The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with discontinuous source functions.

Notation:
If \( \mathcal{L}[f(t)] = F(s) \), then we denote \( \mathcal{L}^{-1}[F(s)] = f(t) \).

Remark: One can show that for a particular type of functions \( f \), that includes all functions we work with in this Section, the notation above is well-defined.

Example
From the Laplace Transform table we know that \( \mathcal{L}[e^{at}] = \frac{1}{s-a} \).
Then also holds that \( \mathcal{L}^{-1} \left[ \frac{1}{s-a} \right] = e^{at} \). 
\( \triangleright \)
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The definition of a step function.

**Definition**

A function \( u \) is called a *step function* at \( t = 0 \) iff holds

\[
    u(t) = \begin{cases} 
        0 & \text{for } t < 0, \\
        1 & \text{for } t \geq 0.
    \end{cases}
\]

**Example**

Graph the step function values \( u(t) \) above, and the translations \( u(t - c) \) and \( u(t + c) \) with \( c > 0 \).

**Solution:**

![Graphs of u(t), u(t-c), and u(t+c)]
The definition of a step function.

Remark: Given any function values \( f(t) \) and \( c > 0 \), then \( f(t - c) \) is a right translation of \( f \) and \( f(t + c) \) is a left translation of \( f \).

Example

\[
\begin{align*}
  f(t) &= e^{at} \\
  f(t) &= u(t)e^{at} \\
  f(t) &= e^{a(t-c)} \\
  f(t) &= u(t-c)e^{a(t-c)}
\end{align*}
\]

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
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- **Piecewise discontinuous functions.**
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Example
Graph of the function $b(t) = u(t - a) - u(t - b)$, with $0 < a < b$.

Solution: The bump function $b$ can be graphed as follows:

Notation: The function values $u(t - c)$ are denoted in the textbook as $u_c(t)$. 

Example
Graph of the function $f(t) = e^{at} [u(t - 1) - u(t - 2)]$.

Solution:
The Laplace Transform of discontinuous functions.

Theorem
Given any real number \( c \geq 0 \), the following equation holds,

\[
\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}, \quad s > 0.
\]

Proof:

\[
\mathcal{L}[u(t - c)] = \int_{0}^{\infty} e^{-st} u(t - c) \, dt = \int_{c}^{\infty} e^{-st} \, dt,
\]

\[
\mathcal{L}[u(t - c)] = \lim_{N \to \infty} \frac{1}{s} (e^{-Ns} - e^{-cs}) = \frac{e^{-cs}}{s}, \quad s > 0.
\]

We conclude that \( \mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}. \)
The Laplace Transform of discontinuous functions.

Example
Compute $\mathcal{L}[3u(t - 2)]$.

Solution: $\mathcal{L}[3u(t - 2)] = 3 \mathcal{L}[u(t - 2)] = 3 \frac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t - 2)] = \frac{3e^{-2s}}{s}$.

Example
Compute $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] = u(t - 3)$.

We conclude: $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] = u(t - 3)$.

The Laplace Transform of step functions (Sect. 6.3).

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Properties of the Laplace Transform.

**Theorem (Translations)**

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and $c \geq 0$, then holds

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs} F(s), \quad s > a.$$  

Furthermore,

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c.$$  

**Remark:**

- $\mathcal{L}\text{ [translation (uf)]} = \text{(exp)} (\mathcal{L}[f]).$
- $\mathcal{L}\text{ [(exp) (f)]} = \text{translation} (\mathcal{L}[f]).$

**Equivalent notation:**

- $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs} \mathcal{L}[f(t)],$
- $\mathcal{L}\{e^{ct}f(t)\} = \mathcal{L}[f](s-c).$

Properties of the Laplace Transform.

**Example**

Compute $\mathcal{L}\{u(t-2) \sin(a(t-2))\}$.

**Solution:**

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad \mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs} \mathcal{L}[f(t)].$$  

$$\mathcal{L}\{u(t-2) \sin(a(t-2))\} = e^{-2s} \mathcal{L}[\sin(at)] = e^{-2s} \frac{a}{s^2 + a^2}.$$  

We conclude: $\mathcal{L}\{u(t-2) \sin(a(t-2))\} = e^{-2s} \frac{a}{s^2 + a^2}$.  

**Example**

Compute $\mathcal{L}\{e^{3t} \sin(at)\}$.

**Solution:**

Recall: $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

We conclude: $\mathcal{L}\{e^{3t} \sin(at)\} = \frac{a}{(s-3)^2 + a^2}$, with $s > 3$.  

Properties of the Laplace Transform.

Example

Find the Laplace transform of \( f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases} \)

Solution: Using step function notation,

\[ f(t) = u(t - 1)(t^2 - 2t + 2). \]

Completing the square we obtain,

\[ t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t - 1)^2 + 1. \]

This is a parabola \( t^2 \) translated to the right by 1 and up by one. This is a discontinuous function.

Properties of the Laplace Transform.

Example

Find the Laplace transform of \( f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases} \)

Solution: Recall: \( f(t) = u(t - 1) [(t - 1)^2 + 1] \).

This is equivalent to

\[ f(t) = u(t - 1)(t - 1)^2 + u(t - 1). \]

Since \( \mathcal{L}[t^2] = 2/s^3 \), and \( \mathcal{L}[u(t - c)g(t - c)] = e^{-cs} \mathcal{L}[g(t)] \), then

\[ \mathcal{L}[f(t)] = \mathcal{L}[u(t - 1)(t - 1)^2] + \mathcal{L}[u(t - 1)] = e^{-s} \frac{2}{s^3} + e^{-s} \frac{1}{s}. \]

We conclude: \( \mathcal{L}[f(t)] = \frac{e^{-s}}{s^3} (2 + s^2). \) \( \triangleq \)
Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

\[ \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c), \]
\[ \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t). \]

Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{3}{s^2 + 9} \right] \).

Recall: \( \mathcal{L}^{-1}\left[ \frac{a}{s^2 + a^2} \right] = \sin(at) \). Then, we conclude that

\[ \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} u(t - 4) \sin(3(t - 4)). \]

Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{s}{s^2 + a^2} \right] = \cos(at) \), \( \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t) \).

We conclude: \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] = e^{2t} \cos(3t) \).

Example

Find \( \mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] \).

Solution: Recall: \( \mathcal{L}^{-1}\left[ \frac{a}{s^2 - a^2} \right] = \sinh(at) \)
and \( \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c) \).
Properties of the Laplace Transform.

Example
Find $L^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right]$.

Solution: Recall:

\[
L^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at), \quad L^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c).
\]

\[
L^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = L^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = L^{-1}\left[\frac{2}{s^2 - 4}\right].
\]

We conclude: $L^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = u(t - 3) \sinh(2(t - 3))$. ◯

Properties of the Laplace Transform.

Example
Find $L^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right]$.

Solution: Find the roots of the denominator:

\[
s_{\pm} = \frac{1}{2} \left[-1 \pm \sqrt{1 + 8}\right] = \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}
\]

Therefore, $s^2 + s - 2 = (s - 1)(s + 2)$.

Use partial fractions to simplify the rational function:

\[
\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},
\]

\[
\frac{1}{s^2 + s - 2} = a(s + 2) + b(s - 1) = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}.
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:
\[
\frac{1}{s^2 + s - 2} = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}
\]
\[a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.
\]

\[
\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} \mathcal{L}^{-1}[e^{-2s}] \frac{1}{s - 1} - \frac{1}{3} \mathcal{L}^{-1}[e^{-2s}] \frac{1}{s + 2}.
\]

Recall: \( \mathcal{L}^{-1}\left[ \frac{1}{s - a} \right] = e^{at} \), \( \mathcal{L}^{-1}[e^{-cs}F(s)] = u(t - c)f(t - c) \),
\[
\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} u(t - 2)e^{(t-2)} - \frac{1}{3} u(t - 2)e^{-2(t-2)}.
\]

Hence:
\[
\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} u(t - 2)\left[ e^{(t-2)} - e^{-2(t-2)} \right]. \quad \triangleq
\]