Overview: Equations with singular points.

Recall: The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

iff holds $P(x_0) = 0$.

Remarks:

- We are interested in finding solutions to the equation above arbitrary close to a singular point $x_0$.
- The order of the differential equation changes in a neighborhood of a singular point.
- In the limit $x \to x_0$ the following could happen:
  1. The two linearly independent solutions remain bounded.
  2. Only one solution remains bounded.
  3. None solution remains bounded.
Overview: Equations with singular points.

Remarks:
- If the singular point of a differential equation is not so singular, in a sense to be made precise later on, then it is known how to find solutions to such equation.
- Singular points where the singular behavior of the solution is somehow mild, in a sense to be made precise later, will be called regular-singular points.
- The main example of a equation with a regular-singular point is the Euler differential equation.

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- **We study the Euler Equation:**
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \(x_0\).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
The Euler equation

Definition
Given real constants \( p_0, q_0 \), the Euler differential equation for the unknown \( y \) with singular point at \( x_0 \in \mathbb{R} \) is given by

\[
(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.
\]

Remarks:
- The Euler equation has variable coefficients.
- Functions \( y(x) = e^{rx} \) are not solutions of the Euler equation.
- The point \( x_0 \in \mathbb{R} \) is a singular point of the equation.
- The particular case \( x_0 = 0 \) is given by
  \[
x^2 y'' + p_0 x y' + q_0 y = 0.
  \]

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[
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  \]
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Solutions to the Euler equation near $x_0$.

Summary of the main idea:

- The main idea to find solution to the constant coefficients equation $y'' + a_1 y' + a_0 y = 0$ was to look for functions of the form $y(x) = e^{rx}$. The exponential cancels out from the equation and we obtain an equation only for $r$ without $x$,

$$ (r^2 + a_1 r + a_0) e^{rx} = 0 \iff (r^2 + a_1 r + a_0) = 0. \quad (1) $$

- In the case of the Euler equation $x^2 y'' + p_0 x y' + q_0 y = 0$ the exponential functions $e^{rx}$ do not have the property given in Eq. (1), since

$$ (x^2 r^2 + p_0 x r + q_0) e^{rx} = 0 \iff x^2 r^2 + p_0 x r + q_0 = 0, $$

but the later equation still involves the variable $x$.

Introduce $y = x^r$ into Euler’s equation $x^2 y'' + p_0 x y' + q_0 y = 0$, for $x \neq 0$ we obtain

$$ [r(r-1) + p_0 r + q_0] x^r = 0 \iff r(r-1) + p_0 r + q_0 = 0. $$

The last equation involves only $r$, not $x$.

This equation is called the indicial equation, and is also called the Euler characteristic equation.
Solutions to the Euler equation near $x_0$.

Theorem (Euler equation)
Given $p_0$, $q_0$, $x_0 \in \mathbb{R}$, consider the Euler equation
\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.\] (2)
Let $r_+$, $r_-$ be solutions of $r(r - 1) + p_0 r + q_0 = 0$.

(a) If $r_+ \neq r_-$, then a real-valued general solution of Eq. (2) is
\[y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R}.\]

(b) If $r_+ = r_-$, then a real-valued general solution of Eq. (2) is
\[y(x) = \left[c_0 + c_1 \ln(|x - x_0|)\right] |x - x_0|^{r_+}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R}.\]

Given $x_0 \neq x_1$, $y_0$, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP
\[(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.\]
Different real roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' + 4x y' + 2 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ x y'(x) = r x^r, \quad x^2 y''(x) = r(r - 1) x^r. \]
Introduce \( y(x) = x^r \) into Euler equation,
\[ \left[ r(r - 1) + 4r + 2 \right] x^r = 0 \iff r(r - 1) + 4r + 2 = 0. \]
The solutions of \( r^2 + 3r + 2 = 0 \) are given by
\[ r_{\pm} = \frac{1}{2} \left[ -3 \pm \sqrt{9 - 8} \right] \Rightarrow r_+ = -1 \quad r_- = -2. \]
The general solution is \( y(x) = c_1 |x|^{-1} + c_2 |x|^{-2}. \)

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[ (x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0. \]
- Solutions to the Euler equation near \( x_0 \).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Repeated roots.

Example
Find the general solution of \( x^2 y'' - 3x y' + 4y = 0 \).

Solution: We look for solutions of the form \( y(x) = x^r \),
\[
x y'(x) = r x^r, \quad x^2 y''(x) = r(r - 1) x^r.
\]
Introduce \( y(x) = x^r \) into Euler equation,
\[
[r(r - 1) - 3r + 4] x^r = 0 \iff r(r - 1) - 3r + 4 = 0.
\]
The solutions of \( r^2 - 4r + 4 = 0 \) are given by
\[
r_\pm = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \Rightarrow r_+ = r_- = 2.
\]
Two linearly independent solutions are
\[
y_1(x) = x^2, \quad y_2 = x^2 \ln(|x|).
\]
The general solution is \( y(x) = c_1 x^2 + c_2 x^2 \ln(|x|) \). \( \triangleright \)

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \( x_0 \).
- **The roots of the indicial polynomial.**
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  - **Different complex roots.**
Different complex roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ x y'(x) = rx^r, \quad x^2 y''(x) = r(r - 1)x^r. \]

Introduce \( y(x) = x^r \) into Euler equation
\[ [r(r - 1) - 3r + 13] x^r = 0 \iff r(r - 1) - 3r + 13 = 0. \]

The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are
\[ r_\pm = \frac{1}{2} [4 \pm \sqrt{16 - 52}] \Rightarrow r_\pm = \frac{1}{2} [4 \pm \sqrt{-36}] \Rightarrow \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases} \]

The general solution is \( y(x) = c_1 |x|^{2+3i} + c_2 |x|^{2-3i} \). ▷

Different complex roots.

Theorem (Real-valued fundamental solutions)
If \( p_0, q_0 \in \mathbb{R} \) satisfy that \( [(p_0 - 1)^2 - 4q_0] < 0 \), then the indicial polynomial \( p(r) = r(r - 1) + p_0r + q_0 \) of the Euler equation
\[ x^2 y'' + p_0xy' + q_0y = 0 \quad (3) \]
has complex roots \( r_+ = \alpha + i\beta \) and \( r_- = \alpha - i\beta \), where
\[ \alpha = -\frac{(p_0 - 1)}{2}, \quad \beta = \frac{1}{2} \sqrt{4q_0 - (p_0 - 1)^2}. \]

Furthermore, a fundamental set of solution to Eq. (3) is
\[ \tilde{y}_1(x) = |x|^{(\alpha+i\beta)}, \quad \tilde{y}_2(x) = |x|^{(\alpha-i\beta)}, \]
while another fundamental set of solutions to Eq. (3) is
\[ y_1(x) = |x|^\alpha \cos(\beta \ln |x|), \quad y_2(x) = |x|^\alpha \sin(\beta \ln |x|). \]
Different complex roots.

**Proof:** Given \( \tilde{y}_1 = |x|^{(\alpha + i\beta)} \) and \( \tilde{y}_2 = |x|^{(\alpha - i\beta)} \), introduce

\[
y_1 = \frac{1}{2} (\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i} (\tilde{y}_1 - \tilde{y}_2).
\]

Use another Euler equation to rewrite \( \tilde{y}_1 \) and \( \tilde{y}_2 \),

\[
\tilde{y}_1 = |x|^{(\alpha + i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^\alpha e^{i\beta \ln(|x|)} = |x|^\alpha e^{i\beta \ln(|x|)}.
\]

\[
\tilde{y}_1 = |x|^\alpha \left[ \cos(\beta \ln |x|) + 1 \sin(\beta \ln |x|) \right],
\]

\[
\tilde{y}_2 = |x|^\alpha \left[ \cos(\beta \ln |x|) - 1 \sin(\beta \ln |x|) \right].
\]

We conclude that

\[
y_1(x) = |x|^{\alpha} \cos(\beta \ln |x|), \quad y_2(x) = |x|^{\alpha} \sin(\beta \ln |x|).
\]

\[\Box\]

Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

\[
x^2 y'' - 3x y' + 13y = 0.
\]

**Solution:** The indicial equation is \( r(r - 1) - 3r + 13 = 0 \).

The solutions of the indicial equations are

\[
r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i.
\]

A complex-valued general solution is

\[
y(x) = \tilde{c}_1 |x|^{(2+3i)} + \tilde{c}_2 |x|^{(2-3i)} \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.
\]

A real-valued general solution is

\[
y(x) = c_1 |x|^2 \cos(3 \ln |x|) + c_2 |x|^2 \sin(3 \ln |x|), \quad c_1, c_2 \in \mathbb{R}.
\]