Area of regions in polar coordinates (Sect. 11.5)

- Review: Few curves in polar coordinates.
- Formula for the area or regions in polar coordinates.
- Calculating areas in polar coordinates.

Transformation rules Polar-Cartesian.

Definition
The **polar coordinates** of a point $P \in \mathbb{R}^2$ is the ordered pair $(r, \theta)$, with $r > 0$ and $\theta \in [0, 2\pi)$ defined by the picture.

Example
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**Formula for the area or regions in polar coordinates**

**Theorem**

*If the functions $r_1, r_2 : [\alpha, \beta] \rightarrow \mathbb{R}$ are continuous and $0 \leq r_1 \leq r_2$, then the area of a region $D \subset \mathbb{R}^2$ given by*

$$D = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_1(\theta), r_2(\theta]], \ \theta \in [\alpha, \beta]\}.$$  

*is given by the integral*

$$A(D) = \int_{\alpha}^{\beta} \frac{1}{2} \left( [r_2(\theta)]^2 - [r_1(\theta)]^2 \right) d\theta.$$  

**Remark:** This result includes the case of $r_1 = 0$, which are fan-shaped regions.
Formula for the area or regions in polar coordinates

Idea of the Proof: Introduce a partition $\theta_k = k \Delta \theta$, with $k = 1, \cdots, n$, and $\Delta \theta = \frac{\beta - \alpha}{n}$

The area of each fan-shaped region on the figure is,

$$A_k = \frac{1}{2} [r(\theta_k)]^2 \Delta \theta.$$

A Riemann sum that approximates the green region area is

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} [r(\theta_k)]^2 \Delta \theta.$$

Refining the partition and taking a limit $n \to \infty$ one can prove that the Riemann sum above converges and the limit is called

$$A(D) = \int_{\alpha}^{\beta} \frac{1}{2} [r(\theta)]^2 \, d\theta.$$

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Calculating areas in polar coordinates

Example
Find the area inside the circle $r = 1$ and outside the cardioid $r = 1 - \sin(\theta)$.

Solution:

The Theorem implies

$$A = \int_{\alpha}^{\beta} \frac{1}{2} \left(1 - \left[1 - \sin(\theta)\right]^2\right) d\theta.$$

We need to find $\alpha$ and $\beta$. They are the intersection of the circle and the cardioid:

$$1 = 1 - \sin(\theta) \quad \Rightarrow \quad \sin(\theta) = 0 \quad \Rightarrow \quad \begin{cases} \alpha = 0, \\ \beta = \pi. \end{cases}$$
Calculating areas in polar coordinates

Example
Find the area of the intersection of the interior of the regions bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first review that these curves are actually circles. 

$$r = \cos(\theta) \iff r^2 = r \cos(\theta) \iff x^2 + y^2 = x.$$ 

Completing the square in $x$ we obtain 

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$ 

Analogously, $r = \sin(\theta)$ is the circle 

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

Also works: 

$$A = \int_0^{\pi/4} \frac{1}{2} \sin^2(\theta) \, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} \cos^2(\theta) \, d\theta.$$