Variable coefficients second order linear ODE (Sect. 3.2).

Summary: The study the main properties of solutions to second order, linear, variable coefficients, ODE.

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel’s theorem on the Wronskian.

Review: Second order linear ODE.

Definition
Given functions \( a_1, a_0, b : \mathbb{R} \rightarrow \mathbb{R} \), the differential equation in the unknown function \( y : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
y'' + a_1(t) y' + a_0(t) y = b(t)
\]

is called a \textit{second order linear} differential equation with \textit{variable coefficients}.

Theorem
\textit{If the functions} \( y_1 \) \textit{and} \( y_2 \) \textit{are solutions to the homogeneous linear equation}

\[
y'' + a_1(t) y' + a_0(t) y = 0,
\]

\textit{then the linear combination} \( c_1 y_1(t) + c_2 y_2(t) \) \textit{is also a solution for any constants} \( c_1, c_2 \in \mathbb{R} \).
Variable coefficients second order linear ODE (Sect. 3.2).

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**Existence and uniqueness of solutions.**

**Theorem (Variable coefficients)**

*If the functions $a, b : (t_1, t_2) \rightarrow \mathbb{R}$ are continuous, the constants $t_0 \in (t_1, t_2)$ and $y_0, y_1 \in \mathbb{R}$, then there exists a unique solution $y : (t_1, t_2) \rightarrow \mathbb{R}$ to the initial value problem*

$$y'' + a_1(t) y' + a_0(t) y = b(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$  

**Remarks:**

- Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.
- **Two integrations** must be done to find solutions to second order linear. Therefore, initial value problems with **two initial conditions** can have a unique solution.
Existence and uniqueness of solutions.

Example
Find the longest interval \( I \in \mathbb{R} \) such that there exists a unique solution to the initial value problem
\[
(t - 1)y'' - 3ty' + 4y = t(t - 1), \quad y(-2) = 2, \quad y'(-2) = 1.
\]

Solution: We first write the equation above in the form given in the Theorem above,
\[
y'' - \frac{3t}{t - 1} y' + \frac{4}{t - 1} y = t.
\]

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are \( I_1 = (-\infty, 1) \) and \( I_2 = (1, \infty) \). Since the initial condition belongs to \( I_1 \), the solution domain is
\[
l_1 = (-\infty, 1).
\]

Existence and uniqueness of solutions.

Remarks:
▶ Every solution of the first order linear equation
\[
y' + a(t) y = 0
\]
is given by \( y(t) = c e^{-A(t)} \), with \( A(t) = \int a(s) \, ds \).
▶ All solutions above are proportional to each other:
\[
y_1(t) = c_1 e^{-A(t)}, \quad y_2(t) = c_2 e^{-A(t)} \quad \Rightarrow \quad y_1(t) = \frac{c_1}{c_2} y_2(t)
\]

Remark: The above statement is not true for solutions of second order, linear, homogeneous equations, \( y'' + a_1(t) y' + a_0(t) y = 0 \). Before we prove this statement we need few definitions:
▶ Proportional functions (linearly dependent).
▶ Wronskian of two functions.
Variable coefficients second order linear ODE (Sect. 3.2).

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Linearly dependent and independent functions.

**Definition**

Two continuous functions $y_1, y_2 : (t_1, t_2) \subset \mathbb{R} \rightarrow \mathbb{R}$ are called **linearly dependent, (ld)**, on the interval $(t_1, t_2)$ iff there exists a constant $c$ such that for all $t \in I$ holds

$$y_1(t) = c \ y_2(t).$$

The two functions are called **linearly independent, (li)**, on the interval $(t_1, t_2)$ iff they are not linearly dependent.

**Remarks:**

- $y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R}$ are ld $\iff$ there exist constants $c_1, c_2$, not both zero, such that $c_1 \ y_1(t) + c_2 \ y_2(t) = 0$ for all $t \in (t_1, t_2)$.
- $y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R}$ are li $\iff$ the only constants $c_1, c_2$, solutions of $c_1 \ y_1(t) + c_2 \ y_2(t) = 0$ for all $t \in (t_1, t_2)$ are $c_1 = c_2 = 0$.
- These definitions are not given in the textbook.
Linearly dependent and independent functions.

Example

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.
(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are li.

Solution:
Case (a): Trivial. $y_2 = 2y_1$.

Case (b): Find constants $c_1$, $c_2$ such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0 \iff (c_1 + c_2 t) \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2} c_2 = 0, \quad c_1 + \frac{3\pi}{2} c_2 = 0 \Rightarrow c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions $y_1$ and $y_2$ are li.

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The Wronskian of two functions.

Remark: The Wronskian is a function that determines whether two functions are ld or li.

Definition
The Wronskian of functions $y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R}$ is the function

$$W_{y_1y_2}(t) = y_1(t)y'_2(t) - y'_1(t)y_2(t).$$

Remark:
- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}$, then $W_{y_1y_2}(t) = \det(A(t)).$
- An alternative notation is: $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}.$

Example
Find the Wronskian of the functions:

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t).$ (ld)
(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t).$ (li)

Solution:
Case (a): $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}.$ Therefore,

$$W_{y_1y_2}(t) = \sin(t)2\cos(t) - \cos(t)2\sin(t) \implies W_{y_1y_2}(t) = 0.$$ 

Case (b): $W_{y_1y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}.$ Therefore,

$$W_{y_1y_2}(t) = \sin(t)[\sin(t) + t\cos(t)] - \cos(t)t\sin(t).$$

We obtain $W_{y_1y_2}(t) = \sin^2(t).$ ◯
The Wronskian of two functions.

Remark: The Wronskian determines whether two functions are linearly dependent or independent.

Theorem (Wronskian and linearly dependence)

The continuously differentiable functions \( y_1, y_2 : (t_1, t_2) \to \mathbb{R} \) are linearly dependent iff \( W_{y_1y_2}(t) = 0 \) for all \( t \in (t_1, t_2) \).

Remark: Importance of the Wronskian:

▶ Sometimes it is not simple to decide whether two functions are proportional to each other.
▶ The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)

Example

Show whether the following two functions form a l.d. or l.i. set:

\[
y_1(t) = \cos(2t) - 2 \cos^2(t), \quad y_2(t) = \cos(2t) + 2 \sin^2(t).
\]

Solution: Compute their Wronskian:

\[
W_{y_1y_2}(t) = y_1y_2' - y_1'y_2.
\]

\[
W_{y_1y_2}(t) = \left[ \cos(2t) - 2 \cos^2(t) \right] \left[ -2 \sin(2t) + 4 \sin(t) \cos(t) \right]
- \left[ -2 \sin(2t) + 4 \sin(t) \cos(t) \right] \left[ \cos(2t) + 2 \sin^2(t) \right].
\]

\[
\sin(2t) = 2 \sin(t) \cos(t) \Rightarrow \left[ -2 \sin(2t) + 4 \sin(t) \cos(t) \right] = 0.
\]

We conclude \( W_{y_1y_2}(t) = 0 \), so the functions \( y_1 \) and \( y_2 \) are l.d.  \( \triangleright \)
The Wronskian of two functions.

Theorem (Variable coefficients)

▶ If $a_1, a_0, b : (t_1, t_2) \rightarrow \mathbb{R}$ are continuous, then there exist two linearly independent solutions $y_1, y_2 : (t_1, t_2) \rightarrow \mathbb{R}$ to the equation

$$y'' + a_1(t)y' + a_0(t)y = b(t). \quad (1)$$

▶ Every other solution $y$ of Eq. (1) can be decomposed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for appropriate constants $c_1, c_2$.

▶ For every constant $t_0 \in (t_1, t_2)$ and $y_0, y_1 \in \mathbb{R}$, there exists a unique solution $y : (t_1, t_2) \rightarrow \mathbb{R}$ to the initial value problem given by Eq. (1) with the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1.$$
General and fundamental solutions.

Remark: The Theorem above justifies the following definitions.

Definition
Two solutions $y_1, y_2$ of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called fundamental solutions iff the functions $y_1, y_2$ are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

Definition
Given any two fundamental solutions $y_1, y_2$, and arbitrary constants $c_1, c_2$, the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is called the general solution of Eq. (1).

Example
Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2 y'' + 3t y' - y = 0.$$

Solution: First show that $y_1$ is a solution:

$$y_1 = t^{1/2}, \quad y_1' = \frac{1}{2} t^{-1/2}, \quad y_1'' = -\frac{1}{4} t^{-3/2},$$

$$2t^2 \left(-\frac{1}{4} t^{-3/2}\right) + 3t \left(\frac{1}{2} t^{-1/2}\right) - t^{1/2} = -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0.$$

Now show that $y_2$ is a solution:

$$y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2'' = 2t^{-3},$$

$$2t^2 (2t^{-3}) + 3t (-t^{-2}) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$
General and fundamental solutions.

Example
Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2 y'' + 3t y' - y = 0.$$ 

Solution: We show that $y_1, y_2$ are linearly independent.

$$W_{y_1y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ 1/2 t^{-1/2} & -t^{-2} \end{vmatrix}.$$ 

$$W_{y_1y_2}(t) = -t^{1/2} t^{-2} - \frac{1}{2} t^{-1/2} t^{-1} = -t^{-3/2} - \frac{1}{2} t^{-3/2}$$ 

$$W_{y_1y_2}(t) = -\frac{3}{3} t^{-3/2} \Rightarrow y_1, y_2 \text{ li.} \quad \triangleleft$$

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Abel’s theorem on the Wronskian.

Theorem (Abel)

If \( a_1, a_0 : (t_1, t_2) \to \mathbb{R} \) are continuous functions and \( y_1, y_2 \) are continuously differentiable solutions of the equation

\[
y'' + a_1(t) y' + a_0(t) y = 0,
\]

then the Wronskian \( W_{y_1y_2} \) is a solution of the equation

\[
W'_{y_1y_2}(t) + a_1(t) W_{y_1y_2}(t) = 0.
\]

Therefore, for any \( t_0 \in (t_1, t_2) \), the Wronskian \( W_{y_1y_2} \) is given by

\[
W_{y_1y_2}(t) = W_{y_1y_2}(t_0) e^{A(t)} \quad A(t) = \int_{t_0}^{t} a_1(s) \, ds.
\]

Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

Abel’s theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

\[
t^2 y'' - t(t + 2) y' + (t + 2) y = 0, \quad t > 0.
\]

Solution: Write the equation as in Abel’s Theorem,

\[
y'' - \left( \frac{2}{t} + 1 \right) y' + \left( \frac{2}{t^2} + \frac{1}{t} \right) y = 0.
\]

Abel’s Theorem says that the Wronskian satisfies the equation

\[
W'_{y_1y_2}(t) - \left( \frac{2}{t} + 1 \right) W_{y_1y_2}(t) = 0.
\]

This is a first order, linear equation for \( W_{y_1y_2} \). The integrating factor method implies

\[
A(t) = -\int_{t_0}^{t} \left( \frac{2}{s} + 1 \right) \, ds = -2 \ln \left( \frac{t}{t_0} \right) - (t - t_0)
\]
Abel’s theorem on the Wronskian.

Example
Find the Wronskian of two solutions of the equation

\[ t^2 y'' - t(t + 2) y' + (t + 2) y = 0, \quad t > 0. \]

Solution: \( A(t) = -2 \ln \left( \frac{t}{t_0} \right) - (t - t_0) = \ln \left( \frac{t_0^2}{t^2} \right) - (t - t_0). \)

The integrating factor is \( \mu = \frac{t_0^2}{t^2} e^{-(t-t_0)}. \) Therefore,

\[
\left[ \mu(t) W_{y_1y_2}(t) \right]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1y_2}(t) - \mu(t_0) W_{y_1y_2}(t_0) = 0
\]

so, the solution is \( W_{y_1y_2}(t) = W_{y_1y_2}(t_0) \frac{t^2}{t_0^2} e^{(t-t_0)}. \)

Denoting \( c = \left( W_{y_1y_2}(t_0) / t_0^2 \right) e^{-t_0}, \) then \( W_{y_1y_2}(t) = c \ t^2 \ e^t. \) \( \triangleq \)