Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.

Convolution solutions (Sect. 6.6).

- **Convolution of two functions.**
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Convolution of two functions.

Definition
The convolution of piecewise continuous functions \( f, g : \mathbb{R} \to \mathbb{R} \) is the function \( f \ast g : \mathbb{R} \to \mathbb{R} \) given by
\[
(f \ast g)(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau.
\]

Remarks:
- \( f \ast g \) is also called the generalized product of \( f \) and \( g \).
- The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac’s delta.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \( (f \ast g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau \).

Integrate by parts twice:
\[
\int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau,
\]
\[
2 \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t,
\]
\[
2(f \ast g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).
\]

We conclude: \( (f \ast g)(t) = \frac{1}{2} [e^{-t} + \sin(t) - \cos(t)] \).
Convolution solutions (Sect. 6.6).

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**Properties of convolutions.**

**Theorem (Properties)**

*For every piecewise continuous functions $f$, $g$, and $h$, hold:*

(i) **Commutativity:** $f * g = g * f$;

(ii) **Associativity:** $f * (g * h) = (f * g) * h$;

(iii) **Distributivity:** $f * (g + h) = f * g + f * h$;

(iv) **Neutral element:** $f * 0 = 0$;

(v) **Identity element:** $f * \delta = f$.

**Proof:**

(v): $(f * \delta)(t) = \int_0^t f(\tau) \delta(t - \tau) d\tau = f(t)$. 
Properties of convolutions.

Proof:
(1): Commutativity: \( f * g = g * f \).

The definition of convolution is,
\[
(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),
\[
(f * g)(t) = \int_t^0 f(t - \hat{\tau}) g(\hat{\tau})(-1) d\hat{\tau}
\]
\[
(f * g)(t) = \int_0^t g(\hat{\tau}) f(t - \hat{\tau}) d\hat{\tau}
\]

We conclude: \( (f * g)(t) = (g * f)(t) \). \qed 

Convolution solutions (Sect. 6.6).

- Convolution of two functions.
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- **Laplace Transform of a convolution.**
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Laplace Transform of a convolution.

Theorem (Laplace Transform)

If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then

\[
\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].
\]

Proof: The key step is to interchange two integrals. We start with the product of the Laplace transforms,

\[
\mathcal{L}[f] \mathcal{L}[g] = \left[ \int_0^\infty e^{-st} f(t) \, dt \right] \left[ \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \, d\tilde{t} \right],
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \left( \int_0^\infty e^{-st} f(t) \, dt \right) d\tilde{t},
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}.
\]

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}. \)

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt; \)

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_\tilde{t}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) d\tilde{t}.
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_\tilde{t}^\infty e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tau \, d\tilde{t}.
\]

The key step: Switch the order of integration.

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \, d\tau.
\]
Laplace Transform of a convolution.

Proof: Recall: \[ \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-st} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \, d\tau. \]

Then, it is straightforward to check that

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-st} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \right) \, d\tau,
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-st} (g * f)(\tau) \, d\tau
\]

We conclude: \[ \mathcal{L}[f * g] = \mathcal{L}[f] \mathcal{L}[g]. \]

Convolution solutions (Sect. 6.6).

- Convolution of two functions.
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- Laplace Transform of a convolution.
- **Impulse response solution.**
- Solution decomposition theorem.
Impulse response solution.

Definition
The impulse response solution is the function $y_\delta$ solution of the IVP

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0, \quad c \in \mathbb{R}.$$ 

Example
Find the impulse response solution of the IVP

$$y_\delta'' + 2 y_\delta' + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$ 

Solution: $L[y_\delta''] + 2 L[y_\delta'] + 2 L[y_\delta] = L[\delta(t - c)].$

$$(s^2 + 2s + 2) L[y_\delta] = e^{-cs} \quad \Rightarrow \quad L[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$ 

Impulse response solution.

Example
Find the impulse response solution of the IVP

$$y_\delta'' + 2 y_\delta' + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$ 

Solution: Recall: $L[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 8} \right]$$ 

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[ s^2 + 2 \left( \frac{2}{2} \right) s + 1 \right] - 1 + 2 = (s + 1)^2 + 1.$$ 

Therefore, $L[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1}.$
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \), and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)] \).

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)].
\]

Since \( e^{-cs} \mathcal{L}[f](s) = \mathcal{L}[u(t - c) f(t - c)] \),

we conclude \( y_\delta(t) = u(t - c) e^{-(t-c)} \sin(t - c). \)

Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- Properties of convolutions.
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- **Solution decomposition theorem.**
Solution decomposition theorem.

Theorem (Solution decomposition)
The solution \( y \) to the IVP
\[
y'' + a_1 y' + a_0 y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,\]
can be decomposed as
\[
y(t) = y_h(t) + (y_\delta * g)(t),\]
where \( y_h \) is the solution of the homogeneous IVP
\[
y_h'' + a_1 y_h' + a_0 y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,\]
and \( y_\delta \) is the impulse response solution, that is,
\[
y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.\]

Example
Use the Solution Decomposition Theorem to express the solution of
\[
y'' + 2 y' + 2 y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.\]

Solution: \( \mathcal{L}[y''] + 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\sin(at)] \), and recall,
\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s (1) - (-1), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1.\]

\[
(s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].\]

\[
\mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)].\]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]

But: \[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \]

and: \[ \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \] So,

\[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t), \]

So: \[ y(t) = e^{-t} \cos(t) + \int_0^t e^{-\tau} \sin(\tau) \sin[a(t - \tau)] d\tau. \]

\[ \triangleright \]

Solution decomposition theorem.

Proof: Compute: \[ \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)], \]

and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0. \]

\[ (s^2 + a_1s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1y_0 = \mathcal{L}[g(t)]. \]

\[ \mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1s + a_0)} + \frac{1}{(s^2 + a_1s + a_0)} \mathcal{L}[g(t)]. \]

Recall: \[ \mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1s + a_0)}, \quad \text{and} \quad \mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1s + a_0)}. \]

Since, \[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)], \] so \[ y(t) = y_h(t) + (y_\delta * g)(t). \]

Equivalently: \[ y(t) = y_h(t) + \int_0^t y_\delta(\tau)g(t - \tau) d\tau. \] \[ \square \]