Limits using L'Hôpital’s Rule (Sect. 7.5)

- Review: L'Hôpital’s rule for indeterminate limits $\frac{0}{0}$.
- Indeterminate limit $\frac{\infty}{\infty}$.
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- Overview of improper integrals (Sect. 8.7).

L'Hôpital’s rule for indeterminate limits $\frac{0}{0}$

Remarks:
- L'Hôpital’s rule applies on limits of the form $L = \lim_{x \to a} \frac{f(x)}{g(x)}$ in the case that both $f(a) = 0$ and $g(a) = 0$.
- These limits are called indeterminate and denoted as $\frac{0}{0}$.

Theorem

If functions $f, g : I \to \mathbb{R}$ are differentiable in an open interval containing $x = a$, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in I - \{a\}$, then holds

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right-hand side exists.
L'Hôpital’s rule for indeterminate limits $\frac{0}{0}$

Example
Evaluate $L = \lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\frac{0}{0}$. But,

$$L = \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - (1/2)}{2x}.$$ 

The limit on the right-hand side is still indeterminate, $\frac{0}{0}$. We use L'Hôpital's rule for a second time,

$$L = \lim_{x \to 0} \frac{(-1/4)(1 + x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$ 

We conclude that $L = -\frac{1}{8}$. △

Remark: L'Hôpital’s rule applies to indeterminate limits only.

Example
Evaluate $L = \lim_{x \to 0} \frac{1 - \cos(x)}{x + x^2}$.

Solution: The limit is indeterminate $\frac{0}{0}$. L'Hôpital’s rule implies,

$$L = \lim_{x \to 0} \frac{1 - \cos(x)}{x + x^2} = \lim_{x \to 0} \frac{\sin(x)}{1 + 2x} = \frac{0}{1} \Rightarrow L = 0.$$

Remark:
- The limit $\frac{0}{1}$ is not indeterminate, since $\frac{0}{1} = 0$.
- Therefore, L'Hôpital’s rule does not hold in this case:

$$\lim_{x \to 0} \frac{\sin(x)}{1 + 2x} \neq \lim_{x \to 0} \frac{(\sin(x))'}{(1 + 2x)'} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$
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Indeterminate limit $\frac{\infty}{\infty}$

**Remark:** L'Hôpital's rule can be generalized to limits $\frac{\infty}{\infty}$, and also to side limits.

**Example**

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{2 + \tan(x)}{3 + \sec(x)}$.

**Solution:** This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital’s rule implies

\[
\lim_{x \to (\frac{\pi}{2})^-} \frac{(2 + \tan(x))'}{(3 + \sec(x))'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x) \tan(x)} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}
\]

Since $\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x) \sin(x)} = \frac{1}{\sin(x)}$, then $L = 1$. △
Remark: Sometimes L’Hôpital’s rule is not useful.

Example
Evaluate \( L = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec(x)}{\tan(x)}. \)

Solution: We know that this limit can be computed simplifying:
\[
\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x) \sin(x)} = \frac{1}{\sin(x)} \quad \Rightarrow \quad L = 1.
\]

We now try to compute this limit using L’Hôpital’s rule.

Example
Evaluate \( L = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec(x)}{\tan(x)}. \)

Solution: This is an indeterminate limit \( \frac{\infty}{\infty}. \) L’Hôpital’s rule implies
\[
L = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{(\sec(x))'}{(\tan(x))'} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec(x) \tan(x)}{\sec^2(x)} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\tan(x)}{\sec(x)}.
\]
The later limit is once again indeterminate, \( \frac{\infty}{\infty}. \) Then
\[
L = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{(\tan(x))'}{(\sec(x))'} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec^2(x)}{\sec(x) \tan(x)} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec(x)}{\tan(x)}.
\]
L’Hôpital’s rule gives us a cycling expression.
Example
Evaluate
\[ L = \lim_{x \to \infty} \frac{3x^2 - 5}{2x^2 - x + 3}. \]

Solution: This is an indeterminate limit \( \frac{\infty}{\infty} \). L’Hospital’s rule implies
\[
L = \lim_{x \to \infty} \frac{(3x^2 - 5)'}{(2x^2 - x + 3)'} = \lim_{x \to \infty} \frac{6x}{4x - 1} = \lim_{x \to \infty} \left( \frac{6}{4 - \frac{1}{x}} \right).
\]
Recalling \( \lim_{x \to \infty} \frac{1}{x} = 0 \), we get that \( L = \frac{6}{4} \). We conclude that
\[
\lim_{x \to \infty} \frac{3x^2 - 5}{2x^2 - x + 3} = \frac{3}{2}.
\]

Limits using L’Hospital’s Rule (Sect. 7.5)

- Review: L’Hospital’s rule for indeterminate limits \( \frac{0}{0} \).
- Indeterminate limit \( \frac{\infty}{\infty} \).
- **Indeterminate limits** \( \infty \cdot 0 \) and \( \infty - \infty \).
- Overview of improper integrals (Sect. 8.7).
Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

**Remark:** Sometimes limits of the form $\infty \cdot 0$ and $(\infty - \infty)$ can be converted by algebraic identities into indeterminate limits $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**Example**

Evaluate $L = \lim_{x \to 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right)$.

**Solution:** This is a limit of the form $(\infty - \infty)$. Since

$$\frac{1}{\sin(x)} - \frac{1}{x} = \frac{x - \sin(x)}{x \sin(x)} \Rightarrow \text{indeterminate} \frac{0}{0}.$$

Then L'Hôpital's rule in this case implies

$$L = \lim_{x \to 0} \frac{(x - \sin(x))'}{(x \sin(x))'} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)}.$$

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

**Example**

Evaluate $L = \lim_{x \to 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right)$.

**Solution:** Recall $L = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)}$.

This limit is still indeterminate $\frac{0}{0}$. Hence

$$L = \lim_{x \to 0} \frac{(1 - \cos(x))'}{(\sin(x) + x \cos(x))'} = \lim_{x \to 0} \frac{\sin(x)}{2 \cos(x) - x \sin(x)}' = \frac{0}{2} = 0.$$

We conclude that $L = 0$.  

\[\blacktriangleleft\]
Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

Example
Evaluate $L = \lim_{x \to \infty} (3x)^{2/x}$.

Solution: This limits is of the form $\infty^0$. So, before using L’Hôpital’s rule we need to rewrite the function above.

$$(3x)^{2/x} = e^{\ln((3x)^{2/x})} = e^{\left(\frac{2}{x} \ln(3x)\right)}.$$  

Since exp is a continuous function, holds

$$\lim_{x \to \infty} (3x)^{2/x} = e^{\lim_{x \to \infty} \left(\frac{2}{x} \ln(3x)\right)} = e^{\lim_{x \to \infty} \left(\frac{2 \ln(3x)}{x}\right)}.$$  

The exponent, is an indeterminate limit $\infty/\infty$. L’Hôpital’s rule implies

$$\lim_{x \to \infty} \frac{2 \ln(3x)}{x} = \lim_{x \to \infty} \frac{\left(2 \ln(3x)\right)'}{(x)'} = \lim_{x \to \infty} \frac{2}{x} = 0.$$  

We conclude that $L = e^0$, that is, $L = 1$.  

Limits using L’Hôpital’s Rule (Sect. 7.5)

- Review: L’Hôpital’s rule for indeterminate limits $0/0$.
- Indeterminate limit $\infty/\infty$.
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- Overview of improper integrals (Sect. 8.7).
Overview of improper integrals (Sect. 8.7)

Remarks:
- L'Hôpital's rule is useful to compute improper integrals.
- Improper integrals are the limit of definite integrals when one endpoint if integration approaches ±∞.

Definition
The improper integral of a continuous function \( f : [a, \infty) \rightarrow \mathbb{R} \) is
\[
\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx.
\]
The improper integral of a continuous function \( f : (-\infty, b] \rightarrow \mathbb{R} \) is
\[
\int_{-\infty}^b f(x) \, dx = \lim_{a \to -\infty} \int_a^b f(x) \, dx.
\]
The improper integral of a continuous function \( f : (-\infty, \infty) \rightarrow \mathbb{R} \),
\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx.
\]

Example
Evaluate \( I = \int_1^{\infty} \frac{\ln(x)}{x^2} \, dx \).

Solution: This is an improper integral:
\[
\int_1^{\infty} \frac{\ln(x)}{x^2} \, dx = \lim_{b \to \infty} \int_1^{b} \frac{\ln(x)}{x^2} \, dx
\]
Integrating by parts, \( u = \ln(x) \), and \( dv = dx/x^2 \),
\[
\int_1^{b} \frac{\ln(x)}{x^2} \, dx = \left( -\frac{1}{x} \right) \ln(x) \bigg|_1^b - \int_1^{b} \left( \frac{1}{x} \right) \left( -\frac{1}{x} \right) \, dx
\]
\[
\int_1^{b} \frac{\ln(x)}{x^2} \, dx = -\frac{\ln(b)}{b} + \int_1^{b} \frac{dx}{x^2} = -\frac{\ln(b)}{b} - \frac{1}{x} \bigg|_1^b.
\]
Overview of improper integrals (Sect. 8.7)

Example
Evaluate \( I = \int_1^\infty \frac{\ln(x)}{x^2} \, dx \).

Solution: Recall: \( \int_1^\infty \frac{\ln(x)}{x^2} \, dx = \lim_{b \to \infty} \left(-\frac{\ln(b)}{b} - \frac{1}{b} + 1\right) \).

The first limit on the right-hand side is indeterminate \( \frac{\infty}{\infty} \).

L’Hôpital’s rule implies
\[
\lim_{b \to \infty} \frac{\ln(b)}{b} = \lim_{b \to \infty} \frac{(\ln(b))'}{(b)'} = \lim_{b \to \infty} \frac{1}{b} = 0.
\]

Therefore, the improper integral is given by
\[
\int_1^\infty \frac{\ln(x)}{x^2} \, dx = 0 - 0 + 1 \Rightarrow \int_1^\infty \frac{\ln(x)}{x^2} \, dx = 1. \triangleq