Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- Example: Method to find solutions.

Recall:
The point \( x_0 \in \mathbb{R} \) is a singular point of the equation
\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]
iff holds that \( P(x_0) = 0 \).

Equations with regular-singular points.

Definition
A singular point \( x_0 \in \mathbb{R} \) of the equation
\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]
is called a regular-singular point iff the following limits are finite,
\[
\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)}, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)},
\]
and both functions
\[
\frac{(x - x_0) Q(x)}{P(x)}, \quad \frac{(x - x_0)^2 R(x)}{P(x)},
\]
admit convergent Taylor series expansions around \( x_0 \).
Equations with regular-singular points.

Remark:
- If \( x_0 \) is a regular-singular point of

\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]

and \( P(x) \simeq (x - x_0)^n \) near \( x_0 \), then near \( x_0 \) holds

\[
Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.
\]

- The main example is an Euler equation, case \( n = 2 \),

\[
(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.
\]

Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

\[
(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0,
\]

where \( p_0, q_0, x_0 \), are real constants. This is an equation

\[
Py'' + Qy' + Ry = 0
\]

with

\[
P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0.
\]

Therefore, we obtain,

\[
\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)} = p_0, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0.
\]

We conclude that \( x_0 \) is a regular-singular point. 

\( \diamond \)
Equations with regular-singular points.

**Remark:** Every equation \( Py'' + Qy' + Ry = 0 \) with a regular-singular point at \( x_0 \) is close to an Euler equation.

**Proof:**
For \( x \neq x_0 \) divide the equation by \( P(x) \),

\[
y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0,
\]
and multiply it by \((x - x_0)^2\),

\[
(x - x_0)^2 y'' + (x - x_0) \left[ \frac{(x - x_0)Q(x)}{P(x)} \right] y' + \left[ \frac{(x - x_0)^2 R(x)}{P(x)} \right] y = 0.
\]

The factors between \([ \ ]\) approach constants, say \( p_0 \), \( q_0 \), as \( x \to x_0 \),

\[
(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0.
\]

\(\square\)

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
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Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Find the singular points of this equation,

\[0 = P(x) = (1 - x^2) = (1 - x)(1 + x) \Rightarrow \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases}\]

Case \(x_0 = 1\): We then have

\[
\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \frac{2x}{1 + x},
\]

\[
\frac{(x - 1)^2 R(x)}{P(x)} = \frac{(x - 1)^2[\alpha(\alpha + 1)]}{(1 - x)(1 + x)} = \frac{(x - 1)[\alpha(\alpha + 1)]}{1 + x};
\]

both functions above have Taylor series around \(x_0 = 1\).

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Recall:

\[
\frac{(x - 1) Q(x)}{P(x)} = \frac{2x}{1 + x}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{(x - 1)[\alpha(\alpha + 1)]}{1 + x}.
\]

Furthermore, the following limits are finite,

\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 1, \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]

We conclude that \(x_0 = 1\) is a regular-singular point.
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]
where \(\alpha\) is a real constant.

Solution:
Case \(x_1 = -1\):
\[
\frac{(x + 1) Q(x)}{P(x)} = \frac{(x + 1)(-2x)}{(1 - x)(1 + x)} = -\frac{2x}{1 - x},
\]
\[
\frac{(x + 1)^2 R(x)}{P(x)} = \frac{(x + 1)^2 [\alpha(\alpha + 1)]}{(1 - x)(1 + x)} = \frac{(x + 1)[\alpha(\alpha + 1)]}{1 - x}.
\]
Both functions above have Taylor series \(x_1 = -1\).

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]
where \(\alpha\) is a real constant.

Solution: Recall:
\[
\frac{(x + 1) Q(x)}{P(x)} = -\frac{2x}{1 - x}, \quad \frac{(x + 1)^2 R(x)}{P(x)} = \frac{(x + 1)[\alpha(\alpha + 1)]}{1 - x}.
\]
Furthermore, the following limits are finite,
\[
\lim_{x \to -1} \frac{(x + 1) Q(x)}{P(x)} = 1, \quad \lim_{x \to -1} \frac{(x + 1)^2 R(x)}{P(x)} = 0.
\]
Therefore, the point \(x_1 = -1\) is a regular-singular point. \(\triangle\)
Example
Find the regular-singular points of the differential equation
\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution:
Recall:
\[
\frac{(x - 1) Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.
\]

Furthermore, the following limits are finite,
\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 0; \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]

Therefore, the point \(x_1 = -1\) is a regular-singular point. \(\triangle\)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x\left[-x \ln(|x|)\right]}{x} = \lim_{x \to 0} -\frac{\ln(|x|)}{x}. \]

Use L'Hôpital's rule: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x^2} = \lim_{x \to 0} x = 0. \)

The other limit is: \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2(3x)}{x} = \lim_{x \to 0} 3x^2 = 0. \)

We conclude that \( x_0 = 0 \) is not a regular-singular point.

\[ \triangle \]

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3x y = 0. \]

Solution: Recall: \( \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \) and \( \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0. \)

However, at the point \( x_0 = 0 \) the function \( xQ/P \) does not have a power series expansion around zero, since

\[ \frac{xQ(x)}{P(x)} = -x \ln(|x|), \]

and the log function does not have a Taylor series at \( x_0 = 0. \)

We conclude that \( x_0 = 0 \) is not a regular-singular point.

\[ \triangle \]
Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
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- **Method to find solutions.**
- Example: Method to find solutions.

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**Method to find solutions.**

Recall: If $x_0$ is a regular-singular point of

$$P(x) y'' + Q(x) y' + R(x) y = 0,$$

with limits $\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0$ and $\lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0$,

then the coefficients of the differential equation above near $x_0$ are close to the coefficients of the Euler equation

$$ (x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0. $$

**Idea:** If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

**Recall:** One solution of an Euler equation is $y(x) = (x - x_0)^r$. 
Method to find solutions.

Summary: Solutions for equations with regular-singular points:

1. Look for a solution $y$ of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

2. Introduce this power series expansion into the differential equation and find both a the exponent $r$ and a recurrence relation for the coefficients $a_n$;

3. First find the solutions for the constant $r$. Then, introduce this result for $r$ into the recurrence relation for the coefficients $a_n$. Only then, solve this latter recurrence relation for the coefficients $a_n$.

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
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Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3) y' + (x+3) y = 0.$$ 

Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}$.

The first and second derivatives are given by

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{(n+r-1)}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r-2)}.$$ 

In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n x^{(n-1)} = \sum_{n=1}^{\infty} n a_n x^{(n-1)},$$

but for $r \neq 0$ this relation is not true.

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3) y' + (x+3) y = 0.$$ 

Solution: We now compute the term $(x+3)y$,

$$(x+3)y = (x+3) \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$(x+3)y = \sum_{n=0}^{\infty} a_n x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)}$$

$$(x+3)y = \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)}.$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: We now compute the term $-x(x + 3) y'$,
\[
-x(x + 3) y' = -(x^2 + 3x) \sum_{n=0}^{\infty} (n + r) a_n x^{(n+r-1)}
\]
\[
-x(x + 3) y' = - \sum_{n=0}^{\infty} (n + r) a_n x^{(n+r+1)} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{(n+r)},
\]
\[
-x(x + 3) y' = - \sum_{n=1}^{\infty} (n + r - 1) a_{n-1} x^{(n+r)} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{(n+r)}.
\]

The guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: The differential equation is given by
\[
\sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r)} - \sum_{n=1}^{\infty} (n + r - 1) a_{(n-1)} x^{(n+r)} \\
- \sum_{n=0}^{\infty} 3(n + r) a_n x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)} = 0.
\]

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geq 1$, that is,
\[
0 = \left[ r(r - 1) - 3r + 3 \right] a_0 x' + \sum_{n=1}^{\infty} \left[ (n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n \right] x^{(n+r)}
\]

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: Therefore, $[ r(r - 1) - 3r + 3 ] = 0$ and
\[
\left[ (n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n \right] = 0.
\]
The last expression can be rewritten as follows,
\[
\left[ ((n + r)(n + r - 1) - 3(n + r) + 3) a_n - (n + r - 1 - 1) a_{(n-1)} \right] = 0,
\]
\[
\left[ ((n + r)(n + r - 1) - 3(n + r - 1)) a_n - (n + r - 2) a_{(n-1)} \right] = 0.
\]
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3) y' + (x+3) y = 0.$$ 

Solution: Hence, the recurrence relation is given by the equations

$$r(r - 1) - 3r + 3 = 0,$$

$$(n + r - 1)(n + r - 3)a_n - (n + r - 2)a_{n-1} = 0.$$ 

First: solve the first equation for $r_{\pm}$.

Second: Introduce the first solution $r_+$ into the second equation above and solve for the $a_n$; the result is a solution $y_+$ of the original differential equation;

Third: Introduce the second solution $r_-$ into into the second equation above and solve for the $a_n$; the result is a solution $y_-$ of the original differential equation;

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3) y' + (x+3) y = 0.$$ 

Solution: We first solve $r(r - 1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ 4 \pm \sqrt{16 - 12} \right] \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introduce $r_+ = 3$ into the equation for $a_n$:

$$(n + 2)n a_n - (n + 1)a_{n-1} = 0.$$ 

One can check that the solution $y_+$ is

$$y_+ = a_0 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right].$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: Introduce $r_- = 1$ into the equation for $a_n$:
\[ n(n - 2)a_n - (n - 1)a_{n-1} = 0. \]

One can also check that the solution $y_-$ is
\[ y_- = a_2 x \left[ x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \cdots \right]. \]

Notice:
\[ y_- = a_2 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right] \Rightarrow y_- = \frac{a_2}{a_1} y_+. \]

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: The solutions $y_+$ and $y_-$ are not linearly independent.

This Example shows that the method does not provide all solutions of a differential equation near a regular-singular point, it only provides at least one solution near a regular-singular point.

Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial $r_+, r_-$ differ by an integer, then the second solution $y_-$, the solution corresponding to the smaller root, is not given by the method above.
This solution involves logarithmic terms.
We do not study this type of solutions in these notes.