) = \frac{2}{t}$ and $b(t) = 4t$, and also $t_0 = 1$ and $y_0 = 2$. 
The integrating factor method.

Example
Find the solution $y$ to the IVP

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$ 

Solution: We first express the ODE as in the Theorem above,

$$y' = \frac{-2}{t}y + 4t.$$ 

Therefore, $a(t) = \frac{2}{t}$ and $b(t) = 4t$, and also $t_0 = 1$ and $y_0 = 2$. We first compute the integrating factor function $\mu = e^{A(t)}$, 

where

$$A(t) = 2 \ln(t) = \ln(t^2).$$
The integrating factor method.

Example
Find the solution \( y \) to the IVP

\[ t \, y' + 2y = 4t^2, \quad y(1) = 2. \]

Solution: We first express the ODE as in the Theorem above,

\[ y' = -\frac{2}{t} y + 4t. \]

Therefore, \( a(t) = \frac{2}{t} \) and \( b(t) = 4t \), and also \( t_0 = 1 \) and \( y_0 = 2 \).

We first compute the integrating factor function \( \mu = e^{A(t)} \), where

\[ A(t) = \int_{t_0}^{t} a(s) \, ds \]
The integrating factor method.

Example
Find the solution $y$ to the IVP

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We first compute the integrating factor function $\mu = e^{A(t)}$, where

$$A(t) = \int_{t_0}^{t} a(s) \, ds = \int_{1}^{t} \frac{2}{s} \, ds.$$
The integrating factor method.

**Example**
Find the solution $y$ to the IVP

$$t \ y' + 2y = 4t^2, \quad y(1) = 2.$$  

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$$A(t) = \int_{t_0}^{t} a(s) \, ds = \int_{1}^{t} \frac{2}{s} \, ds = 2[\ln(t) - \ln(1)]$$
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We conclude that $\mu(t) = t^2$. 
The integrating factor method.

Example
Find the solution $y$ to the IVP

$$t \ y' + 2y = 4t^2, \quad y(1) = 2.$$  

Solution: The integrating factor is $\mu(t) = t^2$. 

The initial condition implies $2 = y(1) = 1 + c$, that is, $c = 1$. 
We conclude that $y(t) = t^2 + 1t^2$. 

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The integrating factor method.

Example

Find the solution $y$ to the IVP

$$t \ y' + 2y = 4t^2, \quad y(1) = 2.$$ 

Solution: The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2 \left( y' + \frac{2}{t} \ y \right) = t^2 (4t)$$
The integrating factor method.

Example
Find the solution $y$ to the IVP

$$t \, y' + 2y = 4t^2, \quad y(1) = 2.$$  

Solution: The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2 \left( y' + \frac{2}{t} y \right) = t^2(4t) \quad \Leftrightarrow \quad t^2 \, y' + 2t \, y = 4t^3$$
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The initial condition implies $2 = y(1)$.
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$\triangleright$
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The initial condition implies $2 = y(1) = 1 + c$, that is, $c = 1$. We conclude that $y(t) = t^2 + \frac{1}{t^2}$. \triangle
Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.
Separable ODE.

Definition
Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a first order ODE on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called \textit{separable} iff the ODE has the form

$$h(y) y'(t) = g(t).$$
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Remark:
A differential equation \( y'(t) = f(t, y(t)) \) is separable iff

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h(y) = N(y) \quad \text{and} \quad g(t) = -M(t).
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Notation:
In lecture: \( t, y(t) \) and \( h(y) y'(t) = g(t) \).
Separable ODE.

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A differential equation $y'(t) = f(t, y(t))$ is separable iff

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Notation:
In lecture: $t, y(t)$ and $h(y) y'(t) = g(t)$.

In textbook: $x, y(x)$ and $M(x) + N(y) y'(x) = 0$. 
Separable ODE.

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In lecture: \( t, y(t) \) and \( h(y) y'(t) = g(t) \).

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Separable ODE.

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In lecture: $t, y(t)$ and $h(y) y'(t) = g(t)$.

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Therefore: $h(y) = N(y)$ and $g(t) = -M(t)$. 
Example

Determine whether the differential equation below is separable,

\[ y'(t) = \frac{t^2}{1 - y^2(t)}. \]
Separable ODE.

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Solution: The differential equation is separable, since it is equivalent to

\[ (1 - y^2) y'(t) = t^2. \]
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Remark: The functions \( g \) and \( h \) are not uniquely defined.
Separable ODE.

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Remark: The functions \( g \) and \( h \) are not uniquely defined. Another choice here is:

\[ g(t) = c t^2, \quad h(y) = c (1 - y^2), \quad c \in \mathbb{R}. \]
Example
Determine whether the differential equation below is separable,

\[ y'(t) + y^2(t) \cos(2t) = 0 \]
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Remark: The functions \( g \) and \( h \) are not uniquely defined. Another choice here is:

\[ g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}. \]
Remark: Not every first order ODE is separable.
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Example

- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
Separable ODE.

Remark: Not every first order ODE is separable.

Example

- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
- The linear differential equation $y'(t) = -\frac{2}{t} y(t) + 4t$ is not separable.
Remark: Not every first order ODE is separable.

Example

- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
- The linear differential equation $y'(t) = -\frac{2}{t} y(t) + 4t$ is not separable.
- The linear differential equation $y'(t) = -a(t) y(t) + b(t)$, with $b(t)$ non-constant, is not separable.
Separable differential equations (Sect. 2.2).

- Separable ODE.
- **Solutions to separable ODE.**
- Explicit and implicit solutions.
- Homogeneous equations.
Solutions to separable ODE.

Theorem (Separable equations)

If the functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, with $h \neq 0$ and with primitives $G$ and $H$, respectively; that is,

$$G'(t) = g(t), \quad H'(u) = h(u),$$

then, the separable ODE

$$h(y) y' = g(t)$$

has infinitely many solutions $y : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the algebraic equation

$$H(y(t)) = G(t) + c,$$

where $c \in \mathbb{R}$ is arbitrary.
Solutions to separable ODE.

Theorem (Separable equations)

If the functions \( g, h : \mathbb{R} \to \mathbb{R} \) are continuous, with \( h \neq 0 \) and with primitives \( G \) and \( H \), respectively; that is,

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Remark: Given functions \( g, h \), find their primitives \( G, H \).
Solutions to separable ODE.

Example

Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to the ODE $y'(t) = \frac{t^2}{1 - y^2(t)}$. 

Solution:

The equation is equivalent to $(1 - y^2(t))y'(t) = t^2$.

Therefore, the functions $g, h$ are given by $g(t) = t^2$, $h(u) = 1 - u^2$.

Their primitive functions, $G$ and $H$, respectively, are given by $g(t) = t^2 \Rightarrow G(t) = \frac{t^3}{3}$, $h(u) = 1 - u^2 \Rightarrow H(u) = u - \frac{u^3}{3}$.

Then, the Theorem above implies that the solution $y$ satisfies the algebraic equation $y(t) - y^3(t) = \frac{t^3}{3} + c$, $c \in \mathbb{R}$. \[\Box\]
Solutions to separable ODE.

Example

Find all solutions \( y : \mathbb{R} \to \mathbb{R} \) to the ODE \( y'(t) = \frac{t^2}{1 - y^2(t)}. \)

Solution: The equation is equivalent to \( (1 - y^2) y'(t) = t^2. \)
Solutions to separable ODE.

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Find all solutions \( y : \mathbb{R} \rightarrow \mathbb{R} \) to the ODE \( y'(t) = \frac{t^2}{1 - y^2(t)} \).

Solution: The equation is equivalent to \((1 - y^2) y'(t) = t^2\). Therefore, the functions \( g, h \) are given by
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g(t) = t^2, \quad h(u) = 1 - u^2.
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Solutions to separable ODE.

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Their primitive functions, $G$ and $H$, respectively,
Solutions to separable ODE.

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Solutions to separable ODE.

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Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to the ODE $y'(t) = \frac{t^2}{1 - y^2(t)}$.

Solution: The equation is equivalent to $(1 - y^2) y'(t) = t^2$. Therefore, the functions $g$, $h$ are given by

$$g(t) = t^2, \quad h(u) = 1 - u^2.$$ 

Their primitive functions, $G$ and $H$, respectively, are given by

$$g(t) = t^2 \quad \Rightarrow \quad G(t) = \frac{t^3}{3},$$

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Solutions to separable ODE.

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Then, the Theorem above implies that the solution \( y \) satisfies the algebraic equation

\[
y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \quad c \in \mathbb{R}.
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Solutions to separable ODE.

Remarks:

- The equation \( y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c \) is algebraic in \( y \), since there is no \( y' \) in the equation.
Solutions to separable ODE.

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- The equation $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is algebraic in $y$, since there is no $y'$ in the equation.

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\[
y'(t) - 3 \left( \frac{y^2(t)}{3} \right) y'(t) = 3 \frac{t^2}{3} \quad \Rightarrow \quad (1 - y^2) y' = t^2.
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Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
  - **Explicit and implicit solutions.**
- Homogeneous equations.
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Remark:
The solution $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is given in implicit form.
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Definition
Assume the notation in the Theorem above. The solution $y$ of a separable ODE is given in \textit{implicit form} iff function $y$ is specified by

$$H(y(t)) = G(t) + c,$$

where $H$ is invertible.
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\[
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Explicit and implicit solutions.

Example

Use the main idea in the proof of the Theorem above to find the solution of the IVP

\[ y'(t) + y^2(t) \cos(2t) = 0, \quad y(0) = 1. \]
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$$\int \frac{du}{u^2} = - \int \cos(2t) \, dt + c \quad \Leftrightarrow \quad -\frac{1}{u} = -\frac{1}{2} \sin(2t) + c.$$
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The initial condition implies that 1 = \( y(0) \)
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We conclude that \( y(t) = \frac{2}{\sin(2t) + 2}. \) \( \triangleq \)
Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- **Homogeneous equations.**
Homogeneous equations.

Definition
The first order ODE $y'(t) = f(t, y(t))$ is called \textit{homogeneous} iff for every numbers $c, t, u \in \mathbb{R}$ the function $f$ satisfies

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- Therefore, a first order ODE is homogeneous iff it has the form

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\[(t - y) y' - 2y + 3t + \frac{y^2}{t} = 0.\]
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and $f(t, y) = F(y/t)$.  \[\triangle\]
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Divide numerator and denominator by \( t^3 \), we obtain

\[
y' = \frac{t^2}{(1 - y^3)} \left( \frac{1}{t^3} \right) \Rightarrow y' = \frac{\left( \frac{1}{t} \right)}{\left( \frac{1}{t^3} \right) - \left( \frac{y}{t} \right)^3}.
\]

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Theorem

If the differential equation $y'(t) = f(t, y(t))$ is homogeneous, then the differential equation for the unknown $v(t) = \frac{y(t)}{t}$ is separable.
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If the differential equation $y'(t) = f(t, y(t))$ is homogeneous, then the differential equation for the unknown $v(t) = \frac{y(t)}{t}$ is separable.

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**Proof:** If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function $F$. 
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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function $F$. Introduce $v = y/t$. This means,

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If the differential equation $y'(t) = f(t, y(t))$ is homogeneous, then the differential equation for the unknown $v(t) = \frac{y(t)}{t}$ is separable.

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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function $F$. Introduce $v = y/t$. This means,

$$y(t) = t \, v(t) \quad \Rightarrow \quad y'(t) = v(t) + t \, v'(t).$$

Introducing all this into the ODE we get

$$v + t \, v' = F(v)$$
Homogeneous equations.

Theorem
If the differential equation \( y'(t) = f(t, y(t)) \) is homogeneous, then the differential equation for the unknown \( v(t) = \frac{y(t)}{t} \) is separable.

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Proof: If \( y' = f(t, y) \) is homogeneous, then it can be written as \( y' = F(y/t) \) for some function \( F \). Introduce \( v = y/t \). This means,

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y(t) = t \, v(t) \quad \Rightarrow \quad y'(t) = v(t) + t \, v'(t).
\]

Introducing all this into the ODE we get

\[
v + t \, v' = F(v) \quad \Rightarrow \quad v' = \frac{F(v) - v}{t}.
\]

This last equation is separable. \( \square \)
Homogeneous equations.

Example

Find all solutions $y$ of the ODE $y' = \frac{t^2 + 3y^2}{2ty}$.
Homogeneous equations.

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Solution: The equation is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \left( \frac{1}{t^2} \right)$$

$$= \frac{1}{t} \left( 1 + \frac{3y^2}{t^2} \right)$$
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Therefore, we introduce the change of unknown $v = y/t$, so $y = t \cdot v$ and $y' = v + t \cdot v'$. Hence

$$v + t \cdot v' = \frac{1 + 3v^2}{2v}.$$
Homogeneous equations.

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Therefore, we introduce the change of unknown \( v = y/t \), so \( y = t \, v \) and \( y' = v + t \, v' \). Hence

\[
v + t \, v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t \, v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}.
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Therefore, we introduce the change of unknown $v = y/t$, so $y = t\, v$ and $y' = v + t\, v'$. Hence

\[ v + t\, v' = \frac{1 + 3v^2}{2v} \Rightarrow t\, v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} \]

We obtain the separable equation $v' = \frac{1}{t} \left( \frac{1 + v^2}{2v} \right)$. 
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Example

Find all solutions $y$ of the ODE $y' = \frac{t^2 + 3y^2}{2ty}$.

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Solution: Recall: $v' = \frac{1}{t} \left( \frac{1 + v^2}{2v} \right)$. We rewrite and integrate it,

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\frac{2v}{1 + v^2} v' = \frac{1}{t}
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The substitution \( u = 1 + v^2(t) \) implies \( du = 2v(t) v'(t) \, dt \),
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But $u = e^{\ln(t)} e^{c_0}$,
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The substitution $u = 1 + v^2(t)$ implies $du = 2v(t) v'(t) \, dt$, so

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But $u = e^{\ln(t) + c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left( \frac{y}{t} \right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t \sqrt{c_1 t - 1}.$$
Modeling with first order equations (Sect. 2.3).

- Main example: Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.
Salt in a water tank.

**Problem:** Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.
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**Main ideas of the test:**

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
Salt in a water tank.

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- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
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- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
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- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
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- The salt in the tank also depends on the water rates coming in and going out of the tank.
- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.
Modeling with first order equations (Sect. 2.3).

- **Main example: Salt in a water tank.**
  - The experimental device.
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  - Analysis of the mathematical model.
  - Predictions for particular situations.
The experimental device.

\[
\begin{align*}
V(t) &\quad r_i \quad q_i(t) \\
Q(t) &\quad r_o \quad q_o(t)
\end{align*}
\]

water
salt

pipe

instantaneously mixed

tank
The experimental device.

Definitions:

- $r_i(t)$, $r_o(t)$: Rates in and out of water entering and leaving the tank at the time $t$. 
- $q_i(t)$, $q_o(t)$: Salt concentration of the water entering and leaving the tank at the time $t$. 
- $V(t)$: Water volume in the tank at the time $t$. 
- $Q(t)$: Salt mass in the tank at the time $t$. 

Units:

- $[r_i(t)] = [r_o(t)] = \text{Volume Time}$. 
- $[q_i(t)] = [q_o(t)] = \text{Mass Volume}$. 
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Modeling with first order equations (Sect. 2.3).

- **Main example: Salt in a water tank.**
  - The experimental device.
  - **The main equations.**
  - Analysis of the mathematical model.
  - Predictions for particular situations.
**Remark:** The mass conservation provides the main equations of the mathematical description for salt in water.
The main equations.

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Main equations:

\[ \frac{d}{dt} V(t) = r_i(t) - r_o(t), \]

Volume conservation, \ (1)
Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

\[
\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)
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\[
\frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \quad (2)
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\[
q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \quad (3)
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Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

\[ \frac{d}{dt} V(t) = r_i(t) - r_o(t), \] Volume conservation, \hspace{1cm} (1)

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\[ r_i, \ r_o : \text{ Constants.} \] \hspace{1cm} (4)
The main equations.

Remarks:

\[
\left[ \frac{dV}{dt} \right] = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o],
\]

\[
\left[ \frac{dQ}{dt} \right] = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o],
\]

\[
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Modeling with first order equations (Sect. 2.3).

- Main example: Salt in a water tank.
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Analysis of the mathematical model.

Eqs. (4) and (1) imply

$$V(t) = (r_i - r_o) t + V_0,$$  \hspace{1cm} (5)

where $V(0) = V_0$ is the initial volume of water in the tank.
Analysis of the mathematical model.

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Eqs. (3) and (2) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \]  

(6)
Analysis of the mathematical model.

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\[ \frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \quad (6) \]

Eqs. (5) and (6) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \quad (7) \]
Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \)
Analysis of the mathematical model.

Recall: \[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t) \].

Notation: \( a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \).
Analysis of the mathematical model.

Recall: \[
\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t).
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Notation: \[a(t) = \frac{r_o}{(r_i - r_o) t + V_0}, \text{ and } b(t) = r_i q_i(t).\]
Analysis of the mathematical model.

Recall: \[
\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t).
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Notation: \[a(t) = \frac{r_o}{(r_i - r_o) t + V_0}, \quad \text{and} \quad b(t) = r_i q_i(t).\]

The main equation of the description is given by

\[ Q'(t) = -a(t) Q(t) + b(t). \]
Analysis of the mathematical model.

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Linear ODE for \( Q \).
Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t) . \)

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Linear ODE for \( Q \). Solution: Integrating factor method.
Analysis of the mathematical model.

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\[ Q'(t) = -a(t) Q(t) + b(t). \]

Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) \, ds \right] \]
Analysis of the mathematical model.

Recall: \[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

Notation: \[ a(t) = \frac{r_o}{(r_i - r_o) t + V_0}, \] and \[ b(t) = r_i q_i(t). \]

The main equation of the description is given by

\[ Q'(t) = -a(t) Q(t) + b(t). \]

Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) \, ds \right] \]

with \( Q(0) = Q_0 \).
Analysis of the mathematical model.

Recall: \[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

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\[ Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) \, ds \right] \]

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Analysis of the mathematical model.

Recall: \[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

Notation: \( a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \), and \( b(t) = r_i q_i(t) \).

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Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) \, ds \right] \]

with \( Q(0) = Q_0 \), where \( \mu(t) = e^{A(t)} \) and \( A(t) = \int_0^t a(s) \, ds \).
Modeling with first order equations (Sect. 2.3).

- **Main example: Salt in a water tank.**
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t) Q(t) + b(t) \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t) Q(t) + b(t) \).
In this case:

\[
a(t) = \frac{r_o}{(r_i - r_o) t + V_0}
\]
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Always holds $Q'(t) = -a(t)Q(t) + b(t)$.
In this case:

$$a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \implies a(t) = \frac{r}{V_0} = a_0,$$
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r \), \( q_i \), \( Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t) Q(t) + b(t) \).
In this case:

\[
a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,
\]

\[
b(t) = r_i q_i(t)
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t) Q(t) + b(t) \).
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\]

\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = rq_i = b_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t) Q(t) + b(t) \). In this case:

\[
a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,
\]

\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = rq_i = b_0.
\]

We need to solve the IVP:
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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a(t) = \frac{r_o}{(r_i - r_o) t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,
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b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = rq_i = b_0.
\]

We need to solve the IVP:

\[
Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 \, Q(t) + b_0, \quad Q(0) = Q_0. \)
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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Integrating factor method:
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t,
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 \, Q(t) + b_0, \quad Q(0) = Q_0. \)
Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t},
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 Q(t) + b_0 \), \( Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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\]

\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1)
\]
Predictions for particular situations.

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If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].
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Predictions for particular situations.

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Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].
\]
\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].
\]
So: \( Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0} \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r \), \( q_i \), \( Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 Q(t) + b_0 \), \( Q(0) = Q_0 \).

Integrating factor method:

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A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].
\]

\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].
\]

So:

\[
Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}. \quad \text{But} \quad \frac{b_0}{a_0} = rq_i \frac{V_0}{r}.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

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A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].
\]

\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].
\]

So: \( Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0} \). But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0 \).
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) = -a_0 Q(t) + b_0$, $Q(0) = Q_0$.

Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].$$

$$\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].$$

So: $Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}$. But $\frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0$.

We conclude: $Q(t) = \left( Q_0 - q_i V_0 \right) e^{-r t / V_0} + q_i V_0$. 
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \).
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \).

Particular cases:

\[ \frac{Q_0}{V_0} > q_i; \]
\[ \frac{Q_0}{V_0} = q_i, \text{ so } Q(t) = Q_0; \]
\[ \frac{Q_0}{V_0} < q_i. \]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \).

Particular cases:
- \( \frac{Q_0}{V_0} > q_i \);
- \( \frac{Q_0}{V_0} = q_i \), so \( Q(t) = Q_0 \);
- \( \frac{Q_0}{V_0} < q_i \).
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.
Predictions for particular situations.

**Example**

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

**Solution:** This problem is a particular case \( q_i = 0 \) of the previous Example.
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \),
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) \, e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 \, e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \)
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2 \text{ liters/min}$, $q_i = 0$, $V_0 = 200 \text{ liters}$, $Q_0/V_0 = 1 \text{ grams/liter}$, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0 e^{-rt/V_0}.$$ 

Since $V(t) = (r_i - r_o) t + V_0$ and $r_i = r_o$, 

$$q(t) = Q(t)/V(t).$$
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \text{ liters/min} \), \( q_i = 0 \), \( V_0 = 200 \text{ liters} \), \( Q_0/V_0 = 1 \text{ grams/liter} \), find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \). So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,

\[
\frac{1}{100} \frac{Q_0}{V_0} = q(t_1)
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

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Predictions for particular situations.

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Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter,
find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1}/V_0 = \frac{1}{100}$. 
Predictions for particular situations.

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Solution: Recall: \( e^{-rt_1/V_0} = \frac{1}{100} \). Then,

\[-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right)\]
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Solution: Recall: \( e^{-rt_1/V_0} = \frac{1}{100} \). Then,

\[- \frac{r}{V_0} t_1 = \ln \left( \frac{1}{100} \right) = - \ln(100)\]
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\[-\frac{r}{V_0} \cdot t_1 = \ln \left( \frac{1}{100} \right) = -\ln(100) \Rightarrow \frac{r}{V_0} \cdot t_1 = \ln(100).\]
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We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \).
Predictions for particular situations.

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

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$$- \frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = - \ln(100) \implies \frac{r}{V_0} t_1 = \ln(100).$$

We conclude that $t_1 = \frac{V_0}{r} \ln(100)$.

In this case: $t_1 = 100 \ln(100)$. $\triangle$
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ are constants. If $r = 5 \times 10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$. 
Predictions for particular situations.

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Solution: Recall: \( Q'(t) = -a(t) Q(t) + b(t) \).
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$$a(t) = \frac{r_o}{(r_i - r_o)t + V_0}$$
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We need to solve the IVP: \( Q'(t) = -a_0 Q(t) + b(t), \ Q(0) = 0 \).
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\[
Q(t) = \frac{1}{\mu(t)} \int_0^t \mu(s) b(s) \, ds,
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Predictions for particular situations.

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Q(t) = \frac{1}{\mu(t)} \int_0^t \mu(s) b(s) \, ds, \quad \mu(t) = e^{a_0 t},
\]

We conclude: \( Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] \, ds. \)