Second order linear homogeneous ODE (Sect. 3.3).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Definition
Any two solutions $y_1, y_2$ of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called fundamental solutions iff the functions $y_1, y_2$ are linearly independent, that is, iff $W_{y_1, y_2} \neq 0$.

Remark: Fundamental solutions are not unique.

Definition
Given any two fundamental solutions $y_1, y_2$, and arbitrary constants $c_1, c_2$, the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is called the general solution of the differential equation above.
Review: On solutions of  $y'' + a_1 y' + a_0 y = 0$.

Theorem (Constant coefficients)

Given real constants $a_1, a_0$, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1 y' + a_0 y = 0. \tag{1}$$

Let $r_+, r_-$ be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let $c_0, c_1$ be arbitrary constants. Then, any solution of Eq. (1) belongs to only one of the following cases:

(a) If $r_+ \neq r_-$, the general solution is $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$.

(b) If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_1 + c_2 t) e^{r_+ t}$.

Furthermore, given real constants $t_0$, $y_1$ and $y_2$, there is a unique solution to the initial value problem given by Eq. (1) and the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Example

Find the general solution of the equation  $y'' - y' - 6y = 0$.

Solution: Since solutions have the form $e^{rt}$, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$, that is,

$$r_{\pm} = \frac{1}{2} (1 \pm \sqrt{1 + 24}) = \frac{1}{2} (1 \pm 5) \quad \Rightarrow \quad r_+ = 3, \quad r_- = -2.$$

So, $r_{\pm}$ are real-valued. A fundamental solution set is formed by

$$y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}.$$

The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is,

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}, \quad c_1, c_2 \in \mathbb{R}.$$

Remark: Since $c_1, c_2 \in \mathbb{R}$, then $y$ is real-valued.
Second order linear homogeneous ODE.

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Two main sets of fundamental solutions.

**Theorem (Complex roots)**

*If the constants \( a_1, a_0 \in \mathbb{R} \) satisfy that \( a_1^2 - 4a_0 < 0 \), then the characteristic polynomial \( p(r) = r^2 + a_1 r + a_0 \) of the equation*

\[
y'' + a_1 y' + a_0 y = 0 \tag{2}
\]

*has complex roots \( r_+ = \alpha + i\beta \) and \( r_- = \alpha - i\beta \), where*

\[
\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}.
\]

*Furthermore, a fundamental set of solutions to Eq. (2) is*

\[
\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},
\]

*while another fundamental set of solutions to Eq. (2) is*

\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Two main sets of fundamental solutions.

Example
Find the general solution of the equation \( y'' - 2y' + 6y = 0 \).

Solution: We first find the roots of the characteristic polynomial,

\[
r^2 - 2r + 6 = 0 \Rightarrow r_{\pm} = \frac{1}{2}(2 \pm \sqrt{4 - 24}) \Rightarrow r_{\pm} = 1 \pm i\sqrt{5}.
\]

A fundamental solution set is

\[
\tilde{y}_1(t) = e^{(1+i\sqrt{5})t}, \quad \tilde{y}_2(t) = e^{(1-i\sqrt{5})t}.
\]

These are complex-valued functions. The general solution is

\[
y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.
\]

Remark:
- The solutions found above include real-valued and complex-valued solutions.
- Since the differential equation is real-valued, it is usually important in applications to obtain the most general real-valued solution. (See RLC circuit below.)
- In the expression above it is difficult to take apart real-valued solutions from complex-valued solutions.
- In other words: It is not simple to see what values of \( \tilde{c}_1 \) and \( \tilde{c}_2 \) make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.
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A real-valued fundamental and general solutions.

**Example**

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$ 

**Solution:** Recall: $y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}, \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$.

The Theorem above says that a real-valued fundamental set is

$$y_1(t) = e^t \cos(\sqrt{5} t), \quad y_2(t) = e^t \sin(\sqrt{5} t).$$

Hence, the complex-valued general solution can also be written as

$$y(t) = [c_1 \cos(\sqrt{5} t) + c_2 \sin(\sqrt{5} t)] e^t, \quad c_1, c_2 \in \mathbb{C}.$$ 

The real-valued general solution is simple to obtain:

$$y(t) = [c_1 \cos(\sqrt{5} t) + c_2 \sin(\sqrt{5} t)] e^t, \quad c_1, c_2 \in \mathbb{R}.$$ 

We just restricted the coefficients $c_1, c_2$ to be real-valued. \(\triangleright\)
A real-valued fundamental and general solutions.

Example
Show that \( y_1(t) = e^t \cos(\sqrt{5} \, t) \) and \( y_2(t) = e^t \sin(\sqrt{5} \, t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

Solution: We start with the complex-valued fundamental solutions, \( \tilde{y}_1(t) = e^{(1+i\sqrt{5}) \, t} \), \( \tilde{y}_2(t) = e^{(1-i\sqrt{5}) \, t} \).

Any linear combination of these functions is solution of the differential equation. In particular,
\[
y_1(t) = \frac{1}{2} [\tilde{y}_1(t) + \tilde{y}_2(t)], \quad y_2(t) = \frac{1}{2i} [\tilde{y}_1(t) - \tilde{y}_2(t)].
\]

Now, recalling \( e^{(1\pm i\sqrt{5}) \, t} = e^t e^{\pm i\sqrt{5} \, t} \)
\[
y_1(t) = \frac{1}{2} [e^t e^{i\sqrt{5} t} + e^t e^{-i\sqrt{5} t}], \quad y_2(t) = \frac{1}{2i} [e^t e^{i\sqrt{5} t} - e^t e^{-i\sqrt{5} t}],
\]

The Euler formula and its complex-conjugate formula
\[
e^{i\sqrt{5} t} = \cos(\sqrt{5} \, t) + i \sin(\sqrt{5} \, t),
\]
\[
e^{-i\sqrt{5} t} = \cos(\sqrt{5} \, t) - i \sin(\sqrt{5} \, t),
\]
imply the inverse relations
\[
e^{i\sqrt{5} t} + e^{-i\sqrt{5} t} = 2 \cos(\sqrt{5} \, t), \quad e^{i\sqrt{5} t} - e^{-i\sqrt{5} t} = 2i \sin(\sqrt{5} \, t).
\]

So functions \( y_1 \) and \( y_2 \) can be written as
\[
y_1(t) = e^t \cos(\sqrt{5} \, t), \quad y_2(t) = e^t \sin(\sqrt{5} \, t).
\]
Example
Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1(t) = e^t \cos(\sqrt{5} t), \; y_2(t) = e^t \sin(\sqrt{5} t)$.

Summary:
- These functions are solutions of the differential equation.
- They are not proportional to each other, Hence li.
- Therefore, $y_1, y_2$ form a fundamental set.
- The general solution of the equation is 
  \[ y(t) = [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)] e^t. \]
- $y$ is real-valued for $c_1, c_2 \in \mathbb{R}$.
- $y$ is complex-valued for $c_1, c_2 \in \mathbb{C}$.

Remark:
- The proof of the Theorem follow exactly the same ideas given in the example above.
- One has to replace the roots of the characteristic polynomial
  \[
  1 + i\sqrt{5} \rightarrow \alpha + i\beta, \quad 1 - i\sqrt{5} \rightarrow \alpha - i\beta.
  \]
- The real-valued fundamental solutions are 
  \[ y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t). \]
A real-valued fundamental and general solutions.

Example
Find real-valued fundamental solutions to the equation
\[ y'' + 2y' + 6y = 0. \]

Solution:
The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \) are
\[
\begin{align*}
  r_\pm &= \frac{1}{2} \left[ -2 \pm \sqrt{4 - 24} \right] = \frac{1}{2} \left[ -2 \pm \sqrt{-20} \right] \Rightarrow r_\pm = -1 \pm i\sqrt{5}.
\end{align*}
\]
These are complex-valued roots, with
\[
\alpha = -1, \quad \beta = \sqrt{5}.
\]
Real-valued fundamental solutions are
\[
y_1(t) = e^{-t} \cos(\sqrt{5} t), \quad y_2(t) = e^{-t} \sin(\sqrt{5} t). \]

Differential equations like the one in this example describe physical processes related to damped oscillations. For example, pendulums with friction.
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of $y'' + 5y = 0$.

Solution: The characteristic polynomial is $p(r) = r^2 + 5$. Its roots are $r_{\pm} = \pm \sqrt{5}i$. This is the case $\alpha = 0$, and $\beta = \sqrt{5}$. Real-valued fundamental solutions are

$$y_1(t) = \cos(\sqrt{5}t), \quad y_2(t) = \sin(\sqrt{5}t).$$

The real-valued general solution is

$$y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t), \quad c_1, c_2 \in \mathbb{R}. \quad \triangleleft$$

Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.

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Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

The electric current flowing in such circuit satisfies:

$$L I'(t) + R I(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.$$  

Derivate both sides above:  

$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0.$$  

Divide by $L$:  

$$I''(t) + \frac{2}{L} \frac{R}{2L} I'(t) + \frac{1}{LC} I(t) = 0.$$  

Introduce $\alpha = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC}$, then  

$$I'' + 2\alpha I' + \omega^2 I = 0.$$  

**Example**

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

**Solution:** The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$  

**Case (a) $R = 0$.** This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. Therefore,  

$$I_1(t) = \cos(\omega t), \quad I_2(t) = \sin(\omega t).$$  

**Remark:** When the circuit has no resistance, the current oscillates without dissipation.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = \frac{R}{2L} \), \( \omega^2 = \frac{1}{LC} \), in the cases (a) (b) below.

Solution: Recall: \( r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \).

Case (b) \( R < \sqrt{4L/C} \). This implies
\[
R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.
\]
Therefore, \( r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2} \). The fundamental solutions are
\[
l_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad l_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).
\]

The resistance \( R \) damps the current oscillations.