Non-homogeneous equations (Sect. 3.6).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

\[ y'' + p(t) y' + q(t) y = f(t). \]

- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.
- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.
Method of variation of parameters.

**Theorem (Variation of parameters)**

Let $p, q, f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, let $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t) y' + q(t) y = 0,$$

and let $W_{y_1y_2}$ be the Wronskian of $y_1$ and $y_2$. If the functions $u_1$ and $u_2$ are defined by

$$u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} dt,$$

then the function $y_p = u_1y_1 + u_2y_2$ is a particular solution to the non-homogeneous equation

$$y'' + p(t) y' + q(t) y = f(t).$$

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t) y' + q(t) y = f(t)$.
- Method of variation of parameters.
- **Using the method in an example.**
- The proof of the variation of parameter method.
- Using the method in another example.
Using the method in an example.

Example
Find the general solution of the inhomogeneous equation
\[ y'' - 5y' + 6y = 2e^t. \]

Solution:
First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is
\[ r^2 - 5r + 6 = 0 \Rightarrow r = \frac{1}{2}(5 \pm \sqrt{25 - 24}) \Rightarrow \begin{cases} r_1 = 3, \\ r_2 = 2. \end{cases} \]

Hence, \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{2t} \). Compute their Wronskian,
\[ W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \Rightarrow W_{y_1y_2}(t) = -e^{5t}. \]

Second: We compute the functions \( u_1 \) and \( u_2 \). By definition,
\[ u_1' = -\frac{y_2f}{W_{y_1y_2}}, \quad u_2' = \frac{y_1f}{W_{y_1y_2}}. \]

\[ u_1' = -e^{2t}(2e^t)(-e^{-5t}) \Rightarrow u_1' = 2e^{-2t} \Rightarrow u_1 = -e^{-2t}, \]
\[ u_2' = e^{3t}(2e^t)(-e^{-5t}) \Rightarrow u_2' = -2e^{-t} \Rightarrow u_2 = 2e^{-t}. \]

Third: The particular solution is
\[ y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \Rightarrow y_p = e^t. \]

The general solution is \( y(t) = c_1 e^{3t} + c_2 e^{2t} + e^t, \ c_1, c_2 \in \mathbb{R}. \)  \( \triangleq \)
Non-homogeneous equations (Sect. 3.6).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- **The proof of the variation of parameter method.**
- Using the method in another example.

The proof of the variation of parameter method.

**Proof:** Denote \( L(y) = y'' + p(t) y' + q(t) y \).

We need to find \( y_p \) solution of \( L(y_p) = f \).

We know \( y_1 \) and \( y_2 \) solutions of \( L(y_1) = 0 \) and \( L(y_2) = 0 \).

**Idea:** The reduction of order method: Find \( y_2 \) proposing \( y_2 = uy_1 \).

**First idea:** Propose that \( y_p \) is given by \( y_p = u_1 y_1 + u_2 y_2 \).

We hope that the equation for \( u_1 \) and \( u_2 \) will be simpler than the original equation for \( y_p \), since \( y_1 \) and \( y_2 \) are solutions to the homogeneous equation. Compute:

\[
\begin{align*}
y'_p &= u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2, \\
y''_p &= u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2.
\end{align*}
\]
The proof of the variation of parameter method.

Proof: Then \( L(y_p) = f \) is given by

\[
\begin{align*}
    u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2'' \\
p(t)[u_1'y_1 + u_1'y_1' + u_2'y_2] + q(t)[u_1y_1 + u_2y_2] = f(t).
\end{align*}
\]

Recall: \( y_1'' + py_1' + qy_1 = 0 \) and \( y_2'' + py_2' + qy_2 = 0 \). Hence,

\[
\begin{align*}
    u_1''y_1 + u_2''y_2 + 2(u_1'y_1' + u_2'y_2') + p(u_1'y_1 + u_2'y_2) \\
    + u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) = f
\end{align*}
\]

Second idea: Look for \( u_1 \) and \( u_2 \) that satisfy the extra equation

\( u_1'y_1 + u_2'y_2 = 0 \).

Summary: If \( u_1 \) and \( u_2 \) satisfy \( u_1'y_1 + u_2'y_2 = 0 \) and \( u_1'y_1' + u_2'y_2' = f \), then \( y_p = u_1y_1 + u_2y_2 \) satisfies \( L(y_p) = f \).
The proof of the variation of parameter method.

Proof: Summary: If $u_1$ and $u_2$ satisfy $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u_2' = -\frac{y_1}{y_2} u_1' \Rightarrow u_1'y_1' - \frac{y_1'y_2}{y_2} u_1' = f \Rightarrow u_1' \left( \frac{y_1'y_2 - y_1'y_2}{y_2} \right) = f.$$ 

Since $W_{y_1y_2} = y_1'y_2 - y_1'y_2$,

$$u_1' = -\frac{y_2f}{W_{y_1y_2}} \Rightarrow u_2' = \frac{y_1f}{W_{y_1y_2}}.$$

Integrating in the variable $t$ we obtain

$$u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt,$$

This establishes the Theorem.

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t)y' + q(t)y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.
Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by \( t^2 \),
\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}. \]

We know that \( y_1 = t^2 \) and \( y_2 = 1/t \). Their Wronskian is
\[ W_{y_1y_2}(t) = (t^2)(-\frac{1}{t^2}) - (2t)(\frac{1}{t}) \quad \Rightarrow \quad W_{y_1y_2}(t) = -3. \]

Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: \( y_1 = t^2, \ y_2 = 1/t, \ f(t) = 3 - \frac{1}{t^2}, \ W_{y_1y_2}(t) = -3. \)

We now compute \( y_1 \) and \( u_2 \),
\[ u_1' = -\frac{1}{t} \left(3 - \frac{1}{t^2}\right) \frac{1}{-3} = \frac{1}{t} - \frac{1}{3} t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6} t^{-2}, \]
\[ u_2' = (t^2) \left(3 - \frac{1}{t^2}\right) \frac{1}{-3} = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3} t^3 + \frac{1}{3} t. \]
Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: The particular solution \( \tilde{y}_p = u_1 y_1 + u_2 y_2 \) is
\[
\tilde{y}_p = \left[ \ln(t) + \frac{1}{6} t^{-2} \right] (t^2) + \frac{1}{3} (-t^3 + t)(t^{-1})
\]
\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{6} - \frac{1}{3} t^2 + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} t^2
\]
\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} y_1(t).
\]
A simpler expression is \( y_p = t^2 \ln(t) + \frac{1}{2} \).

Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: If we do not remember the formulas for \( u_1, u_2 \), we can always solve the system
\[
u_1' y_1 + u_2' y_2 = 0
\]
\[
u_1' y_1' + u_2' y_2' = f.
\]
\[
t^2 u_1' + u_2' \frac{1}{t} = 0, \quad 2t u_1' + u_2' \left( -\frac{1}{t^2} \right) = 3 - \frac{1}{t^2}.
\]
\[
u_2' = -t^3 u_1' \Rightarrow 2t u_1' + t u_1' = 3 - \frac{1}{t^2} \Rightarrow \begin{cases} u_1' = \frac{1}{t} - \frac{1}{3t^3} \\ u_2' = -t^2 + \frac{1}{3}. \end{cases}
\]