Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac’s delta.
- Relation between deltas and steps.
- Dirac’s delta in Physics.
- The Laplace Transform of Dirac’s delta.
- Differential equations with Dirac’s delta sources.
The Dirac delta generalized function.

Definition
Consider the sequence of functions for $n \geq 1,$

$$
\delta_n(t) = \begin{cases} 
0, & t < 0 \\
1, & 0 \leq t \leq \frac{1}{n} \\
0, & t > \frac{1}{n}.
\end{cases}
$$

The Dirac delta generalized function is given by

$$
\lim_{n \to \infty} \delta_n(t) = \delta(t), \quad t \in \mathbb{R}.
$$

Remarks:
(a) There exist infinitely many sequences $\delta_n$ that define the same generalized function $\delta$.
(b) For example, compare with the sequence $\delta_n$ in the textbook.

The Dirac delta generalized function.

Remarks:
(a) The Dirac $\delta$ is a function on the domain $\mathbb{R} \setminus \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} \setminus \{0\}$.
(b) $\delta$ at $t = 0$ is not defined, since $\delta(0) = \lim_{n \to \infty} n = +\infty$.
(c) $\delta$ is not a function on $\mathbb{R}$. 
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**Properties of Dirac's delta.**

*Remark:* The Dirac $\delta$ is not a function.

We define operations on Dirac's $\delta$ as limits $n \to \infty$ of the operation on the sequence elements $\delta_n$.

**Definition**

$$
\delta(t - c) = \lim_{n \to \infty} \delta_n(t - c),
$$

$$
a \delta(t) + b \delta(t) = \lim_{n \to \infty} [a \delta_n(t) + b \delta_n(t)],
$$

$$
f(t) \delta(t) = \lim_{n \to \infty} [f(t) \delta_n(t)],
$$

$$
\int_a^b \delta(t) \, dt = \lim_{n \to \infty} \int_a^b \delta_n(t) \, dt,
$$

$$
\mathcal{L}[\delta] = \lim_{n \to \infty} \mathcal{L}[\delta_n].
$$
Properties of Dirac’s delta.

Theorem

\[ \int_{-a}^{a} \delta(t) \, dt = 1, \quad a > 0. \]

Proof:

\[
\int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(t) \, dt = \lim_{n \to \infty} \int_{0}^{1/n} n \, dt
\]

\[
\int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \left[ n \left( t \left|_{0}^{1/n} \right. \right) \right] = \lim_{n \to \infty} \left[ n \frac{1}{n} \right].
\]

We conclude: \( \int_{-a}^{a} \delta(t) \, dt = 1. \) \( \Box \)

Properties of Dirac’s delta.

Theorem

If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( t_0 \in \mathbb{R} \) and \( a > 0 \), then

\[ \int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t) \, dt = f(t_0). \]

Proof: Introduce the change of variable \( \tau = t - t_0 \),

\[
I = \int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t) \, dt = \int_{-a}^{a} \delta(\tau) f(\tau + t_0) \, d\tau,
\]

\[
I = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(\tau) f(\tau + t_0) \, d\tau = \lim_{n \to \infty} \int_{0}^{1/n} n f(\tau + t_0) \, d\tau
\]

Therefore, \( I = \lim_{n \to \infty} n \int_{0}^{1/n} F'(\tau + t_0) \, d\tau \), where we introduced the primitive \( F(t) = \int f(t) \, dt \), that is, \( f(t) = F'(t) \).
Properties of Dirac's delta.

**Theorem**

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).$$

**Proof:** So, $I = \lim_{n \rightarrow \infty} n \int_{0}^{1/n} F'(\tau + t_0) \, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n \rightarrow \infty} n \left[ F(\tau + t_0) \bigg|_0^{1/n} \right] = \lim_{n \rightarrow \infty} n \left[ F(t_0 + \frac{1}{n}) - F(t_0) \right].$$

$$I = \lim_{n \rightarrow \infty} \frac{F(t_0 + \frac{1}{n}) - F(t_0)}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

We conclude: $\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).$  \qed

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Relation between deltas and steps.

Theorem

The sequence of functions for \( n \geq 1 \),

\[
    u_n(t) = \begin{cases} 
        0, & t < 0 \\ 
        nt, & 0 \leq t \leq \frac{1}{n} \\ 
        1, & t > \frac{1}{n}.
    \end{cases}
\]

satisfies, for \( t \in (\infty, 0) \cup (0, 1/n) \cup (1/n, \infty) \), both equations,

\[
    u'_n(t) = \delta_n(t), \quad \lim_{n \to \infty} u_n(t) = u(t), \quad t \in \mathbb{R}.
\]

Remark:

- If we generalize the notion of derivative as 
  \( u'(t) = \lim_{n \to \infty} \delta_n(t) \), then holds \( u'(t) = \delta(t) \).
- Dirac’s delta is a generalized derivative of the step function.

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Dirac’s delta in Physics.

Remarks:
(a) Dirac’s delta generalized function is useful to describe *impulsive forces* in mechanical systems.
(b) An impulsive force transmits a finite momentum in an infinitely short time.
(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton’s law of motion says,

\[ m v'(t) = F(t), \quad \text{with} \quad F(t) = F_0 \delta(t - t_0). \]

The momentum transfer is:

\[ \Delta I = \lim_{\Delta t \to 0} m v(t) \bigg|_{t_0}^{t_0 + \Delta t} = \lim_{\Delta t \to 0} \int_{t_0 - \Delta t}^{t_0 + \Delta t} F(t) \, dt = F_0. \]

That is, \( \Delta I = F_0. \)

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The Laplace Transform of Dirac’s delta.

Recall: The Laplace Transform can be generalized from functions to \( \delta \), as follows, \( \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)] \).

Theorem
\[
\mathcal{L}[\delta(t - c)] = e^{-cs}.
\]

Proof:
\[
\mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)], \quad \delta_n(t) = n \left[ u(t) - u(t - \frac{1}{n}) \right].
\]
\[
\mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} n \left( \mathcal{L}[u(t - c)] - \mathcal{L}[u(t - c - \frac{1}{n})] \right)
\]
\[
\mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} n \left( \frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s} \right) = e^{-cs} \lim_{n \to \infty} \left( 1 - \frac{e^{-\frac{s}{n}}}{\frac{s}{n}} \right).
\]

This is a singular limit, \( \frac{0}{0} \). Use l’Hôpital rule.

The Laplace Transform of Dirac’s delta.

Proof: Recall: \( \mathcal{L}[\delta(t - c)] = e^{-cs} \lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\frac{s}{n}} \).

\[
\lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\frac{s}{n}} = \lim_{n \to \infty} \frac{-\frac{s}{n^2} e^{-\frac{s}{n}}}{\frac{s}{n}} = \lim_{n \to \infty} e^{-\frac{s}{n}} = 1.
\]

We therefore conclude that \( \mathcal{L}[\delta(t - c)] = e^{-cs} \). \( \square \)

Remarks:
(a) This result is consistent with a previous result:
\[
\int_{t_0 - a}^{t_0 + a} \delta(t - t_0) f(t) \, dt = f(t_0).
\]
(b) \( \mathcal{L}[\delta(t - c)] = \int_0^\infty \delta(t - c) e^{-st} \, dt = e^{-cs} \).
(c) \( \mathcal{L}[\delta(t - c) f(t)] = \int_0^\infty \delta(t - c) e^{-st} f(t) \, dt = e^{-cs} f(c) \).
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Differential equations with Dirac’s delta sources.

**Example**
Find the solution $y$ to the initial value problem

$$y'' - y = -20 \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 0.$$  

**Solution:** Compute:  

$$L[y''] - L[y] = -20 L[\delta(t - 3)].$$

$$L[y''] = s^2 L[y] - s y(0) - y'(0) \quad \Rightarrow \quad (s^2 - 1) L[y] - s = -20 e^{-3s},$$

We arrive to the equation

$$L[y] = \frac{s}{(s^2 - 1)} - 20 e^{-3s} \frac{1}{(s^2 - 1)}.$$  

$$L[y] = L[\cosh(t)] - 20 L[u(t - 3) \sinh(t - 3)],$$

We conclude:

$$y(t) = \cosh(t) - 20 u(t - 3) \sinh(t - 3).$$
Differential equations with Dirac’s delta sources.

Example
Find the solution to the initial value problem
\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Compute:
\[
\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[^\delta(t - \pi)] - \mathcal{L}[^\delta(t - 2\pi)],
\]
\[
(s^2 + 4)\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2 + 4)} - \frac{e^{-2\pi s}}{(s^2 + 4)},
\]
that is,
\[
\mathcal{L}[y] = \frac{e^{-\pi s}}{2} \frac{2}{(s^2 + 4)} - \frac{e^{-2\pi s}}{2} \frac{2}{(s^2 + 4)}.
\]
Recall: \( e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u(t-c)f(t-c)] \). Therefore,
\[
\mathcal{L}[y] = \frac{1}{2} \mathcal{L}\left[u(t-\pi)\sin(2(t-\pi))\right] - \frac{1}{2} \mathcal{L}\left[u(t-2\pi)\sin(2(t-2\pi))\right].
\]

This implies that,
\[
y(t) = \frac{1}{2} u(t - \pi) \sin(2(t - \pi)) - \frac{1}{2} u(t - 2\pi) \sin(2(t - 2\pi)),
\]
We conclude: \( y(t) = \frac{1}{2} [u(t - \pi) - u(t - 2\pi)] \sin(2t). \)