The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
The definition of the Laplace Transform.

Definition
The function $F : D_F \rightarrow \mathbb{R}$ is the Laplace transform of a function $f : [0, \infty) \rightarrow \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt,$$

where $D_F \subset \mathbb{R}$ is the set where the integral converges.

Remark: The domain $D_F$ of $F$ depends on the function $f$.

Notation: We often denote: $F(s) = \mathcal{L}[f(t)]$.

- This notation $\mathcal{L}[]$ emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- Functions are denoted as $t \mapsto f(t)$.
- The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

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Review: Improper integrals.

Recall: Improper integral are defined as a limit.

\[ \int_{t_0}^{\infty} g(t) \, dt = \lim_{N \to \infty} \int_{t_0}^{N} g(t) \, dt. \]

- The integral converges iff the limit exists.
- The integral diverges iff the limit does not exist.

Example

Compute the improper integral \( \int_{0}^{\infty} e^{-at} \, dt \), with \( a > 0 \).

Solution:

\[ \int_{0}^{\infty} e^{-at} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt = \lim_{N \to \infty} -\frac{1}{a} (e^{-aN} - 1). \]

Since \( \lim_{N \to \infty} e^{-aN} = 0 \) for \( a > 0 \), we conclude

\[ \int_{0}^{\infty} e^{-at} \, dt = \frac{1}{a}. \]

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Examples of Laplace Transforms.

**Example**
Compute $\mathcal{L}[1]$.

**Solution:** We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, 1 \, dt = \int_0^\infty e^{-st} \, dt$$

But $\int_0^\infty e^{-at} \, dt = \frac{1}{a}$ for $a > 0$, and diverges for $a \leq 0$.

Therefore $\mathcal{L}[1] = \frac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leq 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \rightarrow \mathbb{R}$ given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty).$$

\[\triangleright\]

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Examples of Laplace Transforms.

**Example**
Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

**Solution:** Following the definition of Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt.$$ 

We have seen that the improper integral is given by

$$\int_0^\infty e^{-(s-a)} \, dt = \frac{1}{(s-a)} \quad \text{for} \quad (s-a) > 0.$$ 

We conclude that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \quad F(s) = \frac{1}{(s-a)}, \quad s > a.$$ 

\[\triangleright\]
Examples of Laplace Transforms.

Example
Compute \( L[\sin(at)] \), where \( a \in \mathbb{R} \).

Solution: In this case we need to compute
\[
L[\sin(at)] = \lim_{N \to \infty} \int_0^N e^{-st} \sin(at) \, dt.
\]
Integrating by parts twice it is not difficult to obtain:
\[
\int_0^N e^{-st} \sin(at) \, dt =
- \frac{1}{s} \left[ e^{-st} \sin(at) \right]_0^N - \frac{a}{s^2} \left[ e^{-st} \cos(at) \right]_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) \, dt.
\]
This identity implies
\[
\left( 1 + \frac{a^2}{s^2} \right) \int_0^N e^{-st} \sin(at) \, dt = - \frac{1}{s} \left[ e^{-st} \sin(at) \right]_0^N - \frac{a}{s^2} \left[ e^{-st} \cos(at) \right]_0^N.
\]
Hence, it is not difficult to see that
\[
\left( \frac{s^2 + a^2}{s^2} \right) \int_0^\infty e^{-st} \sin(at) \, dt = \frac{a}{s^2}, \quad s > 0,
\]
which is equivalent to
\[
L[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0. \quad \triangle
\]
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### A table of Laplace Transforms.

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t) = 1$</td>
<td>$F(s) = \frac{1}{s}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = e^{at}$</td>
<td>$F(s) = \frac{1}{s-a}$</td>
<td>$s &gt; \max{a, 0}$,</td>
</tr>
<tr>
<td>$f(t) = t^n$</td>
<td>$F(s) = \frac{n!}{s^{(n+1)}}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = \sin(at)$</td>
<td>$F(s) = \frac{a}{s^2 + a^2}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = \cos(at)$</td>
<td>$F(s) = \frac{s}{s^2 + a^2}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = \sinh(at)$</td>
<td>$F(s) = \frac{a}{s^2 - a^2}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = \cosh(at)$</td>
<td>$F(s) = \frac{s}{s^2 - a^2}$</td>
<td>$s &gt; 0$,</td>
</tr>
<tr>
<td>$f(t) = t^n e^{at}$</td>
<td>$F(s) = \frac{n!}{(s-a)^{(n+1)}}$</td>
<td>$s &gt; \max{a, 0}$,</td>
</tr>
<tr>
<td>$f(t) = e^{at} \sin(bt)$</td>
<td>$F(s) = \frac{b}{(s-a)^2 + b^2}$</td>
<td>$s &gt; \max{a, 0}$.</td>
</tr>
</tbody>
</table>
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Properties of the Laplace Transform.

Theorem (Sufficient conditions)

*If the function \( f : [0, \infty) \to \mathbb{R} \) is piecewise continuous and there exist positive constants \( k \) and \( a \) such that

\[
|f(t)| \leq k e^{at},
\]

then the Laplace Transform of \( f \) exists for all \( s > a \).*

Theorem (Linear combination)

*If the \( \mathcal{L}[f] \) and \( \mathcal{L}[g] \) are well-defined and \( a, b \) are constants, then

\[
\mathcal{L}[af + bg] = a \mathcal{L}[f] + b \mathcal{L}[g].
\]

Proof: Integration is a linear operation:

\[
\int [af(t) + bg(t)] \, dt = a \int f(t) \, dt + b \int g(t) \, dt.
\]
Properties of the Laplace Transform.

Theorem (Derivatives)

If the $L[f]$ and $L[f']$ are well-defined, then holds,

$$L[f'] = sL[f] - f(0). \quad (1)$$

Furthermore, if $L[f'']$ is well-defined, then it also holds

$$L[f''] = s^2 L[f] - s f(0) - f'(0). \quad (2)$$

Proof of Eq (1): Recall the definition of the Laplace Transform,

$$L[f'] = \int_0^\infty e^{-st} f'(t) \, dt = \lim_{n \to \infty} \int_0^n e^{-st} f'(t) \, dt$$

Integrating by parts,

$$\lim_{n \to \infty} \int_0^n e^{-st} f'(t) \, dt = \lim_{n \to \infty} \left[ \left( e^{-st} f(t) \right) \bigg|_0^n - \int_0^n (-s) e^{-st} f(t) \, dt \right]$$

$$L[f'] = \lim_{n \to \infty} \left[ e^{-sn} f(n) - f(0) \right] + s \int_0^\infty e^{-st} f(t) \, dt = -f(0) + s L[f],$$

where we used that $\lim_{n \to \infty} e^{-sn} f(n) = 0$ for $s$ big enough, and we also used that $L[f]$ is well-defined.

We then conclude that $L[f'] = sL[f] - f(0).$
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Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

Idea of the method:

\[ \mathcal{L} \left[ \begin{array}{l}
\text{Differential Eq. for } y(t). \\
\text{Algebraic Eq. for } \mathcal{L}[y(t)].
\end{array} \right] \quad (1) \quad \rightarrow \quad \begin{array}{l}
\text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)]. \\
\text{Transform back to obtain } y(t). \\
\text{(Using the table.)}
\end{array} \quad (2) \quad \rightarrow \quad (3) \]
Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$  

Solution: We know the solution: $y(t) = 3e^{-2t}$.

(1): Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] \Rightarrow \mathcal{L}[y' + 2y] = 0.$$ 

Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity:

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 0.$$ 

Also recall the property: $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$, that is,

$$\left[ s\mathcal{L}[y] - y(0) \right] + 2\mathcal{L}[y] = 0 \Rightarrow (s + 2)\mathcal{L}[y] = y(0).$$

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.$$ 

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \Rightarrow \frac{3}{s + 2} = 3\mathcal{L}[e^{-2t}] \Rightarrow \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}].$$ 

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \Rightarrow y(t) = 3e^{-2t}$. \hspace{1cm} \triangleleft