Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.

Separable ODE.

**Definition**
Given functions \( h, g : \mathbb{R} \to \mathbb{R} \), a first order ODE on the unknown function \( y : \mathbb{R} \to \mathbb{R} \) is called *separable* iff the ODE has the form

\[
h(y) y'(t) = g(t).
\]

**Remark:**
A differential equation \( y'(t) = f(t, y(t)) \) is separable iff

\[
y' = \frac{g(t)}{h(y)} \iff f(t, y) = \frac{g(t)}{h(y)}.
\]

**Notation:**
In lecture: \( t \), \( y(t) \) and \( h(y) y'(t) = g(t) \).
In textbook: \( x \), \( y(x) \) and \( M(x) + N(y) y'(x) = 0 \).
Therefore: \( h(y) = N(y) \) and \( g(t) = -M(t) \).
**Example**
Determine whether the differential equation below is separable,

\[ y'(t) = \frac{t^2}{1 - y^2(t)}. \]

**Solution:** The differential equation is separable, since it is equivalent to

\[ (1 - y^2) y'(t) = t^2 \quad \Rightarrow \quad \begin{cases} g(t) = t^2, \\ h(y) = 1 - y^2. \end{cases} \]

**Remark:** The functions \( g \) and \( h \) are not uniquely defined.
Another choice here is:

\[ g(t) = c \cdot t^2, \quad h(y) = c \cdot (1 - y^2), \quad c \in \mathbb{R}. \]

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**Example**
Determine whether The differential equation below is separable,

\[ y'(t) + y^2(t) \cos(2t) = 0 \]

**Solution:** The differential equation is separable, since it is equivalent to

\[ \frac{1}{y^2} y'(t) = -\cos(2t) \quad \Rightarrow \quad \begin{cases} g(t) = -\cos(2t), \\ h(y) = \frac{1}{y^2}. \end{cases} \]

**Remark:** The functions \( g \) and \( h \) are not uniquely defined.
Another choice here is:

\[ g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}. \]
Separable ODE.

Remark: Not every first order ODE is separable.

Example

- The differential equation \( y'(t) = e^{y(t)} + \cos(t) \) is not separable.
- The linear differential equation \( y'(t) = -\frac{2}{t} y(t) + 4t \) is not separable.
- The linear differential equation \( y'(t) = -a(t) y(t) + b(t) \), with \( b(t) \) non-constant, is not separable.

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Theorem (Separable equations)

If the functions \( g, h : \mathbb{R} \rightarrow \mathbb{R} \) are continuous, with \( h \neq 0 \) and with primitives \( G \) and \( H \), respectively; that is,

\[
G'(t) = g(t), \quad H'(u) = h(u),
\]

then, the separable ODE

\[
h(y)y' = g(t)
\]

has infinitely many solutions \( y : \mathbb{R} \rightarrow \mathbb{R} \) satisfying the algebraic equation

\[
H(y(t)) = G(t) + c,
\]

where \( c \in \mathbb{R} \) is arbitrary.

Remark: Given functions \( g, h \), find their primitives \( G, H \).

Example

Find all solutions \( y : \mathbb{R} \rightarrow \mathbb{R} \) to the ODE \( y'(t) = \frac{t^2}{1 - y^2(t)} \).

Solution: The equation is equivalent to \( (1 - y^2) \, y'(t) = t^2 \).

Therefore, the functions \( g, h \) are given by

\[
g(t) = t^2, \quad h(u) = 1 - u^2.
\]

Their primitive functions, \( G \) and \( H \), respectively, are given by

\[
g(t) = t^2 \Rightarrow G(t) = \frac{t^3}{3},
\]

\[
h(u) = 1 - u^2 \Rightarrow H(u) = u - \frac{u^3}{3}.
\]

Then, the Theorem above implies that the solution \( y \) satisfies the algebraic equation

\[
y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \quad c \in \mathbb{R}.
\]

\( \triangle \)
Solutions to separable ODE.

Remarks:

- The equation \( y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c \) is algebraic in \( y \), since there is no \( y' \) in the equation.
- Every function \( y \) satisfying the algebraic equation
  \[ y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \]
  is a solution of the differential equation above.
- We now verify the previous statement: Differentiate on both sides with respect to \( t \), that is,
  \[ y'(t) - 3 \left( \frac{y^2(t)}{3} \right) y'(t) = 3 \frac{t^2}{3} \quad \Rightarrow \quad (1 - y^2) y' = t^2. \]
Explicit and implicit solutions.

Remark:
The solution $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is given in implicit form.

Definition
Assume the notation in the Theorem above. The solution $y$ of a separable ODE is given in *implicit form* iff function $y$ is specified by

$$H(y(t)) = G(t) + c,$$

The solution $y$ of a separable ODE is given in *explicit form* iff function $H$ is invertible and $y$ is specified by

$$y(t) = H^{-1}(G(t) + c).$$

Explicit and implicit solutions.

Example
Use the main idea in the proof of the Theorem above to find the solution of the IVP

$$y'(t) + y^2(t) \cos(2t) = 0, \quad y(0) = 1.$$

Solution: The differential equation is separable, with

$$g(t) = -\cos(2t), \quad h(y) = \frac{1}{y^2}.$$

The main idea in the proof of the Theorem above is this: integrate on both sides of the equation,

$$\frac{y'(t)}{y^2(t)} = -\cos(2t) \iff \int \frac{y'(t)}{y^2(t)} \, dt = -\int \cos(2t) \, dt + c.$$

The substitution $u = y(t), \ du = y'(t) \, dt$, implies that

$$\int \frac{du}{u^2} = -\int \cos(2t) \, dt + c \iff -\frac{1}{u} = -\frac{1}{2} \sin(2t) + c.$$
Explicit and implicit solutions.

Example
Use the main idea in the proof of the Theorem above to find the solution of the IVP

\[ y'(t) + y^2(t) \cos(2t) = 0, \quad y(0) = 1. \]

Solution: Recall: \( -\frac{1}{u} = -\frac{1}{2} \sin(2t) + c. \)
Substitute the unknown function \( y \) back in the equation above,

\[ -\frac{1}{y(t)} = -\frac{1}{2} \sin(2t) + c. \]  
(Implicit form.)

\[ y(t) = \frac{2}{\sin(2t) - 2c}. \]  
(Explicit form.)

The initial condition implies that \( 1 = y(0) = \frac{2}{0 - 2c} \), so \( c = -1 \).
We conclude that \( y(t) = \frac{2}{\sin(2t) + 2}. \) \( \triangleleft \)

Separable differential equations (Sect. 2.2).

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Homogeneous equations.

Definition
The first order ODE \( y'(t) = f(t, y(t)) \) is called \textit{homogeneous} iff for every numbers \( c, t, u \in \mathbb{R} \) the function \( f \) satisfies
\[
f(ct, cu) = f(t, u).
\]

Remark:
\begin{itemize}
  \item The function \( f \) is invariant under the change of scale of its arguments.
  \item If \( f(t, u) \) has the property above, it must depend only on \( u/t \).
  \item So, there exists \( F : \mathbb{R} \to \mathbb{R} \) such that \( f(t, u) = F\left(\frac{u}{t}\right) \).
  \item Therefore, a first order ODE is homogeneous iff it has the form
  \[
y'(t) = F\left(\frac{y(t)}{t}\right).
  \]
\end{itemize}

Homogeneous equations.

Example
Show that the equation below is homogeneous,
\[
(t - y) \, y' - 2y + 3t + \frac{y^2}{t} = 0.
\]

Solution: Rewrite the equation in the standard form
\[
(t - y) \, y' = 2y - 3t - \frac{y^2}{t} \quad \Rightarrow \quad y' = \frac{2y - 3t - \frac{y^2}{t}}{(t - y)}.
\]
Divide numerator and denominator by \( t \). We get,
\[
y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{(t - y)} \left(\frac{\frac{1}{t}}{\frac{1}{t}}\right) \quad \Rightarrow \quad y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{1 - \left(\frac{y}{t}\right)}.
\]
Homogeneous equations.

Example
Show that the equation below is homogeneous,
\[(t - y)y' - 2y + 3t + \frac{y^2}{t} = 0.\]

Solution: Recall: \(y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{1 - \left(\frac{y}{t}\right)}\).

We conclude that the ODE is homogeneous, because the right-hand side of the equation above depends only on \(y/t\).
Indeed, in our case:
\[f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y}, \quad F(x) = \frac{2x - 3 - x^2}{1 - x},\]
and \(f(t, y) = F(y/t)\).

Homogeneous equations.

Example
Determine whether the equation below is homogeneous,
\[y' = \frac{t^2}{1 - y^3}.\]

Solution:
Divide numerator and denominator by \(t^3\), we obtain
\[y' = \frac{t^2}{(1 - y^3)} \left(\frac{\frac{1}{t^3}}{1}\right) \Rightarrow y' = \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t^3}\right) - \left(\frac{y}{t}\right)^3}.\]

We conclude that the differential equation is not homogeneous.
Homogeneous equations.

Theorem
If the differential equation $y'(t) = f(t, y(t))$ is homogeneous, then the differential equation for the unknown $v(t) = \frac{y(t)}{t}$ is separable.

Remark: Homogeneous equations can be transformed into separable equations.

Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function $F$. Introduce $v = y/t$. This means,

$$y(t) =tv(t) \implies y'(t) = v(t) + tv'(t).$$

Introducing all this into the ODE we get

$$v + tv' = F(v) \implies v' = \frac{F(v) - v}{t}.$$

This last equation is separable.

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Homogeneous equations.

Example
Find all solutions $y$ of the ODE $y' = \frac{t^2 + 3y^2}{2ty}$.

Solution: The equation is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \left(\frac{1}{t^2}\right) \left(\frac{1}{t^2}\right) \implies y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Therefore, we introduce the change of unknown $v = y/t$, so $y = tv$ and $y' = v + tv'$. Hence

$$v + tv' = \frac{1 + 3v^2}{2v} \implies tv' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}.$$

We obtain the separable equation $v' = \frac{1}{t} \left(\frac{1 + v^2}{2v}\right)$. 
Homogeneous equations.

Example
Find all solutions $y$ of the ODE $y' = \frac{t^2 + 3y^2}{2ty}$.

Solution: Recall: $v' = \frac{1}{t} \left( \frac{1 + v^2}{2v} \right)$. We rewrite and integrate it,

$$\frac{2v}{1 + v^2} v' = \frac{1}{t} \Rightarrow \int \frac{2v}{1 + v^2} v' \, dt = \int \frac{1}{t} \, dt + c_0.$$

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t) v'(t) \, dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \Rightarrow \ln(u) = \ln(t) + c_0 \Rightarrow u = e^{\ln(t) + c_0}.$$

But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \Rightarrow 1 + \left( \frac{y}{t} \right)^2 = c_1 t \Rightarrow y(t) = \pm t \sqrt{c_1 t - 1}.$$