Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.
  - Solving differential equations (7.4).
  - Inverse trigonometric functions (7.6).
  - Hyperbolic functions (7.7).
  - Integration techniques (8-IT).
  - Integration by parts (8.1).
  - Trigonometric integrals (8.2).
- Section not covered:
  - Trigonometric substitutions (8.3).
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  - **Solving differential equations (7.4).**
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Remark: Typical problems in this section:

(1) Find the function $y$ solution of $y' = \frac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.
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(1) Find the function $y$ solution of $y' = \frac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

(2) The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?
Example

Find the function $y$ solution of $y' = \frac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

Solution:

$$4y y' = \sin(x) \Rightarrow \int 4y \, dy = \int \sin(x) \, dx.$$ 

The substitution $u = y(x)$, with $du = y'(x) \, dx$, implies

$$2u^2 = -\cos(x) + c.$$ 

Therefore, $y(x) = \frac{-\cos(x) + c}{\sqrt{2}}$.

The condition $y(0) < 0$, implies $y(x) = -\sqrt{5} - \cos(x) / \sqrt{2}$.

Furthermore, $-\sqrt{2} = -\sqrt{5 - 1} \sqrt{2} \Rightarrow 2 = \sqrt{5 - 1} \Rightarrow c = 5$.

We conclude that $y = -\sqrt{5} - \cos(x) / \sqrt{2}$. \(\triangleq\)
Example
Find the function \( y \) solution of \( y' = \frac{\sin(x)}{4y} \) and \( y(0) = -\sqrt{2} \).

Solution:

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4y \, y' = \sin(x)
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Example

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$$4y \ y' = \sin(x) \quad \Rightarrow \quad \int 4y(x) \ y'(x) \, dx = \int \sin(x) \, dx.$$ 

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Therefore, $y^2(x) = (-\cos(x) + c)/2$. 

\[ \text{\textcopyright} \]
Solving differential equations (7.4)

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◁
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4 \int u \, du = \int \sin(x) \, dx \quad \Rightarrow \quad 2u^2 = -\cos(x) + c,
\]
Therefore, \( y^2(x) = (\frac{-\cos(x) + c}{2})/2 \). The condition \( y(0) < 0 \), implies \( y(x) = -\sqrt{c - \cos(x)}/\sqrt{2} \). Furthermore,
\[
-\sqrt{2} = -\frac{\sqrt{c - 1}}{\sqrt{2}}
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Solution:

$$ 4y \cdot y' = \sin(x) \quad \Rightarrow \quad \int 4y(x) \cdot y'(x) \, dx = \int \sin(x) \, dx. $$

The substitution $u = y(x)$, with $du = y'(x) \, dx$, implies

$$ 4 \int u \, du = \int \sin(x) \, dx \quad \Rightarrow \quad 2u^2 = -\cos(x) + c, $$

Therefore, $y^2(x) = (\cos(x) + c)/2$. The condition $y(0) < 0$, implies $y(x) = -\sqrt{c - \cos(x)}/\sqrt{2}$. Furthermore,

$$ -\sqrt{2} = -\frac{\sqrt{c - 1}}{\sqrt{2}} \quad \Rightarrow \quad 2 = \sqrt{c - 1} \quad \Rightarrow \quad c = 5. $$

We conclude that $y = -\sqrt{5 - \cos(x)}/\sqrt{2}$. △
Example

The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?
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$$L(x) = L_0 e^{-kx}.$$
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Solution: Recall: \( L(x) = L_0 e^{-kx} \). Now the first condition implies

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Solution: Recall: $L(x) = L_0 e^{-kx}$. Now the first condition implies

$$\frac{L_0}{2} = L(15) = L_0 e^{-15k} \implies e^{-15k} = \frac{1}{2}$$
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\[ \]
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Using the value $k = \ln(2)/15$,
Solving differential equations (7.4)

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Using the value $k = \ln(2)/15$, we get

$$x_1 = \ln(8) \frac{15}{\ln(2)} \Rightarrow x_1 = 3(15).$$
Example

The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

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$$\frac{L_0}{8} = L_0 e^{-kx_1} \Rightarrow e^{-kx_1} = \frac{1}{8} \Rightarrow -kx_1 = -\ln(8).$$

Using the value $k = \ln(2)/15$, we get

$$x_1 = \ln(8) \frac{15}{\ln(2)} \Rightarrow x_1 = 3(15) \Rightarrow x_1 = 45.$$
Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.

- Solving differential equations (7.4).
- **Inverse trigonometric functions (7.6).**
- Hyperbolic functions (7.7).
- Integration techniques (8-IT).
- Integration by parts (8.1).
- Trigonometric integrals (8.2).

Section not covered:

- Trigonometric substitutions (8.3).
Inverse trigonometric functions (7.6)

**Notation:** In the literature is common the notation \( \sin^{-1} = \text{arcsin} \), and similar for the rest of the trigonometric functions.

Do not confuse \( \frac{1}{\sin(x)} \neq \sin^{-1}(x) = \text{arcsin}(x) \).

**Remark:** \( \sin, \cos \) have simple values at particular angles.

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Inverse trigonometric functions (7.6)

Remark: On certain domains the trigonometric functions are invertible.
Inverse trigonometric functions (7.6)

Remark: The graph of the inverse function is a reflection of the original function graph about the \( y = x \) axis.

\[
\begin{align*}
  y &= \arcsin(x) \\
  y &= \arccos(x) \\
  y &= \arctan(x) \\
  y &= \text{arccsc}(x) \\
  y &= \text{arcsec}(x) \\
  y &= \text{arccot}(x)
\end{align*}
\]
Inverse trigonometric functions (7.6)

Theorem

*The derivative of inverse trigonometric functions are:*

\[
\begin{align*}
\text{arcsin}'(x) &= \frac{1}{\sqrt{1 - x^2}}, \\
\text{arccos}'(x) &= -\frac{1}{\sqrt{1 - x^2}}, \quad |x| \leq 1, \\
\text{arctan}'(x) &= \frac{1}{1 + x^2}, \\
\text{arccot}'(x) &= -\frac{1}{1 + x^2}, \quad x \in \mathbb{R}, \\
\text{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2 - 1}}, \\
\text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| \geq 1.
\end{align*}
\]
Inverse trigonometric functions (7.6)

Theorem

The derivative of inverse trigonometric functions are:

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\begin{align*}
\arcsin'(x) &= \frac{1}{\sqrt{1 - x^2}}, \\
\arccos'(x) &= -\frac{1}{\sqrt{1 - x^2}}, \quad |x| \leq 1, \\
\arctan'(x) &= \frac{1}{1 + x^2}, \\
\arccot'(x) &= -\frac{1}{1 + x^2}, \quad x \in \mathbb{R}, \\
\text{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2 - 1}}, \\
\text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| \geq 1.
\end{align*}
\]

Recall \( \arctan'(x) = \frac{1}{\tan'(\arctan(x))} \),
Inverse trigonometric functions (7.6)

Theorem

The derivative of inverse trigonometric functions are:

\[ \text{arcsin}'(x) = \frac{1}{\sqrt{1 - x^2}}, \quad \text{arccos}'(x) = -\frac{1}{\sqrt{1 - x^2}}, \quad |x| \leq 1, \]

\[ \text{arctan}'(x) = \frac{1}{1 + x^2}, \quad \text{arccot}'(x) = -\frac{1}{1 + x^2}, \quad x \in \mathbb{R}, \]

\[ \text{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad \text{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| \geq 1. \]

Recall \( \text{arctan}'(x) = \frac{1}{\tan'(\text{arctan}(x))}, \quad \tan'(y) = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} \).
Inverse trigonometric functions (7.6)

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The derivative of inverse trigonometric functions are:

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Recall \( \arctan'(x) = \frac{1}{\tan'(\arctan(x))} \), \( \tan'(y) = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} \)

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Inverse trigonometric functions (7.6)

Theorem

The derivative of inverse trigonometric functions are:

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\begin{align*}
\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}}, & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}, & |x| \leq 1, \\
\arctan'(x) &= \frac{1}{1+x^2}, & \arccot'(x) &= -\frac{1}{1+x^2}, & x \in \mathbb{R}, \\
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Inverse trigonometric functions (7.6)

Theorem

*The derivative of inverse trigonometric functions are:*

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\begin{align*}
arcsin'(x) &= \frac{1}{\sqrt{1 - x^2}}, & \text{arccos}'(x) &= -\frac{1}{\sqrt{1 - x^2}}, & x \leq 1, \\
arctan'(x) &= \frac{1}{1 + x^2}, & \text{arccot}'(x) &= -\frac{1}{1 + x^2}, & x \in \mathbb{R}, \\
arcsec'(x) &= \frac{1}{|x|\sqrt{x^2 - 1}}, & \text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2 - 1}}, & |x| \geq 1.
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Recall \( \arctan'(x) = \frac{1}{\tan'(\arctan(x))} \), \( \tan'(y) = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} \), \( \tan'(y) = 1 + \tan^2(y) \), \( y = \arctan(x) \), \( \Rightarrow \arctan'(x) = \frac{1}{1 + x^2} \).
Remark: Typical problems in this section:

(1) Sketch the graphs of

\[ y(x) = \sec(x), \quad z(x) = \text{arcsec}(x). \]

State the respective domains and ranges.

(2) Evaluate \( \cos(\arcsin(1/\sqrt{2})) \).

(3) Evaluate \( \sec(\arctan(-2/3)) \).

(4) Find \( y' \) for \( y(x) = \arctan(3x^2) \).

(5) Find \( I = \int \frac{dx}{\sqrt{2 - x^2}} \).
Inverse trigonometric functions (7.6)

Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution:
We only need the relation between \( \sec \) and \( \tan \),
\[
\sec^2(\theta) = \tan^2(\theta) + 1.
\]
Then holds \( \sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1} \).

We need to find the correct sign:
\( \theta = \arctan(-2/3) \in (-\pi/2, 0) \).
Since \( \sec(\theta) = 1/\cos(\theta) \),
we conclude that \( \sec(\theta) > 0 \).
Hence
\[
\sec(\arctan(-2/3)) = \sqrt{\tan^2(\arctan(-2/3)) + 1} = \sqrt{4/9 + 1} = \sqrt{13/9}.
\]
We conclude that
\[
\sec(\arctan(-2/3)) = \sqrt{13}/3.
\]
\( \triangleright \)
Inverse trigonometric functions (7.6)

Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution: We only need the relation between sec and tan,

\[
\sec^2(\theta) = \tan^2(\theta) + 1
\]

Then holds \( \sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1} \).

We need to find the correct sign:
\( \theta = \arctan(-2/3) \in (-\pi/2, 0) \).
Since \( \sec(\theta) = 1/\cos(\theta) \), we conclude that \( \sec(\theta) > 0 \).
Hence \( \sec(\arctan(-2/3)) = \sqrt{\frac{-4}{9} + 1} = \sqrt{\frac{13}{9}} \).

We conclude that
\( \sec(\arctan(-2/3)) = \sqrt{\frac{13}{3}} \).
Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution: We only need the relation between \( \sec \) and \( \tan \),

\[
\sec^2(\theta) = \tan^2(\theta) + 1.
\]
Inverse trigonometric functions (7.6)

Example
Evaluate $\sec(\arctan(-2/3))$.

Solution: We only need the relation between sec and tan,

$$\sec^2(\theta) = \tan^2(\theta) + 1.$$  

Then holds $\sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1}$. 

We conclude that $\sec(\arctan(-2/3)) = \sqrt{13}/3$. 

◁
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Evaluate \( \sec(\arctan(-2/3)) \).

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Example
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Inverse trigonometric functions (7.6)

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Inverse trigonometric functions (7.6)

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Evaluate $\sec(\arctan(-2/3))$.

Solution: We only need the relation between sec and tan,

$$\sec^2(\theta) = \tan^2(\theta) + 1.$$ 

Then holds $\sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1}$. We need to find the correct sign: $\theta = \arctan(-2/3) \in (-\pi/2, 0)$. Since $\sec(\theta) = 1/\cos(\theta)$, we conclude that $\sec(\theta) > 0$. 

$\therefore$
Inverse trigonometric functions (7.6)

Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution: We only need the relation between sec and tan,

\[
\sec^2(\theta) = \tan^2(\theta) + 1.
\]

Then holds \( \sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1} \). We need to find the correct sign: \( \theta = \arctan(-2/3) \in (-\pi/2, 0) \). Since \( \sec(\theta) = 1/\cos(\theta) \), we conclude that \( \sec(\theta) > 0 \). Hence

\[
\sec(\arctan\left(-\frac{2}{3}\right)) = \sqrt{\tan^2(\arctan\left(-\frac{2}{3}\right)) + 1}
\]

\( \triangleq \)
Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution: We only need the relation between \( \sec \) and \( \tan \),

\[
\sec^2(\theta) = \tan^2(\theta) + 1.
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Then holds \( \sec(\theta) = \pm \sqrt{\tan^2(\theta) + 1} \). We need to find the correct sign: \( \theta = \arctan(-2/3) \in (-\pi/2, 0) \). Since \( \sec(\theta) = 1/\cos(\theta) \), we conclude that \( \sec(\theta) > 0 \). Hence

\[
\sec(\arctan\left(-\frac{2}{3}\right)) = \sqrt{\tan^2(\arctan\left(-\frac{2}{3}\right)) + 1} = \sqrt{\frac{4}{9} + 1}
\]
Inverse trigonometric functions (7.6)

Example
Evaluate \( \sec(\arctan(-2/3)) \).

Solution: We only need the relation between sec and tan,

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\sec(\arctan(-\frac{2}{3})) = \sqrt{\tan^2(\arctan(-\frac{2}{3})) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.
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Inverse trigonometric functions (7.6)

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Evaluate \( \sec(\arctan(-2/3)) \).

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\[
\sec(\arctan\left(-\frac{2}{3}\right)) = \sqrt{\tan^2(\arctan\left(-\frac{2}{3}\right)) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.
\]

We conclude that \( \sec(\arctan(-2/3)) = \sqrt{13}/3 \). \( \triangleleft \)
Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.
  ▶ Solving differential equations (7.4).
  ▶ Inverse trigonometric functions (7.6).
  ▶ **Hyperbolic functions (7.7).**
  ▶ Integration techniques (8-IT).
  ▶ Integration by parts (8.1).
  ▶ Trigonometric integrals (8.2).

Section not covered:
  ▶ Trigonometric substitutions (8.3).
Hyperbolic functions (7.7)

Definition
The complete set of *hyperbolic trigonometric functions* is given by

\[
\begin{align*}
\cosh(x) &= \frac{e^x + e^{-x}}{2}, \\
\sinh(x) &= \frac{e^x - e^{-x}}{2}, \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)}, \\
\coth(x) &= \frac{\cosh(x)}{\sinh(x)}, \\
\csch(x) &= \frac{1}{\sinh(x)}, \\
\sech(x) &= \frac{1}{\cosh(x)}.
\end{align*}
\]

Theorem
*The following identities hold,*

\[
\begin{align*}
\cosh^2(x) - \sinh^2(x) &= 1, \\
\sinh(2x) &= 2\sinh(x)\cosh(x), \\
\cosh(2x) &= \cosh^2(x) + \sinh^2(x), \\
\cosh^2(x) &= \frac{1}{2}[1 + \cosh(2x)], \\
\sinh^2(x) &= \frac{1}{2}[-1 + \cosh(2x)].
\end{align*}
\]
Remark: Typical problems in this section:

(1) Prove the identities: \( \cosh^2(x) - \sinh^2(x) = 1 \), and

\[
\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad \sinh(2x) = 2 \sinh(x) \cosh(x),
\]

\[
\cosh^2(x) = \frac{1}{2} (1 + \cosh(2x)), \quad \sinh^2(x) = \frac{1}{2} (-1 + \cosh(2x)).
\]

(2) Know the derivatives and integrals of hyperbolic functions.
Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.

- Solving differential equations (7.4).
- Inverse trigonometric functions (7.6).
- Hyperbolic functions (7.7).
- Integration techniques (8-IT).
- Integration by parts (8.1).
- Trigonometric integrals (8.2).

Section not covered:

- Trigonometric substitutions (8.3).
Remark: Evaluate the following integrals:

(1) \[ \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \]
(2) \[ \int_{1}^{8} \frac{dx}{x^2 - 2x + 50}. \]
(3) \[ \int x^3 \ln(x) \, dx. \]
(4) \[ \int x^2 \, e^{2x} \, dx. \]
(5) \[ \int \frac{dx}{\sqrt{8x - x^2}}. \]
(6) \[ \int \frac{dx}{\sqrt{25 - x^2}}, \quad |x| < 5. \]
(7) \[ \int \cot^3(x) \, dx. \]
(8) \[ \int \sin^4(x) \, dx. \]
(9) \[ \int x^3 \cos(x) \, dx. \]
(10) \[ \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx. \]
(11) \[ \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx. \]
(12) \[ \int \frac{2\ln(x)}{x} \, dx. \]
Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) \[ \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \]
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Remark: Evaluate the following integrals:

(1) \[ \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \] Split the integral and do two substitutions.

(2) \[ \int_{1}^{8} \frac{dx}{x^2 - 2x + 50}. \] Complete the square

(3) \[ \int x^3 \ln(x) \, dx. \] Three integrations by parts.

(4) \[ \int x^2 e^{2x} \, dx. \] Two integrations by parts.

(5) \[ \int dx \sqrt{8x - x^2}. \] Complete the square and recall arcsin.

(6) \[ \int dx \sqrt{25 - x^2}, \ |x| < 5. \] Substitution and recall arcsin.
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(1) \[ \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \] Split the integral and do two substitutions.

(2) \[ \int_{1}^{8} \frac{dx}{x^2 - 2x + 50}. \] Complete the square and recall the arctan’.
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(1) $\int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}$. Split the integral and do two substitutions.

(2) $\int_{1}^{8} \frac{dx}{x^2 - 2x + 50}$. Complete the square and recall the arctan'.

(3) $\int x^3 \ln(x) \, dx$. Three integrations by parts.

(4) $\int x^2 e^{2x} \, dx$. Two integrations by parts.

(5) $\int \frac{dx}{\sqrt{8x - x^2}}$. Complete the square and recall arcsin'.

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(2) \[ \int_1^8 \frac{dx}{x^2 - 2x + 50} \]. Complete the square and recall the arctan’.

(3) \[ \int x^3 \ln(x) \, dx \]. Three integrations by parts.

(4) \[ \int x^2 e^{2x} \, dx \]. Two integrations by parts.

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(3) \( \int x^3 \ln(x) \, dx \). Three integrations by parts.

(4) \( \int x^2 \, e^{2x} \, dx \). Two integrations by parts.

(5) \( \int \frac{dx}{\sqrt{8x - x^2}} \). Complete the square and recall arcsin'.

(6) \( \int \frac{dx}{\sqrt{25 - x^2}}, \quad |x| < 5. \).
Remark: Evaluate the following integrals:

(1) \[ \int \frac{(1 + x)}{\sqrt{1 - 2x^2}} \, dx \] Split the integral and do two substitutions.

(2) \[ \int_1^8 \frac{dx}{x^2 - 2x + 50} \] Complete the square and recall the arctan'.

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Remark: Evaluate the following integrals:

(7) $\int \cot^3(x) \, dx$. 

(8) $\int \sin^4(x) \, dx$.

(9) $\int x^3 \cos(x) \, dx$.

(10) $\int_{\pi/2}^{\pi/4} \sqrt{1 - \cos(2x)} \, dx$.

(11) $\int_{\pi/3}^{\pi/4} \sec^2(x) \tan(x) \, dx$.

(12) $\int 2 \ln(x) \, x \, dx$. 

Write using $\sin$, $\cos$, and substitution.

Double angle formula, twice.

Integrations by parts, three times.

Double angle formula, cancel $\sqrt{}$.

Write using $\sin$ and $\cos$, and substitution.

Substitution.
Remark: Evaluate the following integrals:

(7) $\int \cot^3(x) \, dx$. Write using sin, cos

(8) $\int \sin^4(x) \, dx$. Double angle formula, twice.

(9) $\int x^3 \cos(x) \, dx$. Integrations by parts, three times.

(10) $\int_{\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx$. Double angle formula, cancel $\sqrt{\cdots}$.

(11) $\int_{\pi/3}^{\pi/4} \sec^2(x) \tan(x) \, dx$. Write using sin and cos, and substitution.

(12) $\int 2 \ln(x) x \, dx$. Substitution.
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Double angle formula, twice.

(9) $\int x^3 \cos(x) \, dx$. Integrations by parts, three times.

(10) $\int_{\pi/2}^{\pi} \sqrt{1 - \cos(2x)} \, dx$. Double angle formula, cancel $\sqrt{\cdot}$.

(11) $\int_{\pi/3}^{\pi/4} \sec^2(x) \tan(x) \, dx$. Write using sin and cos, and substitution.

(12) $\int 2 \ln(x) x \, dx$. Substitution.
Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2
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(8) $\int \sin^4(x) \, dx$. Double angle formula, twice.

(9) $\int x^3 \cos(x) \, dx$. Integrations by parts, three times.

(10) $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx$. Double angle formula, cancel $\sqrt{}$.
Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(7) \[ \int \cot^3(x) \, dx. \] Write using sin, cos and substitution.

(8) \[ \int \sin^4(x) \, dx. \] Double angle formula, twice.

(9) \[ \int x^3 \cos(x) \, dx. \] Integrations by parts, three times.

(10) \[ \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx. \] Double angle formula,
Remark: Evaluate the following integrals:

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(11) $\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx$. Write using sin and cos, and substitution.
Remark: Evaluate the following integrals:

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Remark: Evaluate the following integrals:

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Remark: Evaluate the following integrals:

(7) \( \int \cot^3(x) \, dx \). Write using sin, cos and substitution.

(8) \( \int \sin^4(x) \, dx \). Double angle formula, twice.

(9) \( \int x^3 \cos(x) \, dx \). Integrations by parts, three times.

(10) \( \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \). Double angle formula, cancel \( \sqrt{ } \).

(11) \( \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx \). Write using sin and cos, and substitution.

(12) \( \int \frac{2\ln(x)}{x} \, dx \).
Remark: Evaluate the following integrals:

(7) \( \int \cot^3(x) \, dx \). Write using sin, cos and substitution.

(8) \( \int \sin^4(x) \, dx \). Double angle formula, twice.

(9) \( \int x^3 \cos(x) \, dx \). Integrations by parts, three times.

(10) \( \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \). Double angle formula, cancel \( \sqrt{\cdot} \).

(11) \( \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx \). Write using sin and cos, and substitution.

(12) \( \int \frac{2 \ln(x)}{x} \, dx \). Substitution.
Example

Evaluate $I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}$. 

Solution:

Split the integral:

$I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int x \, dx$.

For the first integral substitute $y = \sqrt{2x}$, then $dy = \sqrt{2} \, dx$.

$I_1 = \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}x) + C$.

For the second integral substitute $u = 1 - 2x^2$, then $du = -4x \, dx$.

$I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \sqrt{1 - 2x^2} + C$.

We conclude:

$I = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}x) - \frac{1}{2} \sqrt{1 - 2x^2} + C$.

$\blacksquare$
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \)

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}. \)

For the first integral substitute \( y = \sqrt{2} x, \)
Example

Evaluate $I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}$.

Solution: Split the integral: $I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}$.

For the first integral substitute $y = \sqrt{2} x$, then $dy = \sqrt{2} \, dx$.

$I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}}$
Example
Evaluate \[ I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \]

Solution: Split the integral: 
\[ I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}. \]

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).
\[ I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} \]
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral substitute \( y = \sqrt{2} \, x \), then \( dy = \sqrt{2} \, dx \).

\[ I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \, x) + c. \]
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \)

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}. \)

For the first integral substitute \( y = \sqrt{2} x, \) then \( dy = \sqrt{2} \, dx. \)

\[
I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} x) + c.
\]

For the second integral
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).

\[
I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}x) + c.
\]

For the second integral substitute \( u = 1 - 2x^2 \),
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).

\[
I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} x) + c.
\]

For the second integral substitute \( u = 1 - 2x^2 \), then \( du = -4x \, dx \).

\[
I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}}
\]
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).

\[
I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} x) + c.
\]

For the second integral substitute \( u = 1 - 2x^2 \), then \( du = -4x \, dx \).

\[
I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{4} (2\sqrt{u}) + c.
\]
Example

Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}} \).

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}} \).

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).

\[
l_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} x) + c.
\]

For the second integral substitute \( u = 1 - 2x^2 \), then \( du = -4x \, dx \).

\[
l_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{4} (2\sqrt{u}) + c = -\frac{1}{2} \sqrt{1 - 2x^2} + c.
\]
Example

Evaluate \[ I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \]

Solution: Split the integral: \[ I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}. \]

For the first integral substitute \( y = \sqrt{2} x \), then \( dy = \sqrt{2} \, dx \).

\[ I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \, x) + c. \]

For the second integral substitute \( u = 1 - 2x^2 \), then \( du = -4x \, dx \).

\[ I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{4} (2\sqrt{u}) + c = -\frac{1}{2} \sqrt{1 - 2x^2} + c. \]

We conclude: \[ I = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \, x) - \frac{1}{2} \sqrt{1 - 2x^2} + c. \]
Example

Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}} \).
Example
Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}}. \)

Solution: Complete the square and recall arcsin'.
Example
Evaluate \[ I = \int \frac{dx}{\sqrt{8x - x^2}}. \]

Solution: Complete the square and recall \( \text{arcsin}' \).

\[ I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} \]
Example

Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}}. \)

Solution: Complete the square and recall \( \arcsin' \).

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2}} = \int \frac{dx}{\sqrt{1 - \left(\frac{x - 4}{4}\right)^2}}.
\]

Substitute \( u = \frac{x - 4}{4} \), then \( du = \frac{dx}{4} \).

\[
I = \frac{1}{4} \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + c = \arcsin\left(\frac{x - 4}{4}\right) + c.
\]
Example
Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}}. \)

Solution: Complete the square and recall arcsin'.

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]
\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}}.
\]
Example
Evaluate \[ I = \int \frac{dx}{\sqrt{8x - x^2}}. \]

Solution: Complete the square and recall arcsin'.

\[ I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}}, \]

\[ I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}. \]
Example

Evaluate \[ I = \int \frac{dx}{\sqrt{8x - x^2}}. \]

Solution: Complete the square and recall arcsin'.

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]

\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}
\]

\[
I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x - 4)/4]^2}}.
\]
Example

Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}} \).

Solution: Complete the square and recall arcsin'.

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]

\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}
\]

\[
I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x - 4)/4]^2}}.
\]

Substitute \( u = (x - 4)/4 \),
Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int \frac{dx}{\sqrt{8x - x^2}}$.

Solution: Complete the square and recall arcsin'.

$$I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},$$

$$I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}$$

$$I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x - 4)/4]^2}}.$$

Substitute $u = (x - 4)/4$, then $du = dx/4$. 
Example
Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}} \).

Solution: Complete the square and recall arcsin'.

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]

\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}.
\]

\[
I = \frac{1}{4} \int \frac{du}{\sqrt{1 - [(x - 4)/4]^2}}.
\]

Substitute \( u = (x - 4)/4 \), then \( du = dx/4 \).

\[
I = I \int \frac{du}{\sqrt{1 - u^2}}
\]
Example

Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}} \).

Solution: Complete the square and recall \( \arcsin' \).

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]
\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}
\]
\[
I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x - 4)/4]^2}}.
\]

Substitute \( u = (x - 4)/4 \), then \( du = dx/4 \).

\[
I = I \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + c
\]
Example
Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}} \).

Solution: Complete the square and recall arcsin'.

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]

\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}
\]

\[
I = \frac{1}{4} \int \frac{du}{\sqrt{1 - [(x - 4)/4]^2}}.
\]

Substitute \( u = (x - 4)/4 \), then \( du = dx/4 \).

\[
I = \frac{1}{4} \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + c \Rightarrow I = \arcsin\left(\frac{x - 4}{4}\right) + c.
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdots} \).
Sections 8-IT, 8.1, 8.2

Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{} \).

Recall: \( \sin^2(\theta) = [1 - \cos(2\theta)]/2 \).
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]

Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),
Example

Evaluate  \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{[1 - \cos(2\theta)]}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]

Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),

\[
I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x) \, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x) \, dx.
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{ \cdot } \).

Recall: \( \sin^2(\theta) = \frac{[1 - \cos(2\theta)]}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]

Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),

\[
I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x) \, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x) \, dx.
\]

\[
I = \sqrt{2} \cos(x) \bigg|_{-\pi/2}^{0} - \sqrt{2} \cos(x) \bigg|_{0}^{\pi/2}
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{[1 - \cos(2\theta)]}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]
Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),

\[
I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x) \, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x) \, dx.
\]

\[
I = \sqrt{2} \cos(x) \bigg|_{-\pi/2}^{0} - \sqrt{2} \cos(x) \bigg|_{0}^{\pi/2} = \sqrt{2}(1-0) - \sqrt{2}(0-1)
\]
Example

Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\phantom{1}} \).

Recall: \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \). Hence,

\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2 \sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]

Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),

\[
I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x) \, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x) \, dx.
\]

\[
I = \sqrt{2} \cos(x) \bigg|_{-\pi/2}^{0} - \sqrt{2} \cos(x) \bigg|_{0}^{\pi/2} = \sqrt{2}(1-0) - \sqrt{2}(0-1) = 2\sqrt{2}.
\]
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \quad k < n. \)

- The method of partial fractions.
  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \quad r_1 \neq r_2 \) (Non-repeated roots).
  - The case \( \frac{p_{(n-1)}(x)}{(x - r_1)^n} \) (Repeated roots).
  - The case \( \frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}, \quad b^2 - 4c < 0 \) (Complex roots).
  - The general case.
Integrating rational functions

Remark:
The problem is to integrate rational functions $f(x) = \frac{p_m(x)}{q_n(x)}$, where $p_m(x), q_m(x)$ are polynomials degree $m$, and $n$.

Example
Evaluate $I = \int (5x - 3) \left( x^2 - 2x - 3 \right) dx$.

Solution:
It can be proven that $(5x - 3) \left( x^2 - 2x - 3 \right) = 2x + 1 + 3x - 3$.
Then, integration is simple:
$I = 2 \ln |x + 1| + 3 \ln |x - 3| + c$.

Remark:
We now present a method to simplify functions $f(x) = \frac{p_m(x)}{q_n(x)}$, as additions of functions simpler to integrate.
Integrating rational functions

Remark:
The problem is to integrate rational functions \( f(x) = \frac{p_m(x)}{q_n(x)} \), where \( p_m(x), q_m(x) \) are polynomials degree \( m \), and \( n \).

Example
Evaluate \( I = \int \frac{(5x - 3)}{(x^2 - 2x - 3)} \, dx \).
Integrating rational functions

Remark:
The problem is to integrate rational functions $f(x) = \frac{p_m(x)}{q_n(x)}$, where $p_m(x), q_m(x)$ are polynomials degree $m$, and $n$.

Example
Evaluate $I = \int \frac{(5x - 3)}{(x^2 - 2x - 3)} \, dx$.

Solution:
It can be proven that $\frac{(5x - 3)}{(x^2 - 2x - 3)} = \frac{2}{x + 1} + \frac{3}{x - 3}$. 
Integrating rational functions

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Remark: We now present a method to simplify functions \( f(x) = \frac{p_m(x)}{q_n(x)} \), as additions of functions simpler to integrate.
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\frac{p_m(x)}{q_n(x)}$.

- **Polynomial division:** $\frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \ k < n$.

- The method of partial fractions.
  - The case $\frac{p_1(x)}{(x - r_1)(x - r_2)} \ \ r_1 \neq r_2$ (Non-repeated roots).
  - The case $\frac{p_{(n-1)}(x)}{(x - r_1)^n}$. (Repeated roots).
  - The case $\frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}, \ b^2 - 4c < 0$ (Complex roots).
  - The general case.
Polynomial division

Remark:
Before start any integration, use long division to simplify the rational function:

\[ f(x) = \frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}, \quad k < n. \]
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Before start any integration, use long division to simplify the rational function:

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Example
Verify that \( 4x^2 - 7 + 3 = 2x - 3 + 2x + 3 \).

Solution:
\( 2x - 3 + 2x + 3 = (2x - 3)(2x + 3) + 2x + 3 \).
Polynomial division

Remark:
Before start any integration, use long division to simplify the rational function:

\[ f(x) = \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \quad k < n. \]
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Example
Verify that \( \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3} \).
Polynomial division

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Example
Verify that \( \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3} \).

Solution:
\[
2x - 3 + \frac{2}{2x + 3} = \frac{(2x - 3)(2x + 3) + 2}{2x + 3}
\]
Polynomial division

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Before start any integration, use long division to simplify the rational function:
\[ f(x) = \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \quad k < n. \]

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Example
Verify that \( \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3} \).

Solution:
\[ 2x - 3 + \frac{2}{2x + 3} = \frac{(2x - 3)(2x + 3) + 2}{2x + 3} = \frac{4x^2 - 9 + 2}{2x + 3}. \]
Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3} \, dx$. 

Solution:
The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

$$\frac{4x^2 - 7}{2x + 3}$$

$$= \frac{2x}{2x + 3} + \frac{3}{2x + 3}$$

$$= \frac{2x}{2x + 3} + \frac{6}{2x + 3} - \frac{3}{2x + 3}$$

$$= \frac{2x}{2x + 3} + \frac{3}{2x + 3}$$

$$= \frac{4x^2 - 7}{2x + 3}$$

$$= \frac{2x}{2x + 3} + \frac{3}{2x + 3}$$

$$= \int \frac{4x^2 - 7}{2x + 3} \, dx = \int \frac{2x}{2x + 3} \, dx + \int \frac{3}{2x + 3} \, dx$$

$$= x - 3 \ln |2x + 3| + C.$$
Polynomial division

Example
Evaluate $I = \int \frac{4x^2 - 7}{2x + 3} \, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
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Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

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Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3} \, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

$$2x + 3) \overline{4x^2 - 7}$$
Polynomial division

Example

Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

\[
\begin{array}{c}
2x \\
\hline
2x + 3)
\end{array}
\begin{array}{c}
4x^2 \\
- 7
\end{array}
\Rightarrow \frac{2x}{2x + 3} \\ \int \frac{4x^2 - 7}{2x + 3} \, dx + \int \frac{2x}{2x + 3} \, dx
\Rightarrow I = x^2 - 3x + \ln(2x + 3) + c.
Example
Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

\[
2x \\
2x + 3 \overbrace{\quad 4x^2} - 7 \\
\quad - 4x^2 - 6x
\]

\( 4x - 3 + 2 \ln(2x + 3) + C \).
Polynomial division

Example
Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

\[
\begin{array}{c}
2x \\
2x + 3) \quad 4x^2 - 7 \\
\quad - 4x^2 - 6x \\
\quad - 6x - 7
\end{array}
\]
Polynomial division

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Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

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\[
\begin{array}{c|cc}
& 2x - 3 \\
\hline
2x + 3) & 4x^2 & - 7 \\
& - 4x^2 - 6x & \\
& - 6x - 7 & \\
\end{array}
\]
Example

Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

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\[
\begin{array}{c|c|c|c|c|c|c}
& 2x & - & 3 \\
\hline
2x & + & 3 & ) & 4x^2 & - & 7 \\
& & - & 4x^2 & - & 6x & \\
& & & - & 6x & - & 7 \\
& & & & & 6x & + & 9 \\
\end{array}
\]

\[ I = \frac{2x - 3}{2x + 3} + 2 + \ln(2x + 3) + c. \]
Polynomial division

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Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator. In this case it is convenient to do the division:

\[
\begin{array}{c}
2x + 3) \hspace{1cm} 4x^2 - 7 \\
\hspace{1cm} - 4x^2 - 6x \\
\hspace{1cm} \hspace{1cm} 6x - 7 \\
\hspace{1cm} \hspace{1cm} 6x + 9 \\
\hspace{1cm} \hspace{1cm} \hspace{1cm} 2
\end{array}
\]
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In this case it is convenient to do the division:

\[
\begin{array}{c}
2x + 3) 4x^2 - 7 \\
\underline{2x + 3 \times 4x^2} \\
- 6x - 7 \\
- 6x - 7 \\
\underline{\phantom{2x + 3 \times 4x^2}} \\
2
\end{array}
\]

\[ \Rightarrow \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}. \]
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\hline
4x^2 & - 7 \\
- 4x^2 - 6x & \\
\hline
- 6x - 7 & 6x + 9 \\
\hline
& 2 \\
\end{array}
\]

\[ \Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}. \]

\[ I = \int (2x - 3) \, dx + \int \frac{2 \, dx}{2x + 3} \]
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Evaluate \( I = \int \frac{4x^2 - 7}{2x + 3} \, dx \).

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\quad - 4x^2 - 6x \\
\quad \underline{- 6x - 7} \\
\quad 6x + 9 \\
\quad \underline{2} \\
\end{array}
\]

\[
\Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.
\]

\[
I = \int (2x - 3) \, dx + \int \frac{2 \, dx}{2x + 3} \quad \Rightarrow \quad I = x^2 - 3x + \ln(2x + 3) + c.
\]
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \ k < n. \)

- **The method of partial fractions.**
  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \) \( r_1 \neq r_2 \) (Non-repeated roots).
  - The case \( \frac{p_{(n-1)}(x)}{(x - r_1)^n} \). (Repeated roots).
  - The case \( \frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}, \ b^2 - 4c < 0 \) (Complex roots).
  - The general case.
The method of partial fractions

Remarks:

- We study rational functions \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).
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- Example: \( \frac{(5x - 3)}{(x + 1)(x - 3)} \)
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  \[
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  \]
The method of partial fractions

Remarks:

- We study rational functions \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).

- Example: \[ \frac{(5x - 3)}{(x + 1)(x - 3)} = \frac{2}{(x + 1)} + \frac{3}{(x - 3)}. \]

- The method is called of *partial fractions*
The method of partial fractions

Remarks:

▶ We study rational functions $\frac{r_k(x)}{q_n(x)}$, with $k < n$.

▶ Example: $\frac{(5x - 3)}{(x + 1)(x - 3)} = \frac{2}{(x + 1)} + \frac{3}{(x - 3)}$.

▶ The method is called of *partial fractions* because the denominators on the right-hand side above contain only part of the denominator on the left-hand side.
The method of partial fractions

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- We present the method through examples.
The method of partial fractions

Remarks:

▶ We study rational functions \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).

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▶ The method is called of \textit{partial fractions} because the denominators on the right-hand side above contain only part of the denominator on the left-hand side.

▶ We present the method through examples.

▶ We go from simpler to more complicated situations.
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \ k < n. \)

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  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \), \( r_1 \neq r_2 \) (Non-repeated roots).
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  - The general case.
The method of partial fractions (Non-repeated roots)

Example

Evaluate

\[ I = \int \frac{1}{(x - 1)(x + 2)} \, dx. \]
The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{1}{(x - 1)(x + 2)} \, dx$.

Solution: Denote $r_1 = 1$, $r_2 = -2$. 

The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1, \ r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)}
\]
The method of partial fractions (Non-repeated roots)

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Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1 \), \( r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

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\]

\[1 = a_1(x + 2) + a_2(x - 1).\]
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1, \ r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that
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\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]

\[ 1 = a_1(x + 2) + a_2(x - 1). \]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 1 \),
The method of partial fractions (Non-repeated roots)

Example
Evaluate \[ I = \int \frac{1}{(x - 1)(x + 2)} \, dx. \]

Solution: Denote \( r_1 = 1, \ r_2 = -2. \) Find \( a_1 \) and \( a_2 \) such that
\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{x - 1} + \frac{a_2}{x + 2} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]

\[ 1 = a_1(x + 2) + a_2(x - 1). \]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 1, \)
\[ 1 = a_1(3) \]
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1 \), \( r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]

1 = \( a_1(x + 2) + a_2(x - 1) \).

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 1 \),

\[
1 = a_1(3) \quad \Rightarrow \quad a_1 = \frac{1}{3}.
\]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1, \ r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]

\[ 1 = a_1(x + 2) + a_2(x - 1). \]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 1 \),

\[ 1 = a_1(3) \quad \Rightarrow \quad a_1 = \frac{1}{3}. \]

To find \( a_2 \) evaluate the equation above at the root \( r_2 = -2 \),
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Denote \( r_1 = 1, \ r_2 = -2 \). Find \( a_1 \) and \( a_2 \) such that

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.
\]

1 = \(a_1\)(x + 2) + \(a_2\)(x − 1).

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 1, \)

\[1 = a_1(3) \Rightarrow a_1 = \frac{1}{3}.\]

To find \( a_2 \) evaluate the equation above at the root \( r_2 = -2, \)

\[1 = a_2(-3).\]
The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{1}{(x - 1)(x + 2)} \, dx$.

Solution: Denote $r_1 = 1$, $r_2 = -2$. Find $a_1$ and $a_2$ such that

$$\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)} = \frac{a(x + 2) + b(x - 1)}{(x - 1)(x + 2)}.$$  

$$1 = a_1(x + 2) + a_2(x - 1).$$

To find $a_1$ evaluate the equation above at the root $r_1 = 1$,

$$1 = a_1(3) \quad \Rightarrow \quad a_1 = \frac{1}{3}.$$  

To find $a_2$ evaluate the equation above at the root $r_2 = -2$,

$$1 = a_2(-3) \quad \Rightarrow \quad a_2 = -\frac{1}{3}.$$
Example

Evaluate $I = \int \frac{1}{(x - 1)(x + 2)} \, dx$.

Solution: Recall: 

$$\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{x - 1} + \frac{a_2}{x + 2},$$

with $a_1 = \frac{1}{3}$, $a_2 = -\frac{1}{3}$. 

The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Recall: \( \frac{1}{(x - 1)(x + 2)} = \frac{a_1}{x - 1} + \frac{a_2}{x + 2} \),

with \( a_1 = \frac{1}{3}, \ a_2 = -\frac{1}{3} \). The integral is now simple to evaluate,
The method of partial fractions (Non-repeated roots)

Example

Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Recall: \( \frac{1}{(x - 1)(x + 2)} = \frac{a_1}{x - 1} + \frac{a_2}{x + 2} \),

with \( a_1 = \frac{1}{3}, \ a_2 = -\frac{1}{3} \). The integral is now simple to evaluate,

\[
I = \int \frac{1}{(x - 1)(x - 2)} \, dx = \int \frac{1}{3} \frac{1}{x - 1} \, dx - \int \frac{1}{3} \frac{1}{x + 2} \, dx
\]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \( I = \int \frac{1}{(x - 1)(x + 2)} \, dx \).

Solution: Recall:

\[
\frac{1}{(x - 1)(x + 2)} = \frac{a_1}{(x - 1)} + \frac{a_2}{(x + 2)},
\]

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We conclude that

\[
I = \frac{1}{3} \ln |x - 1| - \frac{1}{3} \ln |x + 2| + c.
\]
The method of partial fractions (Non-repeated roots)

Example
Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

Solution: First, find the zeros of the denominator,
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx \).

Solution: First, find the zeros of the denominator,

\[ x^2 - x - 2 = 0 \]
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx \).

Solution: First, find the zeros of the denominator,

\[ x^2 - x - 2 = 0 \implies x_{\pm} = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 8} \right] \]
The method of partial fractions (Non-repeated roots)

Example
Evaluate
\[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

Solution: First, find the zeros of the denominator,
\[ x^2 - x - 2 = 0 \quad \Rightarrow \quad x_{\pm} = \frac{1}{2} [1 \pm \sqrt{1 + 8}] \quad \Rightarrow \quad \begin{cases} x_+ = 2, \\ x_- = -1, \end{cases} \]
The method of partial fractions (Non-repeated roots)

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Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

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Therefore, we rewrite: \[ I = \int \frac{(x - 1)}{(x - 2)(x + 1)} \, dx. \]
The method of partial fractions (Non-repeated roots)

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Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

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Partial fraction problem: Find constants \( a_1 \) and \( a_2 \) such that
\[ \frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1}{(x - 2)} + \frac{a_2}{(x + 1)}, \quad r_1 = 2, \quad r_2 = -1. \]
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Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

Solution: First, find the zeros of the denominator,

\[ x^2 - x - 2 = 0 \quad \Rightarrow \quad x_\pm = \frac{1}{2} [1 \pm \sqrt{1 + 8}] \quad \Rightarrow \quad \begin{cases} 
    x_+ = 2, \\
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Partial fraction problem: Find constants \( a_1 \) and \( a_2 \) such that

\[ \frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1}{(x - 2)} + \frac{a_2}{(x + 1)}, \quad r_1 = 2, \quad r_2 = -1. \]

Do the addition on the right-hand side above:

\[ \frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)}. \]
The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx$.

Solution: Recall: 

$$\frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)}.$$
The method of partial fractions (Non-repeated roots)

Example

Evaluate \( \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx \).

Solution: Recall: \( \frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)} \).

The equation above implies:

\( x - 1 = a_1(x + 1) + a_2(x - 2) \)
The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{x - 1}{x^2 - x - 2} \, dx \).

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To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),
The method of partial fractions (Non-repeated roots)

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To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),

\[ 1 = a_1(3) \]
The method of partial fractions (Non-repeated roots)

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Evaluate \( I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx \).

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To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),

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The method of partial fractions (Non-repeated roots)

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x - 1 = a_1(x + 1) + a_2(x - 2)
\]
To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),
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The method of partial fractions (Non-repeated roots)

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Evaluate \( I = \int \frac{x - 1}{(x^2 - x - 2)} \, dx \).

Solution: Recall:

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\frac{x - 1}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)}.
\]

The equation above implies:

\[x - 1 = a_1(x + 1) + a_2(x - 2)\]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),

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The method of partial fractions (Non-repeated roots)

Example
Evaluate \( I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx \).

Solution: Recall: \( \frac{(x - 1)}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)} \).

The equation above implies:

\[ x - 1 = a_1(x + 1) + a_2(x - 2) \]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),

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To find \( a_2 \) evaluate the equation above at the root \( r_2 = -1 \),

\[ -2 = a_2(-3) \quad \Rightarrow \quad a_2 = \frac{2}{3}. \]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \[ I = \int \frac{x - 1}{x^2 - x - 2} \, dx. \]

Solution: Recall: \[ \frac{x - 1}{(x - 2)(x + 1)} = \frac{a_1(x + 1) + a_2(x - 2)}{(x - 2)(x + 1)}. \]

The equation above implies:
\[ x - 1 = a_1(x + 1) + a_2(x - 2) \]

To find \( a_1 \) evaluate the equation above at the root \( r_1 = 2 \),
\[ 1 = a_1(3) \Rightarrow a_1 = \frac{1}{3}. \]

To find \( a_2 \) evaluate the equation above at the root \( r_2 = -1 \),
\[ -2 = a_2(-3) \Rightarrow a_2 = \frac{2}{3}. \]

We obtain \[ \frac{x - 1}{(x - 2)(x + 1)} = \frac{1}{3} \frac{1}{x - 2} + \frac{2}{3} \frac{1}{x + 1}. \]
The method of partial fractions (Non-repeated roots)

Example

Evaluate \[ I = \int \frac{(x - 1)}{(x^2 - x - 2)} \, dx. \]

Solution: Recall: \[
\frac{(x - 1)}{(x - 2)(x + 1)} = \frac{1}{3} \frac{1}{x - 2} + \frac{2}{3} \frac{1}{x + 1}.
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The integral is now simple to evaluate,
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\]
We conclude that 
\[
I = \frac{1}{3} \ln |x - 2| + \frac{2}{3} \ln |x + 1| + c.
\]
The method of partial fractions (Non-repeated roots)

Theorem (Non-repeated roots - Heaviside cover method)

The rational function \( \frac{p_k(x)}{(x - r_1) \cdots (x - r_n)} \), with \( n > k \) and all roots \( r_1, \cdots, r_n \) different, can be written as

\[
\frac{p_k(x)}{(x - r_1) \cdots (x - r_n)} = \frac{a_1}{(x - r_1)} + \cdots + \frac{a_n}{(x - r_n)},
\]

where the constants \( a_1, \cdots, a_n \) are given by

\[
a_1 = \frac{p_k(r_1)}{\prod_{j \neq 1}(r_1 - r_j)}, \quad \cdots \quad a_n = \frac{p_k(r_n)}{\prod_{j \neq n}(r_n - r_j)}.
\]
The method of partial fractions (Non-repeated roots)

**Theorem (Non-repeated roots - Heaviside cover method)**

The rational function \[ \frac{p_k(x)}{(x - r_1) \cdots (x - r_n)} \], with \( n > k \) and all roots \( r_1, \ldots, r_n \) different, can be written as

\[ \frac{p_k(x)}{(x - r_1) \cdots (x - r_n)} = \frac{a_1}{(x - r_1)} + \cdots + \frac{a_n}{(x - r_n)}, \]

where the constants \( a_1, \ldots, a_n \) are given by

\[ a_1 = \frac{p_k(r_1)}{\prod_{j \neq 1} (r_1 - r_j)}, \quad \ldots \quad a_n = \frac{p_k(r_n)}{\prod_{j \neq n} (r_n - r_j)}. \]

**Proof:** \( p_k(x) = a_1 \left[ \prod_{j \neq 1} (x - r_j) \right] + \cdots + a_n \left[ \prod_{j \neq n} (x - r_j) \right]. \) \( \square \)
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \ k < n. \)

- **The method of partial fractions.**
  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \) \( r_1 \neq r_2 \) (Non-repeated roots).
  - **The case** \( \frac{p_{(n-1)}(x)}{(x - r_1)^n} \) (Repeated roots).
  - The case \( \frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}, \ b^2 - 4c < 0 \) (Complex roots).
  - The general case.
The method of partial fractions (Repeated roots)

Example

Evaluate \[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx. \]
The method of partial fractions (Repeated roots)

Example

Evaluate \[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx. \]

Solution: First, find the zeros of the denominator,
The method of partial fractions (Repeated roots)

Example

Evaluate \[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx. \]

Solution: First, find the zeros of the denominator,

\[ x^2 - 6x + 9 = 0 \]
The method of partial fractions (Repeated roots)

Example
Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

Solution: First, find the zeros of the denominator,

\[ x^2 - 6x + 9 = 0 \quad \Rightarrow \quad x_{\pm} = \frac{1}{2} [6 \pm \sqrt{36 - 36}] \]
The method of partial fractions (Repeated roots)

Example

Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

Solution: First, find the zeros of the denominator,

\[ x^2 - 6x + 9 = 0 \quad \Rightarrow \quad x_{\pm} = \frac{1}{2} [6 \pm \sqrt{36 - 36}] \quad \Rightarrow \quad x_{\pm} = 3. \]
The method of partial fractions (Repeated roots)

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Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

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Partial fraction problem: Find constants \( a_1 \) and \( a_2 \) such that

\[ \frac{(2x - 1)}{(x - 3)^2} = \frac{a_1}{x - 3} + \frac{a_2}{(x - 3)^2}. \]
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Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

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Partial fraction problem: Find constants \( a_1 \) and \( a_2 \) such that

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\frac{(2x - 1)}{(x - 3)^2} = \frac{a_1}{(x - 3)} + \frac{a_2}{(x - 3)^2}.
\]

Do the addition on the right-hand side above:

\[
\frac{(2x - 1)}{(x - 3)^2} = \frac{a_1(x - 3) + a_2}{(x - 3)^2}.
\]
The method of partial fractions (Repeated roots)

Example

Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

Solution: Recall: \( \frac{(2x - 1)}{(x - 3)^2} = \frac{a_1(x - 3) + a_2}{(x - 3)^2} \).
The method of partial fractions (Repeated roots)

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Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

Solution: Recall: \( \frac{(2x - 1)}{(x - 3)^2} = \frac{a_1(x - 3) + a_2}{(x - 3)^2} \). Then,

\[
2x - 1 = a_1(x - 3) + a_2.
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The method of partial fractions (Repeated roots)

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To compute \( a_2 \) evaluate the expression above at \( r = 3 \),

\[ 5 = a_2. \]
The method of partial fractions (Repeated roots)

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The method of partial fractions (Repeated roots)

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\]

To compute \( a_2 \) evaluate the expression above at \( r = 3 \),

\[
5 = a_2.
\]

To compute \( a_1 \) derivate the expression above, then evaluate at \( r = 3 \),
The method of partial fractions (Repeated roots)

Example

Evaluate \[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx. \]

Solution: Recall: \[ \frac{(2x - 1)}{(x - 3)^2} = \frac{a_1(x - 3) + a_2}{(x - 3)^2}. \] Then,

\[ 2x - 1 = a_1(x - 3) + a_2. \]

To compute \( a_2 \) evaluate the expression above at \( r = 3 \),

\[ 5 = a_2. \]

To compute \( a_1 \) derivate the expression above, then evaluate at \( r = 3 \), (the evaluation at \( r = 3 \) is not needed in this case),

\[ 2 = a_1. \]
The method of partial fractions (Repeated roots)

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Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

Solution: Recall: \( \frac{(2x - 1)}{(x - 3)^2} = \frac{a_1(x - 3) + a_2}{(x - 3)^2} \). Then,

\[ 2x - 1 = a_1(x - 3) + a_2. \]

To compute \( a_2 \) evaluate the expression above at \( r = 3 \),

\[ 5 = a_2. \]

To compute \( a_1 \) derivate the expression above, then evaluate at \( r = 3 \), (the evaluation at \( r = 3 \) is not needed in this case),

\[ 2 = a_1. \]

We conclude: \( \frac{(2x - 1)}{(x - 3)^2} = \frac{2}{(x - 3)} + \frac{5}{(x - 3)^2} \).
The method of partial fractions (Repeated roots)

Example

Evaluate \[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx. \]

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The method of partial fractions (Repeated roots)

Example
Evaluate \( I = \int \frac{2x - 1}{x^2 - 6x + 9} \, dx \).

Solution: Recall: \( \frac{2x - 1}{(x - 3)^2} = \frac{2}{x - 3} + \frac{5}{(x - 3)^2} \).

The integral is now simple to evaluate,
The method of partial fractions (Repeated roots)

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\[ I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx = \int \frac{2}{x - 3} \, dx + \int \frac{5}{(x - 3)^2} \, dx \]
The method of partial fractions (Repeated roots)

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Evaluate \( I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx \).

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I = \int \frac{(2x - 1)}{(x^2 - 6x + 9)} \, dx = \int \frac{2}{x - 3} \, dx + \int \frac{5}{(x - 3)^2} \, dx
\]

We conclude that

\[
I = 2 \ln|x - 3| - \frac{5}{(x - 3)} + c.
\]
The method of partial fractions (Repeated roots)

Theorem (Repeated roots)

The rational function \( \frac{p_k(x)}{(x - r)^n} \), with \( n > k \), can be written as

\[
\frac{p_k(x)}{(x - r)^n} = \frac{a_1}{(x - r)} + \cdots + \frac{a_n}{(x - r)^n},
\]

where \( a_i, \) for \( i = 1, \cdots, n \), is given by \( a_i = \frac{p_k^{(n-i)}(r)}{(n-i)!} \),
The method of partial fractions (Repeated roots)

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where \( a_i \), for \( i = 1, \cdots, n \), is given by \( a_i = \frac{p_k^{(n-i)}(r)}{(n-i)!} \).

Proof: Taking common denominator on the right-hand side above,

\[
p_k(x) = a_1 (x - r)^{(n-1)} + a_2(x - r)^{(n-2)} + \cdots a_{(n-1)}(x - r) + a_n,
\]
The method of partial fractions (Repeated roots)

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The rational function \( \frac{p_k(x)}{(x - r)^n} \), with \( n > k \), can be written as

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Proof: Taking common denominator on the right-hand side above,

\[
p_k(x) = a_1 (x - r)^{(n-1)} + a_2(x - r)^{(n-2)} + \cdots a_{(n-1)}(x - r) + a_n,
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\( a_n = p_k(r) \),
The method of partial fractions (Repeated roots)

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where \( a_i \), for \( i = 1, \cdots, n \), is given by \( a_i = \frac{p_k^{(n-i)}(r)}{(n-i)!} \).

Proof: Taking common denominator on the right-hand side above,

\[
p_k(x) = a_1 (x - r)^{(n-1)} + a_2(x - r)^{(n-2)} + \cdots a_{(n-1)}(x - r) + a_n,
\]

\( a_n = p_k(r) \), \( a_{(n-1)} = p'(r) \),
The method of partial fractions (Repeated roots)

Theorem (Repeated roots)

The rational function \( \frac{p_k(x)}{(x - r)^n} \), with \( n > k \), can be written as

\[
\frac{p_k(x)}{(x - r)^n} = \frac{a_1}{(x - r)} + \cdots + \frac{a_n}{(x - r)^n},
\]

where \( a_i \), for \( i = 1, \cdots, n \), is given by \( a_i = \frac{p_k^{(n-i)}(r)}{(n - i)!} \),

Proof: Taking common denominator on the right-hand side above,

\[
p_k(x) = a_1 (x - r)^{(n-1)} + a_2(x - r)^{(n-2)} + \cdots + a_{(n-1)}(x - r) + a_n,
\]

\[
a_n = p_k(r), \quad a_{(n-1)} = p'(r), \quad \cdots \quad a_2 = \frac{p^{(n-2)}(r)}{(n - 2)!}.
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The method of partial fractions (Repeated roots)

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p_k(x) = a_1 (x - r)^{(n-1)} + a_2 (x - r)^{(n-2)} + \cdots + a_{(n-1)}(x - r) + a_n,
\]

\[
a_n = p_k(r), \quad a_{(n-1)} = p'(r), \quad \cdots \quad a_2 = \frac{p^{(n-2)}(r)}{(n-2)!}, \quad a_1 = \frac{p^{(n-1)}(r)}{(n-1)!}.
\]
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)} \), \( k < n \).

- **The method of partial fractions.**
  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \) \( r_1 \neq r_2 \) (Non-repeated roots).
  - The case \( \frac{p_{(n-1)}(x)}{(x - r_1)^n} \). (Repeated roots).
  - **The case** \( \frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n} \), \( b^2 - 4c < 0 \) (Complex roots).
  - The general case.
The method of partial fractions (Complex roots)

Example

Evaluate \( I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx \).
The method of partial fractions (Complex roots)

Example

Evaluate \( I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx \).

Solution: Find constants \( a_1, b_1 \) and \( a_2, b_2 \) such that

\[
\frac{(x + 1)^2}{(x^2 + 1)^2} = \frac{(a_1 x + b_1)}{(x^2 + 1)} + \frac{(a_2 x + b_2)}{(x^2 + 1)^2}.
\]
The method of partial fractions (Complex roots)

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\frac{(x + 1)^2}{(x^2 + 1)^2} = \frac{a_1 x + b_1}{x^2 + 1} + \frac{a_2 x + b_2}{(x^2 + 1)^2}.
\]

\[
\frac{(x + 1)^2}{(x^2 + 1)^2} = \frac{(a_1 x + b_1)(x^2 + 1) + (a_2 x + b_2)}{(x^2 + 1)^2},
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The method of partial fractions (Complex roots)

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\]

\[
x^2 + 2x + 1 = a_1x^3 + a_1x + b_1x^2 + b_1 + a_2x + b_2.
\]
The method of partial fractions (Complex roots)

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\]

\[x^2 + 2x + 1 = a_1x^3 + b_1x^2 + (a_1 + a_2)x + (b_1 + b_2).
\]
The method of partial fractions (Complex roots)

Example

Evaluate $I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx$.

Solution: Recall:

$x^2 + 2x + 1 = a_1x^3 + b_1x^2 + (a_1 + a_2)x + (b_1 + b_2)$. 
The method of partial fractions (Complex roots)

Example

Evaluate \( I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx \).

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We conclude: \( a_1 = 0, \)
The method of partial fractions (Complex roots)

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Evaluate \( I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx \).

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\]
We conclude: \( a_1 = 0, \ b_1 = 1, \)
The method of partial fractions (Complex roots)

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The method of partial fractions (Complex roots)

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We conclude:  \( a_1 = 0, b_1 = 1, a_2 = 2, \) and \( b_2 = 0. \) Hence,

\[ I = \int \frac{(a_1 x + b_1)}{(x^2 + 1)} \, dx + \int \frac{(a_2 x + b_2)}{(x^2 + 1)^2} \, dx. \]
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Evaluate \[ I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx. \]

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\[ I = \int \frac{(a_1x + b_1)}{(x^2 + 1)} \, dx + \int \frac{(a_2x + b_2)}{(x^2 + 1)^2} \, dx. \]

\[ I = \int \frac{dx}{x^2 + 1} + \int \frac{2x \, dx}{(x^2 + 1)^2}. \]
The method of partial fractions (Complex roots)

Example

Evaluate \( I = \int \frac{(x + 1)^2}{(x^2 + 1)^2} \, dx \).

Solution: Recall:

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x^2 + 2x + 1 = a_1 x^3 + b_1 x^2 + (a_1 + a_2) x + (b_1 + b_2).
\]

We conclude: \( a_1 = 0, \ b_1 = 1, \ a_2 = 2, \) and \( b_2 = 0 \). Hence,

\[
I = \int \frac{(a_1 x + b_1)}{(x^2 + 1)} \, dx + \int \frac{(a_2 x + b_2)}{(x^2 + 1)^2} \, dx.
\]

\[
I = \int \frac{dx}{x^2 + 1} + \int \frac{2x \, dx}{(x^2 + 1)^2}.
\]

We conclude that \( I = \arctan(x) - \frac{1}{(x^2 + 1)} + c \). \( \triangle \)
The method of partial fractions (Complex roots)

**Theorem (Repeated roots)**

The rational function \( \frac{p_{(2n-1)}(x)}{(x + bx + c)^n} \), with \( b^2 - 4c < 0 \), can be written as

\[
p_{(2n-1)}(x) = a_1 x + b_1 + \cdots + a_n x + b_n \]

for appropriate constants \( a_i, b_1 \) for \( i = 1, \cdots, n \).
The method of partial fractions (Complex roots)

Theorem (Repeated roots)

The rational function \( \frac{p_{(2n-1)}(x)}{(x + bx + c)^n} \), with \( b^2 - 4c < 0 \), can be written as

\[
\frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n} = \frac{a_1x + b_1}{(x^2 + bx + c)} + \cdots + \frac{a_nx + b_n}{(x^2 + bx + c)^n}
\]

for appropriate constants \( a_i, b_1 \) for \( i = 1, \cdots, n \).

Idea of the Proof:
Taking common denominator on the right-hand side above,

\[
p_{(2n-1)}(x) = (a_1x + b_1)(x^2 + bx + c)^{(n-1)} + \cdots + (a_nx + b_n).
\]
The method of partial fractions (Complex roots)

Theorem (Repeated roots)

The rational function \( \frac{p(2n-1)(x)}{(x + bx + c)^n} \), with \( b^2 - 4c < 0 \), can be written as

\[
p(2n-1)(x) = \frac{a_1x + b_1}{(x^2 + bx + c)^n} + \cdots + \frac{a_nx + b_n}{(x^2 + bx + c)^n}
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for appropriate constants \( a_i, b_1 \) for \( i = 1, \cdots, n \).

Idea of the Proof:
Taking common denominator on the right-hand side above,

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\]

Expanding the equation above one can find a system of equations for the coefficients.
Integrating rational functions (Sect. 8.4)

- Integrating rational functions, \( \frac{p_m(x)}{q_n(x)} \).

- Polynomial division: \( \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \ k < n. \)

- The method of partial fractions.
  - The case \( \frac{p_1(x)}{(x - r_1)(x - r_2)} \), \( r_1 \neq r_2 \) (Non-repeated roots).
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  - The case \( \frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}, \ b^2 - 4c < 0 \) (Complex roots).
  - The general case.
Remarks:

- Consider a general rational function \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).
The method of partial fractions (General case)

Remarks:

- Consider a general rational function \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).

- Express the denominator, \( q \), as a product of factors \( (x - r_i)^{m_i} \) and \( (x^2 + b_i x + c_i)^{\ell_i} \), with \( r_i \) roots of \( q_n \), and \( b_i^2 - 4c_i < 0 \).
The method of partial fractions (General case)

Remarks:

▸ Consider a general rational function \( \frac{r_k(x)}{q_n(x)} \), with \( k < n \).

▸ Express the denominator, \( q \), as a product of factors \( (x - r_i)^{m_i} \) and \( (x^2 + b_i x + c_i)^{\ell_i} \), with \( r_i \) roots of \( q_n \), and \( b_i^2 - 4c_i < 0 \).

▸ The partial fraction decomposition for \( \frac{r_k}{q_n} \) is the addition of the partial fraction decomposition for each factor in \( q \).
The method of partial fractions (General case)

Example

Evaluate \[ I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx. \]
The method of partial fractions (General case)

Example
Evaluate \( I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \ dx. \)

Solution: The partial fraction decomposition is:

\[
\frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} = \frac{(ax + b)}{(x^2 + 1)} + \frac{c}{(x - 2)} + \frac{d}{x}
\]
The method of partial fractions (General case)

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\]

\[
6x^3 - 8x^2 + 5x - 6 = (ax + b)(x - 2)x + c(x^2 + 1)x + d(x^2 + 1)(x - 2)
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The method of partial fractions (General case)

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\]

\[
= ax^3 - 2ax^2 + bx^2 - 2bx + cx^3 + cx + dx^3 - 2dx^2 + dx - 2d
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The method of partial fractions (General case)

Example

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\[ = (a + c + d)x^3 + (-2a + b - 2d)x^2 + (-2b + c + d)x - 2d \]
The method of partial fractions (General case)

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Evaluate

\[ I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx. \]

Solution: The partial fraction decomposition is:

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\[ = (a + c + d)x^3 + (-2a + b - 2d)x^2 + (-2b + c + d)x - 2d \]

\[ a + c + d = 6, \quad -2a + b - 2d = -8, \quad 5 = -2b + c + d \quad d = 3. \]
The method of partial fractions (General case)

Example

Evaluate

\[ I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx. \]

Solution: Recall:

\[ a + c + d = 6, \quad -2a + b - 2d = -8, \quad 5 = -2b + c + d \quad d = 3. \]
The method of partial fractions (General case)

Example

Evaluate \( I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx. \)

Solution: Recall:

\[ a + c + d = 6, \quad -2a + b - 2d = -8, \quad 5 = -2b + c + d \quad d = 3. \]

\[ a + c = 3, \quad 2a - b = 2, \quad -2b + c = 2. \]
The method of partial fractions (General case)

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Evaluate \( I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx \).

Solution: Recall:
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\begin{align*}
    a + c + d &= 6, & -2a + b - 2d &= -8, & 5 &= -2b + c + d & d &= 3. \\
    a + c &= 3, & 2a - b &= 2, & -2b + c &= 2. \\
    c &= 3 - a
\end{align*}
\]

\( I = \frac{1}{2} \ln(x^2 + 1) + 2 \ln|x - 2| + 3 \ln|x| + c \).
The method of partial fractions (General case)

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Evaluate \( I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx \).

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\[
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  a + c &= 3, \\
  2a - b &= 2, \\
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\end{align*}
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\[
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Hence \( b = 0 \), and then \( a = 1 \) and \( c = 2 \).
The method of partial fractions (General case)

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\end{align*}
\]

Hence \( b = 0 \), and then \( a = 1 \) and \( c = 2 \). We conclude,

\[
I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx = \int \left[ \frac{x}{(x^2 + 1)} + \frac{2}{x - 2} + \frac{3}{x} \right] \, dx
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Evaluate \[ I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx. \]

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Hence \( b = 0 \), and then \( a = 1 \) and \( c = 2 \). We conclude,
\[ I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2 + 1)(x - 2)x} \, dx = \int \left[ \frac{x}{(x^2 + 1)} + \frac{2}{(x - 2)} + \frac{3}{x} \right] \, dx \]
\[ I = \frac{1}{2} \ln(x^2 + 1) + 2 \ln |x - 2| + 3 \ln |x| + c. \]