Limits using L'Hôpital's Rule (Sect. 7.5)

- Review: L'Hôpital's rule for indeterminate limits $\frac{0}{0}$.
- Indeterminate limit $\frac{\infty}{\infty}$.
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- Overview of improper integrals (Sect. 8.7).

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Remarks:

- L'Hôpital's rule applies on limits of the form $L = \lim_{x \to a} \frac{f(x)}{g(x)}$ in the case that both $f(a) = 0$ and $g(a) = 0$.
- These limits are called indeterminate and denoted as $\frac{0}{0}$.

Theorem

If functions $f, g : I \to \mathbb{R}$ are differentiable in an open interval containing $x = a$, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in I - \{a\}$, then holds

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right-hand side exists.

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Example

Evaluate $L = \lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\dfrac{0}{0}$. But,

$$L = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x}.$$

The limit on the right-hand side is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a second time,

$$L = \lim_{x\to 0} \frac{(-1/4)(1+x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$

We conclude that $L = -\dfrac{1}{8}$. ◁

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Remark: L'Hôpital's rule applies to indeterminate limits only.

Example

Evaluate $L = \lim_{x\to 0} \dfrac{1-\cos(x)}{x+x^2}$.

Solution: The limit is indeterminate $\dfrac{0}{0}$. L'Hôpital's rule implies,

$$L = \lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} = \lim_{x\to 0} \frac{\sin(x)}{1+2x} = \frac{0}{1} \quad \Rightarrow \quad L = 0.$$ ◁

Remark:

- The limit $\dfrac{0}{1}$ is not indeterminate, since $\dfrac{0}{1} = 0$.
- Therefore, L'Hôpital's rule does not hold in this case:

$$\lim_{x\to 0} \frac{\sin(x)}{1+2x} \neq \lim_{x\to 0} \frac{(\sin(x))'}{(1+2x)'} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

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Indeterminate limit $\frac{\infty}{\infty}$

Remark: L'Hôpital's rule can be generalized to limits $\frac{\infty}{\infty}$, and also to side limits.

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{2 + \tan(x)}{3 + \sec(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$\lim_{x \to (\frac{\pi}{2})^-} \frac{\big(2 + \tan(x)\big)'}{\big(3 + \sec(x)\big)'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$$

Since $\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x)} \frac{\cos(x)}{\sin(x)} = \frac{1}{\sin(x)}$, then $L = 1$. $\triangleleft$

Indeterminate limit $\frac{\infty}{\infty}$

Remark: Sometimes L'Hôpital's rule is not useful.

Example

Evaluate $L = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: We know that this limit can be computed simplifying:

$$\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x)}\frac{\cos(x)}{\sin(x)} = \frac{1}{\sin(x)} \quad \Rightarrow \quad L = 1. \qquad \triangleleft$$

We now try to compute this limit using L'Hôpital's rule.

Indeterminate limit $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$L = \lim_{x\to(\frac{\pi}{2})^-} \frac{(\sec(x))'}{(\tan(x))'} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\tan(x)}{\sec(x)}.$$

The later limit is once again indeterminate, $\frac{\infty}{\infty}$. Then

$$L = \lim_{x\to(\frac{\pi}{2})^-} \frac{(\tan(x))'}{(\sec(x))'} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}.$$

L'Hôpital's rule gives us a cycling expression. $\triangleleft$

Indeterminate limit $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x\to\infty} \frac{3x^2 - 5}{2x^2 - x + 3}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$L = \lim_{x\to\infty} \frac{(3x^2 - 5)'}{(2x^2 - x + 3)'} = \lim_{x\to\infty} \frac{6x}{4x - 1} = \lim_{x\to\infty} \Big(\frac{6}{4 - \frac{1}{x}}\Big).$$

Recalling $\lim_{x\to\infty} \frac{1}{x} = 0$, we get that $L = \frac{6}{4}$. We conclude that

$$\lim_{x\to\infty} \frac{3x^2 - 5}{2x^2 - x + 3} = \frac{3}{2}. \qquad \triangleleft$$

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- Overview of improper integrals (Sect. 8.7).

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

Remark: Sometimes limits of the form $\infty \cdot 0$ and $(\infty - \infty)$ can be converted by algebraic identities into indeterminate limits $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to 0} \Big(\frac{1}{\sin(x)} - \frac{1}{x} \Big)$.

Solution: This is a limit of the form $(\infty - \infty)$. Since

$$\frac{1}{\sin(x)} - \frac{1}{x} = \frac{x - \sin(x)}{x \sin(x)} \Rightarrow \text{ indeterminate } \frac{0}{0}.$$

Then L'Hôpital's rule in this case implies

$$L = \lim_{x \to 0} \frac{\big(x - \sin(x)\big)'}{\big(x \sin(x)\big)'} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)}$$

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

Example

Evaluate $L = \lim_{x \to 0} \Big(\frac{1}{\sin(x)} - \frac{1}{x} \Big)$.

Solution: Recall $L = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)}$.

This limit is still indeterminate $\frac{0}{0}$. Hence

$$L = \lim_{x \to 0} \frac{\big(1 - \cos(x)\big)'}{\big(\sin(x) + x \cos(x)\big)'} = \lim_{x \to 0} \frac{\sin(x)}{2\cos(x) - x \sin(x)\big)'} = \frac{0}{2} = 0.$$

We conclude that $L = 0$. $\lhd$

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

Example

Evaluate $L = \lim_{x\to\infty} (3x)^{2/x}$.

Solution: This limits is of the form ∞^0. So, before using L'Hôpital's rule we need to rewrite the function above.

$$(3x)^{2/x} = e^{\ln\left((3x)^{2/x}\right)} = e^{\left(\frac{2}{x}\,\ln(3x)\right)}.$$

Since exp is a continuous function, holds

$$\lim_{x\to\infty} (3x)^{2/x} = e^{\lim_{x\to\infty}\left(\frac{2}{x}\,\ln(3x)\right)} = e^{\lim_{x\to\infty}\left(\frac{2\ln(3x)}{x}\right)}.$$

The exponent, is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$\lim_{x\to\infty} \frac{2\ln(3x)}{x} = \lim_{x\to\infty} \frac{\big(2\ln(3x)\big)'}{(x)'} = \lim_{x\to\infty} \frac{2/x}{1} = 0.$$

We conclude that $L = e^0$, that is, $L = 1$. ◁

Limits using L'Hôpital's Rule (Sect. 7.5)

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- **Overview of improper integrals (Sect. 8.7).**

Overview of improper integrals (Sect. 8.7)

Remarks:

- L'Hôpital's rule is useful to compute improper integrals.
- Improper integrals are the limit of definite integrals when one endpoint if integration approaches $\pm\infty$.

Definition

The improper integral of a continuous function $f : [a, \infty) \to \mathbb{R}$ is

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function $f : (-\infty, b] \to \mathbb{R}$ is

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function $f : (-\infty, \infty) \to \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx.$$

Overview of improper integrals (Sect. 8.7)

Example

Evaluate $I = \displaystyle\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx.$

Solution: This is an improper integral:

$$\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}$$

Integrating by parts, $u = \ln(x)$, and $dv = dx/x^2$,

$$\int_1^b \frac{\ln(x)}{x^2}\,dx = \Big(-\frac{1}{x}\Big)\,\ln(x)\Big|_1^b - \int_1^x \Big(\frac{1}{x}\Big)\Big(-\frac{1}{x}\Big)\,dx$$

$$\int_1^b \frac{\ln(x)}{x^2}\,dx = -\frac{\ln(b)}{b} + \int_1^b \frac{dx}{x^2} = -\frac{\ln(b)}{b} - \frac{1}{x}\Big|_1^b.$$

Overview of improper integrals (Sect. 8.7)

Example

Evaluate $I = \int_1^{\infty} \frac{\ln(x)}{x^2}\, dx.$

Solution: Recall: $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x^2}\, dx = \lim_{b\to\infty} \Bigl(-\frac{\ln(b)}{b} - \frac{1}{b} + 1\Bigr).$

The first limit on the right-hand side is indeterminate $\frac{\infty}{\infty}$.

L'Hôpital's rule implies

$$\lim_{b\to\infty} \frac{\ln(b)}{b} = \lim_{b\to\infty} \frac{(\ln(b))'}{(b)'} = \lim_{b\to\infty} \frac{(1/b)}{1} = 0.$$

Therefore, the improper integral is given by

$$\int_1^{\infty} \frac{\ln(x)}{x^2}\, dx = 0 - 0 + 1 \quad \Rightarrow \quad \int_1^{\infty} \frac{\ln(x)}{x^2}\, dx = 1. \qquad \triangleleft$$