Even, odd functions.

Definition
A function $f : [-L, L] \rightarrow \mathbb{R}$ is even iff for all $x \in [-L, L]$ holds

$$f(-x) = f(x).$$

A function $f : [-L, L] \rightarrow \mathbb{R}$ is odd iff for all $x \in [-L, L]$ holds

$$f(-x) = -f(x).$$

Remarks:
- The only function that is both odd and even is $f = 0$.
- Most functions are neither odd nor even.
Even, odd functions.

Example
Show that the function $f(x) = x^2$ is even on $[-L, L]$.

Solution: The function is even, since

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Even, odd functions.

Example
Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

Solution: The function is odd, since

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$
Even, odd functions.

Example

1. The function \( f(x) = \cos(ax) \) is even on \([-L, L]\);
2. The function \( f(x) = \sin(ax) \) is odd on \([-L, L]\);
3. The functions \( f(x) = e^x \) and \( f(x) = (x - 2)^2 \) are neither even nor odd.

Sine and Cosine Series (Sect. 10.4).

▶ Even, odd functions.
▶ **Main properties of even, odd functions.**
▶ Sine and cosine series.
▶ Even-periodic, odd-periodic extensions of functions.
Main properties of even, odd functions.

Theorem

(1) A linear combination of even (odd) functions is even (odd).
(2) The product of two odd functions is even.
(3) The product of two even functions is even.
(4) The product of an even function by an odd function is odd.

Proof:
(1) Let \( f \) and \( g \) be even, that is, \( f(-x) = f(x) \), \( g(-x) = g(x) \). Then, for all \( a, b \in \mathbb{R} \) holds,

\[
(af + bg)(-x) = af(-x) + bg(-x) = af(x) + bg(x) = (af + bg)(x).
\]

Case "odd" is similar.

Main properties of even, odd functions.

Theorem

(1) A linear combination of even (odd) functions is even (odd).
(2) The product of two odd functions is even.
(3) The product of two even functions is even.
(4) The product of an even function by an odd function is odd.

Proof:
(2) Let \( f \) and \( g \) be odd, that is, \( f(-x) = -f(x) \), \( g(-x) = -g(x) \). Then, for all \( a, b \in \mathbb{R} \) holds,

\[
(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x).
\]

Cases (3), (4) are similar.
Main properties of even, odd functions.

Theorem

If \( f : [-L, L] \rightarrow \mathbb{R} \) is even, then \( \int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx \).

If \( f : [-L, L] \rightarrow \mathbb{R} \) is odd, then \( \int_{-L}^{L} f(x) \, dx = 0 \).

Proof:

\[
I = \int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx \quad y = -x, \ dy = -dx.
\]

\[
I = \int_{0}^{0} f(-y) (-dy) + \int_{0}^{L} f(x) \, dx = \int_{0}^{L} f(-y) \, dy + \int_{0}^{L} f(x) \, dx.
\]

Even case: \( f(-y) = f(y) \), therefore,

\[
I = \int_{0}^{L} f(y) \, dy + \int_{0}^{L} f(x) \, dx \Rightarrow \int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx.
\]

Odd case: \( f(-y) = -f(y) \), therefore,

\[
I = -\int_{0}^{L} f(y) \, dy + \int_{0}^{L} f(x) \, dx \Rightarrow \int_{-L}^{L} f(x) \, dx = 0. \quad \Box
\]
Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- **Sine and cosine series.**
- Even-periodic, odd-periodic extensions of functions.

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Sine and cosine series.

**Theorem (Cosine and Sine Series)**

Consider the function $f : [-L, L] \rightarrow \mathbb{R}$ with Fourier expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].$$

1. If $f$ is even, then $b_n = 0$ for $n = 1, 2, \cdots$, and the Fourier series

   $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right)$$

   is called a **Cosine Series**.

2. If $f$ is odd, then $a_n = 0$ for $n = 0, 1, \cdots$, and the Fourier series

   $$f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)$$

   is called a **Sine Series**.
Sine and cosine series.

Proof:
If $f$ is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

If $f$ is odd, and since the Cosine function is even, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- **Even-periodic, odd-periodic extensions of functions.**
(1) Even-periodic case:
A function \( f : [0, L] \rightarrow \mathbb{R} \) can be extended as an even function \( f : [-L, L] \rightarrow \mathbb{R} \) requiring for \( x \in [0, L] \) that

\[ f(-x) = f(x). \]

This function \( f : [-L, L] \rightarrow \mathbb{R} \) can be further extended as a periodic function \( f : \mathbb{R} \rightarrow \mathbb{R} \) requiring for \( x \in [-L, L] \) that

\[ f(x + 2nL) = f(x). \]

Example
Sketch the graph of the even-periodic extension of \( f(x) = x^5 \), with \( x \in [0, 1] \).

Solution:
Even-periodic, odd-periodic extensions of functions.

(2) Odd-periodic case:
A function \( f : (0, L) \to \mathbb{R} \) can be extended as an odd function \( f : (-L, L) \to \mathbb{R} \) requiring for \( x \in (0, L) \) that
\[
f(-x) = -f(x), \quad f(0) = 0.
\]
This function \( f : (-L, L) \to \mathbb{R} \) can be further extended as a periodic function \( f : \mathbb{R} \to \mathbb{R} \) requiring for \( x \in (-L, L) \) and \( n \) integer that
\[
f(x + 2nL) = f(x), \quad \text{and} \quad f(nL) = 0.
\]

Remark: At \( x = \pm L \), the extension \( f \) must satisfy:
(a) \( f \) is odd, hence \( f(-L) = -f(L) \);
(b) \( f \) is periodic, hence \( f(-L) = f(-L + 2L) = f(L) \).
We then conclude that \( -f(L) = f(L) \), hence \( f(L) = 0 \).

Example
Sketch the graph of the odd-periodic extension of \( f(x) = x^5 \), with \( x \in (0, 1) \).

Solution:
Even-periodic, odd-periodic extensions of functions.

Example
Sketch the graph of the even-periodic extension of \( f(x) = x \), with \( x \in [0, 1] \), and then find its Fourier Series.

Solution:

Since \( f \) is even and periodic, then the Fourier Series is a Cosine Series, that is, \( b_n = 0 \). From the graph: \( a_0 = 1 \).

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx.
\]

\[
a_n = 2 \int_{0}^{1} x \cos(n \pi x) \, dx = 2 \left[ \frac{x \sin(n \pi x)}{n \pi} + \frac{\cos(n \pi x)}{(n \pi)^2} \right]_{0}^{1},
\]

\[
a_n = \frac{2}{(n \pi)^2} [\cos(n \pi) - 1] \quad \Rightarrow \quad a_n = \frac{2}{(n \pi)^2} [(-1)^n - 1].
\]
Even-periodic, odd-periodic extensions of functions.

Example
Sketch the graph of the even-periodic extension of \( f(x) = x \), with \( x \in [0, 1] \), and then find its Fourier Series.

Solution: Recall: \( b_n = 0 \), and \( a_n = \frac{2}{(n\pi)^2} [(-1)^n - 1] \).

\[
\begin{align*}
n = 2k & \implies a_{2k} = \frac{2}{((2k)\pi)^2} [(-1)^{2k} - 1] \implies a_{2k} = 0. \\
n = 2k - 1 & \implies a_{2k-1} = \frac{2 [-1 - 1]}{((2k - 1)\pi)^2} \implies a_{2k-1} = \frac{-4}{((2k - 1)\pi)^2}.
\end{align*}
\]

\[
f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2} \cos((2k - 1)\pi x). \quad \blacktriangleleft
\]

Even-periodic, odd-periodic extensions of functions.

Example
Sketch the graph of the odd-periodic extension of \( f(x) = x \), with \( x \in (0, 1) \), and then find its Fourier Series.

Solution:
**Example**

Sketch the graph of the odd-periodic extension of \( f(x) = x \), with \( x \in (0, 1) \), and then find its Fourier Series.

**Solution:** Since \( f \) is odd and periodic, then the Fourier Series is a Sine Series, that is, \( a_n = 0 \).

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx.
\]

\[
b_n = 2 \int_{0}^{1} x \sin(n \pi x) \, dx = 2 \left[ -\frac{x \cos(n \pi x)}{n \pi} + \frac{\sin(n \pi x)}{(n \pi)^2} \right]_0^1,
\]

\[
b_n = -\frac{2}{n \pi} [\cos(n \pi) - 0] \quad \Rightarrow \quad b_n = \frac{-2 (-1)^n}{n \pi}.
\]

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**Even-periodic, odd-periodic extensions of functions.**

**Example**

Sketch the graph of the odd-periodic extension of \( f(x) = x \), with \( x \in (0, 1) \), and then find its Fourier Series.

**Solution:** Recall: \( a_n = 0 \), and \( b_n = \frac{2 (-1)^{n+1}}{n \pi} \). Therefore,

\[
f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n \pi x).
\]