Autonomous systems (Sect. 2.5).

- Definition and examples.
- Qualitative analysis of the solutions.
- Equilibrium solutions and stability.
- Population growth equation.

Definition and examples

Definition
A first order ODE on the unknown function \( y : \mathbb{R} \to \mathbb{R} \) is called autonomous iff the ODE has the form

\[
\frac{dy}{dt} = f(y).
\]

Remark:
- The independent variable, \( t \), does not appear explicitly in an autonomous ODE.

- Autonomous systems are a particular case of separable equations,

\[
h(y) y' = g(t), \quad g(t) = 1, \quad f(y) = \frac{1}{h(y)}.
\]

- It is simple to study the qualitative properties of solutions to autonomous systems.
Autonomous systems (Sect. 2.5).

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**Qualitative analysis of the solutions**

**Remark:** It is simple to study the qualitative properties of solutions to autonomous systems.

**Example**

Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

**Solution:** One way: Find the exact solutions and then graph them.

The equation is separable, then

$$\frac{y'(t)}{\sin[y(t)]]} = 1 \quad \Rightarrow \quad \int_0^t \frac{y'(t)}{\sin[y(t)]]} = t$$

Use the substitution $u = y(t), \quad du = y'(t) \, dt$,

$$\int_{y_0}^{y(t)} \frac{du}{\sin(u)} = t \quad \Rightarrow \quad \ln\left[\frac{\sin(u)}{1 + \cos(u)}\right]_{y_0}^{y(t)} = t.$$
Example
Sketch a qualitative graph of solutions to \( y' = \sin(y) \), for different initial data conditions \( y(0) = y_0 \).

Solution: Recall: \[ \ln\left(\frac{\sin(u)}{1 + \cos(u)}\right)_{y_0}^{y(t)} = t. \]

\[ \ln\left(\frac{\sin(y)}{1 + \cos(y)}\right) - \ln\left(\frac{\sin(y_0)}{1 + \cos(y_0)}\right) = t. \]

\[ \ln\left(\frac{\sin(y)}{[1 + \cos(y)]}\right) - \ln\left(\frac{[1 + \cos(y_0)]}{\sin(y_0)}\right) = t. \]

The implicit expression of the solution is

\[ \frac{\sin(y)}{[1 + \cos(y)]} = \frac{\sin(y_0)}{[1 + \cos(y_0)]} e^t. \]

Without a computer it is difficult to graph the solution.

Qualitative analysis of the solutions

Example
Sketch a qualitative graph of solutions to \( y' = \sin(y) \), for different initial data conditions \( y(0) = y_0 \).

Solution: Recall: \[ \frac{\sin(y)}{[1 + \cos(y)]} = \frac{\sin(y_0)}{[1 + \cos(y_0)]} e^t. \]

Another way:
(1) Plot the function \( f(y) = \sin(y) \).

(2) Find the zeros of \( f \). Since \( f(y) = \sin(y) = 0 \), then \( y = m\pi \).

The constants \( y = m\pi \), are solutions of \( y' = \sin(y) \).
They are called equilibrium solutions.
Qualitative analysis of the solutions

Example
Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution:

Equilibrium solutions: $y(t) = n \pi$.

(3) The solution is:
Increasing for $y' = \sin(y) > 0$,
Decreasing for $y' = \sin(y) < 0$.

Autonomous systems (Sect. 2.5).

- Definition and examples.
- Qualitative analysis of the solutions.
- Equilibrium solutions and stability.
- Population growth equation.
Equilibrium solutions and stability

Definition
The constant \( y_0 \) is an equilibrium solution of the autonomous system \( y' = f(y) \) iff hold that \( f(y_0) = 0 \).

The equilibrium solution \( y_0 \) is asymptotically stable iff there exists \( I = (y_0 - \epsilon, y_0 + \epsilon) \) such that every solution \( y \) with \( y(0) \in I \) satisfies
\[
\lim_{t \to \infty} y(t) = y_0.
\]

Definition
The equilibrium solution \( y_0 \) is asymptotically unstable iff there exists \( I = (y_0 - \epsilon, y_0 + \epsilon) \) such that for every solution \( y \) with \( y(0) \in I \) holds
\[
\lim_{t \to \infty} y(t) \neq y_0.
\]

Example
Sketch a qualitative graph of solutions to \( y' = \sin(y) \), for different initial data conditions \( y(0) = y_0 \).

Solution: Summary:

\[ f(y) = \sin(y) \]
Definition and examples.
Qualitative analysis of the solutions.
Equilibrium solutions and stability.
Population growth equation.

Population growth equation (Logistic equation)

Example
Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r \left(1 - \frac{y}{K}\right) y$, where $r$ and $K$ are given positive constants.

Solution:
(1) Plot the function $f(y) = r \left(1 - \frac{y}{K}\right) y$.
(2) Find the zeros of $f$.
$y_0 = 0$, $y_0 = K$.

The constants $y_0 = 0$ and $y_0 = K$ are the equilibrium solutions.
The solution $y_0$ is unstable, while $y_0 = K$ is stable.
Population growth equation (Logistic equation)

**Example**

Sketch a qualitative graph of solutions for different initial data conditions \( y(0) = y_0 \) to the population growth equation

\[
y' = r \left( 1 - \frac{y}{K} \right) y,
\]

where \( r \) and \( K \) are given positive constants.

**Solution:**

\[
f(y) = r \left( 1 - \frac{y}{K} \right) y
\]

(3) For \( y_0 \in (0, K) \) the solution is *Increasing*.

For \( y_0 \in (K, \infty) \) the solution is *Decreasing*.

Population growth equation (Logistic equation)

**Remark:** The curvature of the solution \( y \) depends on \( f'(y) f(y) \).

**Theorem**

*If the function \( y \) is a solution of the autonomous system \( y' = f(y) \), then the graph of \( y \) has positive curvature iff \( f'(y) f(y) > 0 \), and negative curvature iff \( f'(y) f(y) < 0 \).*

**Proof:**

\[
\frac{d^2y}{dt^2} = \frac{df}{dy}(y) \frac{dy}{dt}, \quad \frac{dy}{dt} = f(y) \quad \Rightarrow \quad y'' = f'(y) f(y).
\]
Population growth equation (Logistic equation)

Example
Find the exact expression for the solutions to the population growth equation $y' = r \left(1 - \frac{y}{K}\right) y$, with $y(0) = y_0$.

Solution: This is a separable equation,

$$
\frac{K}{r} \int \frac{y' \, dt}{(K - y)y} = t + c_0.
$$

Substitution: $u = y(t)$, then $du = y' \, dt$,

$$
\frac{K}{r} \int \frac{du}{(K - u)u} = t + c_0.
$$

Partial fraction decomposition:

$$
\frac{K}{r} \int \left[ \frac{1}{K(K - u)} + \frac{1}{u} \right] \, du = t + c_0.
$$

\[\begin{align*}
-\ln(|K - y|) + \ln(|y|) &= rt + rc_0, \\
\ln\left(\frac{|y|}{|K - y|}\right) &= rt + rc_0 \quad \Rightarrow \quad \frac{y}{K - y} = c e^{rt}, \quad c = e^{rc_0}.
\end{align*}\]

$$
y(t) = \frac{cK e^{rt}}{1 + c e^{rt}}, \quad c = \frac{y_0}{K - y_0}
$$

We conclude that $y(t) = \frac{KY_0}{y_0 + (K - y_0) e^{-rt}}$. \(\triangleright\)
Population growth equation (Logistic equation)

Example
Sketch a qualitative graph of solutions for different initial data conditions \( y(0) = y_0 \) to the population growth equation \( y' = r \left(1 - \frac{y}{K}\right) y \), where \( r \) and \( K \) are given positive constants.

Solution: 

The solution is 

\[
y(t) = \frac{K y_0}{y_0 + (K - y_0) e^{-r t}}.
\]