Infinite sequences (Sect. 10.1)

Today’s Lecture:
- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- The Sandwich Theorem for sequences.

Next Lecture:
- The Continuous Function Theorem for sequences.
- Using L'Hôpital’s rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

Overview: Sequences, series, and calculus

Remarks:
- We have defined the $\int_a^b f(x) \, dx$ as a limit of partial sums. That is, as an infinite sum of numbers (areas of rectangles).
- In the next section we define, precisely, what is an infinite sum. Infinite sums are called series.
- In this section we introduce the idea of an infinite sequence of numbers. We will use sequences to define series.
- Later on, the idea of infinite sums will be generalized from numbers to functions.
- We will express differentiable functions as infinite sums of polynomials (Taylor series expansions).
- Then we will be able to compute integrals like $\int_a^b e^{-x^2} \, dx$. 
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Definition and geometrical representations

Definition
An infinite sequence of numbers is an ordered set of real numbers.

Remark: A sequence is denoted as
\[\{a_1, a_2, a_3, \ldots, a_n, \ldots\}\text{, or } \{a_n\}_{n=1}^\infty, \text{ or } \{a_n\}.\]

Example
\[\left\{\frac{n}{n+1}\right\}_{n=1}^\infty, \quad a_n = \frac{n}{n+1}, \quad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}.\]
\[\left\{(-1)^n \sqrt{n}\right\}_{n=3}^\infty, \quad a_n = (-1)^n \sqrt{n}, \quad \{-\sqrt{3}, \sqrt{4}, -\sqrt{5}, \ldots\}.\]
\[\left\{\cos(n\pi/6)\right\}_{n=0}^\infty, \quad a_n = \cos(n\pi/6), \quad \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots\right\}.\]
Definition and geometrical representations

Example
Find a formula for the general term of the sequence
\[ \left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots \right\}. \]

Solution: We know that:
\[ a_1 = \frac{3}{5}, \quad a_2 = -\frac{4}{25}, \quad a_3 = \frac{5}{125}, \quad a_4 = -\frac{6}{625}. \]
\[ a_1 = \frac{(1 + 2)}{5}, \quad a_2 = -\frac{(2 + 2)}{5^2}, \quad a_3 = \frac{(3 + 2)}{5^3}, \quad a_4 = -\frac{(4 + 2)}{5^4}. \]

We conclude that \( a_n = (-1)^{n-1} \frac{(n+2)}{5^n}. \)

Remark:
Infinite sequences can be represented on a line or on a plane.

Example
Graph the sequence \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \) on a line and on a plane.

Solution:
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The limit of a sequence, convergence, divergence

**Remark:**
- As it happened in the example above, the numbers $a_n$ in a sequence may approach a single value as $n$ increases.

$$
\left\{ a_n = \frac{1}{n} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right\} \rightarrow 0.
$$

- This is not the case for every sequence. The sequence elements may grow unbounded:

$$
\left\{ n^2 \right\}_{n=1}^{\infty} = \left\{ 1, 4, 9, 16, \cdots \right\}.
$$

The sequence numbers may oscillate:

$$
\left\{ (-1)^n \right\}_{n=0}^{\infty} = \left\{ 1, -1, 1, -1, 1, \cdots \right\}.
$$
The limit of a sequence, convergence, divergence

Definition
An infinite sequence \( \{a_n\} \) has limit \( L \) iff for every number \( \epsilon > 0 \) there exists a positive integer \( N \) such that
\[
N < n \implies |a_n - L| < \epsilon.
\]

A sequence is called convergent iff it has a limit, otherwise it is called divergent.

Remark: We use the notation \( \lim_{n \to \infty} a_n = L \) or \( a_n \to L \).

Example
Find the limit of the sequence \( \{a_n = 1 + \frac{3}{n^2}\}_{n=1}^{\infty} \).

Solution: Since \( \frac{1}{n^2} \to 0 \), we will prove that \( L = 1 \).

\[
|a_n - 1| < \epsilon \iff \left| \frac{3}{n^2} \right| < \epsilon \iff \frac{3}{\epsilon} < n^2 \iff \sqrt{\frac{3}{\epsilon}} < n.
\]

Therefore, given \( \epsilon > 0 \), choose \( N = \sqrt{\frac{3}{\epsilon}} \).

We then conclude that for all \( n > N \) holds,
\[
\sqrt{\frac{3}{\epsilon}} < n \iff \frac{3}{\epsilon} < n^2 \iff \left| \frac{3}{n^2} \right| < \epsilon \iff |a_n - 1| < \epsilon.
\]
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Properties of sequence limits

Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.

Theorem (Limit properties)

If the sequence \(\{a_n\} \to A\) and \(\{b_n\} \to B\), then holds,

(a) \(\lim_{n \to \infty} \{a_n + b_n\} = A + B\);

(b) \(\lim_{n \to \infty} \{a_n - b_n\} = A - B\);

(c) \(\lim_{n \to \infty} \{ka_n\} = kA\);

(d) \(\lim_{n \to \infty} \{a_nb_n\} = AB\);

(e) If \(B \neq 0\), then \(\lim_{n \to \infty} \left\{ \frac{a_n}{b_n} \right\} = \frac{A}{B}\).
Properties of sequence limits

Example
Find the limit of the sequence \( \left\{ a_n = \frac{1 - 2n}{2 + 3n} \right\}_{n=1}^{\infty} \).

Solution: We use the properties above to find the limit.

Rewrite the sequence as follows,

\[
a_n = \frac{(1 - 2n) \left( \frac{1}{n} \right)}{(2 + 3n) \left( \frac{1}{n} \right)} = \frac{\frac{1}{n} - 2}{\frac{2}{n} + 3}.
\]

Since \( \frac{1}{n} \to 0 \) as \( n \to \infty \), then

\[
\frac{1}{n} - 2 \to -2, \quad \frac{2}{n} \to 0, \quad \frac{2}{n} + 3 \to 3.
\]

Hence, the quotient property implies \( a_n \to -\frac{2}{3} \). \( \triangleright \)

Properties of sequence limits

Example
Find the limit of the sequence \( \left\{ a_n = \frac{3n^3 - 2n + 1}{2n^2 + 4} \right\}_{n=1}^{\infty} \).

Solution: Rewrite the sequence as follows,

\[
a_n = \frac{(3n^3 - 2n + 1) \left( \frac{1}{n^2} \right)}{(2n^2 + 4) \left( \frac{1}{n^2} \right)} = \frac{3n - \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{4}{n^2}}.
\]

Since \( \frac{1}{n} \to 0 \) as \( n \to \infty \), then

\[
\frac{1}{n^2} = \left( \frac{1}{n} \right)^2 \to 0, \quad \frac{2}{n} \to 0, \quad 2 + \frac{4}{n^2} \to 2.
\]

Hence, the quotient property implies \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3n}{2} \).

We conclude that \( a_n \) diverges. \( \triangleright \)
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The Sandwich Theorem for sequences

**Theorem (Sandwich-Squeeze)**

*If the sequences* \( \{a_n\}, \{b_n\}, \text{ and } \{c_n\} \) *satisfy*

\[
a_n \leq b_n \leq c_n, \quad \text{for } n > N,
\]

*and if both* \( a_n \to L \) *and* \( c_n \to L \), *then holds*

\[
b_n \to L.
\]

**Example**

Find the limit of the sequence \( \{a_n = \frac{\sin(3n)}{n^2}\}_{n=1}^{\infty} \).

**Solution:** Since \( |\sin(3n)| \leq 1 \), then

\[
|a_n| = \left| \frac{\sin(3n)}{n^2} \right| \leq \left| \frac{1}{n^2} \right| = \frac{1}{n^2} \quad \Rightarrow \quad -\frac{1}{n^2} \leq a_n \leq \frac{1}{n^2}.
\]

Since \( \pm \frac{1}{n^2} \to 0 \), we conclude that \( a_n \to 0 \). \( \triangleq \)