Graphing in polar coordinates (Sect. 11.4)

- Review: Polar coordinates.
- Review: Transforming back to Cartesian.
- Computing the slope of tangent lines.
- Using symmetry to graph curves.
- Examples:
  - Circles in polar coordinates.
  - Graphing the Cardiod.
  - Graphing the Lemniscate.

Transformation rules Polar-Cartesian.

Definition
The **polar coordinates** of a point $P \in \mathbb{R}^2$ is the ordered pair $(r, \theta)$, with $r \geq 0$ and $\theta \in [0, 2\pi)$ defined by the picture.

Theorem (Cartesian-polar transformations)
The Cartesian coordinates of a point $P = (r, \theta)$ are given by

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ in the first and fourth quadrants are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \left( \frac{y}{x} \right).$$
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**Review: Transforming back to Cartesian**

**Example**

Find the equation of the curve in Cartesian coordinates for $r = 4 \cos(\theta)$, for $\theta \in [-\pi/2, \pi/2]$.

**Solution:** Multiply by $r$ the whole equation, $r^2 = 4r \cos(\theta)$.

Recall: $x = r \cos(\theta)$, and $y = r \sin(\theta)$, therefore $x^2 + y^2 = r^2$,

$$x^2 + y^2 = 4x \quad \Rightarrow \quad x^2 - 4x + y^2 = 0.$$  

Complete the square:

$$[x^2 - 2\left(\frac{4}{2}\right)x + 4] - 4 + y^2 = 0$$

$$(x - 2)^2 + y^2 = 4.$$ 

This is the equation of a circle radius $r = 2$ with center at $(2, 0)$.  △
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Computing the slope of tangent lines

Recall: The slope of the line tangent to the curve \( y = f(x) \), can be written in terms of \((x(t), y(t))\) as follows

\[
\frac{df}{dx} = \frac{dy/dt}{dx/dt}.
\]

Remark: If the curve is given in polar coordinates, \( r = r(\theta) \), then

\[
x(\theta) = r(\theta) \cos(\theta) \quad y(\theta) = r(\theta) \sin(\theta).
\]

The formula for the slope is then

\[
\frac{df}{dx} = \frac{y'(\theta)}{x'(\theta)} \Rightarrow \frac{df}{dx} = \frac{r'(\theta) \sin(\theta) + r(\theta) \cos(\theta)}{r'(\theta) \cos(\theta) - r(\theta) \sin(\theta)}.
\]

If the curve passes through the origin, \( r(\theta_0) = 0 \), then

\[
\frac{df}{dx} \bigg|_{\theta_0} = \frac{r'(\theta_0) \sin(\theta_0)}{r'(\theta_0) \cos(\theta_0)}.
\]
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Using symmetry to graph curves

**Remark:** If a curve is symmetric under reflections about the **x-axis**, or the **y-axis**, or the **origin**, then the work needed to graph of the curve can be reduced.

- **x-axis symmetry:** \((r, \theta)\) and \((r, -\theta)\) belong to the graph.
- **Origin symmetry:** \((r, \theta)\) and \((-r, \theta)\) belong to the graph.
- **y-axis symmetry:** \((r, \theta)\) and \((-r, -\theta)\) belong to the graph.
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Circles in polar coordinates

**Remark:** Circles centered at the origin are trivial to graph.

**Example**
Graph the curve \( r = 2, \quad \theta \in [0, 2\pi) \).

**Solution:** Back to Cartesian:

**Remark:** Circles not centered at the origin are more complicated to graph.

**Example**
Graph the curve \( r = 4 \cos(\theta), \quad \theta \in [0, 2\pi) \).

**Solution:** Back to Cartesian:
Circles in polar coordinates

Remark: We now use the graph of the function \( r = 4 \cos(\theta) \) to
graph the curve \( r = 4 \cos(\theta) \) in the \( xy \)-plane.

Example
Graph the curve \( r = 4 \cos(\theta) \), \( \theta \in [0, 2\pi) \).

Solution:
Notice that \( r(\theta) = r(-\theta) \).
(Reflection about \( x \)-axis symmetry.)
The graph of \( r = 4 \cos(\theta) \) is

The graph above helps to do the curve on the
\( xy \)-plane. We actually cover the circle twice!

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Graphing the Cardiod

Example
Graph on the $xy$-plane the curve $r = 1 - \cos(\theta)$, $\theta \in [0, 2\pi)$.

Solution: We first graph the function $r = 1 - \cos(\theta)$.

From the previous graph we obtain the curve: on the $xy$-plane:

Graphing the Cardiod

Example
Graph on the $xy$-plane the curve $r = 1 + \cos(\theta)$, $\theta \in [0, 2\pi)$.

Solution: We first graph the function $r = 1 + \cos(\theta)$.

From the previous graph we obtain the curve: on the $xy$-plane:
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Graphing the Lemniscate

Example
Graph on the xy-plane the curve \( r^2 = \sin(2\theta), \ \theta \in [0, 2\pi) \).

Solution: We first graph the function \( r = \pm \sqrt{\sin(2\theta)} \).

From the previous graph we obtain the curve: on the xy-plane: