Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.
  - Solving differential equations (7.4).
  - Inverse trigonometric functions (7.6).
  - Hyperbolic functions (7.7).
  - Integration techniques (8-IT).
  - Integration by parts (8.1).
  - Trigonometric integrals (8.2).
- Section not covered:
  - Trigonometric substitutions (8.3).
Solving differential equations (7.4)

Remark: Typical problems in this section:

(1) Find the function $y$ solution of $y' = \frac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

(2) The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

Example

Find the function $y$ solution of $y' = \frac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

Solution:

$$4y \ y' = \sin(x) \quad \Rightarrow \quad \int 4y(x) \ y'(x) \ dx = \int \sin(x) \ dx.$$ 

The substitution $u = y(x)$, with $du = y'(x) \ dx$, implies

$$4 \int u \ du = \int \sin(x) \ dx \quad \Rightarrow \quad 2u^2 = -\cos(x) + c,$$

Therefore, $y^2(x) = (\frac{-\cos(x) + c}{2})$. The condition $y(0) < 0$, implies $y(x) = -\sqrt{c - \cos(x)/\sqrt{2}}$. Furthermore,

$$-\sqrt{2} = -\frac{\sqrt{c - 1}}{\sqrt{2}} \quad \Rightarrow \quad 2 = \sqrt{c - 1} \quad \Rightarrow \quad c = 5.$$

We conclude that $y = -\sqrt{5 - \cos(x)/\sqrt{2}}$. ◯
Solving differential equations (7.4)

Example
The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

Solution: Integrate the differential equation,

$$\int \frac{L'(x)}{L(x)} \, dx = -k \int dx, \quad u = L(x), \quad du = L'(x) \, dx$$

$$\int \frac{du}{u} = -k \int dx \Rightarrow \ln(u) = -kx + c \Rightarrow L(x) = e^{-kx+c}.$$ 

Since $L(x) = e^{-kx}e^c$, calling $L_0 = e^c$, we get the solution

$$L(x) = L_0 e^{-kx}.$$ 

Solving differential equations (7.4)

Example
The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

Solution: Recall: $L(x) = L_0 e^{-kx}$. Now the first condition implies

$$\frac{L_0}{2} = L(15) = L_0 e^{-15k} \Rightarrow e^{-15k} = \frac{1}{2} \Rightarrow -15k = -\ln(2)$$

so we conclude that $k = \ln(2)/15$. The second condition implies

$$\frac{L_0}{8} = L_0 e^{-kx_1} \Rightarrow e^{-kx_1} = \frac{1}{8} \Rightarrow -kx_1 = -\ln(8).$$

Using the value $k = \ln(2)/15$, we get

$$x_1 = \ln(8) \frac{15}{\ln(2)} \Rightarrow x_1 = 3(15) \Rightarrow x_1 = 45. \quad \triangle$$
Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.

- Solving differential equations (7.4).
- **Inverse trigonometric functions (7.6).**
- Hyperbolic functions (7.7).
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Section not covered:
- Trigonometric substitutions (8.3).

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**Inverse trigonometric functions (7.6)**

**Notation:** In the literature is common the notation $\sin^{-1} = \arcsin$, and similar for the rest of the trigonometric functions.

Do not confuse $\frac{1}{\sin(x)} \neq \sin^{-1}(x) = \arcsin(x)$.

**Remark:** $\sin$, $\cos$ have simple values at particular angles.

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<th>$\theta$</th>
<th>$\sin(\theta)$</th>
<th>$\cos(\theta)$</th>
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<td>$0$</td>
<td>$1$</td>
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<td>$1/2$</td>
<td>$\sqrt{3}/2$</td>
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<td>$\pi/2$</td>
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</tr>
</tbody>
</table>
Inverse trigonometric functions (7.6)

Remark: On certain domains the trigonometric functions are invertible.

Inverse trigonometric functions (7.6)

Remark: The graph of the inverse function is a reflection of the original function graph about the $y = x$ axis.
Inverse trigonometric functions (7.6)

Theorem
The derivative of inverse trigonometric functions are:

\[
\begin{align*}
\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}}, & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}, & |x| \leq 1, \\
\arctan'(x) &= \frac{1}{1+x^2}, & \arccot'(x) &= -\frac{1}{1+x^2}, & x \in \mathbb{R}, \\
\text{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}}, & \text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}, & |x| \geq 1.
\end{align*}
\]

Recall \( \arctan'(x) = \frac{1}{\tan'(\arctan(x))} \), \( \tan'(y) = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} \)
\[
\tan'(y) = 1 + \tan^2(y), \quad y = \arctan(x), \quad \Rightarrow \quad \arctan'(x) = \frac{1}{1+x^2}.
\]

Inverse trigonometric functions (7.6)

Remark: Typical problems in this section:

(1) Sketch the graphs of

\[
y(x) = \sec(x), \quad z(x) = \text{arcsec}(x).
\]

State the respective domains and ranges.

(2) Evaluate \( \cos(\arcsin(1/\sqrt{2})) \).

(3) Evaluate \( \sec(\arctan(-2/3)) \).

(4) Find \( y' \) for \( y(x) = \arctan(3x^2) \).

(5) Find \( I = \int \frac{dx}{\sqrt{2-x^2}} \).
Inverse trigonometric functions (7.6)

Example
Evaluate sec(arctan(−2/3)).

Solution: We only need the relation between sec and tan,

$$
\sec^2(\theta) = \tan^2(\theta) + 1.
$$

Then holds sec(θ) = ±√tan^2(θ) + 1. We need to find the correct sign: θ = arctan(−2/3) ∈ (−π/2, 0). Since sec(θ) = 1/ cos(θ), we conclude that sec(θ) > 0. Hence

$$
\sec(\arctan\left(-\frac{2}{3}\right)) = \sqrt{\tan^2(\arctan\left(-\frac{2}{3}\right)) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.
$$

We conclude that \( \sec(\arctan(−2/3)) = \sqrt{13}/3. \)
Hyperbolic functions (7.7)

Definition
The complete set of *hyperbolic trigonometric functions* is given by

\[
\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \\
\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}, \\
\text{csch}(x) = \frac{1}{\sinh(x)}, \quad \text{sech}(x) = \frac{1}{\cosh(x)}.
\]

Theorem
*The following identities hold,*

\[
\cosh^2(x) - \sinh^2(x) = 1, \\
\sinh(2x) = 2 \sinh(x) \cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x), \\
\cosh^2(x) = \frac{1}{2} [1 + \cosh(2x)], \quad \sinh^2(x) = \frac{1}{2} [-1 + \cosh(2x)].
\]

Remark: Typical problems in this section:

(1) Prove the identities: \(\cosh^2(x) - \sinh^2(x) = 1\), and

\[
\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad \sinh(2x) = 2 \sinh(x) \cosh(x), \\
\cosh^2(x) = \frac{1}{2} (1 + \cosh(2x)), \quad \sinh^2(x) = \frac{1}{2} (-1 + \cosh(2x)).
\]

(2) Know the derivatives and integrals of hyperbolic functions.
Review for Exam 2.

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Section not covered:

- Trigonometric substitutions (8.3).

Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

1. \[ \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \]
2. \[ \int_{1}^{8} \frac{dx}{x^2 - 2x + 50}. \]
3. \[ \int x^3 \ln(x) \, dx. \]
4. \[ \int x^2 e^{2x} \, dx. \]
5. \[ \int \frac{dx}{\sqrt{8x - x^2}}. \]
6. \[ \int \frac{dx}{\sqrt{25 - x^2}}, \ |x| < 5. \]
7. \[ \int \cot^3(x) \, dx. \]
8. \[ \int \sin^4(x) \, dx. \]
9. \[ \int x^3 \cos(x) \, dx. \]
10. \[ \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx. \]
11. \[ \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx. \]
12. \[ \int \frac{2\ln(x)}{x} \, dx. \]
Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) $\int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}$. Split the integral and do two substitutions.

(2) $\int_1^8 \frac{dx}{x^2 - 2x + 50}$. Complete the square and recall the arctan'.$

(3) $\int x^3 \ln(x) \, dx$. Three integrations by parts.

(4) $\int x^2 e^{2x} \, dx$. Two integrations by parts.

(5) $\int \frac{dx}{\sqrt{8x - x^2}}$. Complete the square and recall arcsin'.$

(6) $\int \frac{dx}{\sqrt{25 - x^2}}, \, |x| < 5$. Substitution and recall arcsin'.$

(7) $\int \cot^3(x) \, dx$. Write using sin, cos and substitution.

(8) $\int \sin^4(x) \, dx$. Double angle formula, twice.

(9) $\int x^3 \cos(x) \, dx$. Integrations by parts, three times.

(10) $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx$. Double angle formula, cancel $\sqrt{}$.

(11) $\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)} \, dx$. Write using sin and cos, and substitution.

(12) $\int \frac{2\ln(x)}{x} \, dx$. Substitution.
Example
Evaluate \( I = \int \frac{(1 + x) \, dx}{\sqrt{1 - 2x^2}}. \)

Solution: Split the integral: \( I = \int \frac{dx}{\sqrt{1 - 2x^2}} + \int \frac{x \, dx}{\sqrt{1 - 2x^2}}. \)

For the first integral substitute \( y = \sqrt{2} \, x \), then \( dy = \sqrt{2} \, dx. \)

\[
I_1 = \int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \, x) + c.
\]

For the second integral substitute \( u = 1 - 2x^2 \), then \( du = -4x \, dx. \)

\[
I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{4} (2\sqrt{u}) + c = -\frac{1}{2} \sqrt{1 - 2x^2} + c.
\]

We conclude: \( I = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \, x) - \frac{1}{2} \sqrt{1 - 2x^2} + c. \)

Example
Evaluate \( I = \int \frac{dx}{\sqrt{8x - x^2}}. \)

Solution: Complete the square and recall \( \arcsin' \).

\[
I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},
\]

\[
I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x - 4)^2}}
\]

\[
I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x - 4)/4]^2}}.
\]

Substitute \( u = (x - 4)/4 \), then \( du = dx/4. \)

\[
I = \frac{1}{4} \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + c \Rightarrow I = \arcsin \left( \frac{x - 4}{4} \right) + c.
\]
Example
Evaluate \( I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)} \, dx \).

Solution: Double angle formula, cancel \( \sqrt{\cdot} \).
Recall: \( \sin^2(\theta) = \frac{[1 - \cos(2\theta)]}{2} \). Hence,
\[
I = \int_{-\pi/2}^{\pi/2} \sqrt{2\sin^2(x)} \, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)| \, dx.
\]
Since \( \sin(x) < 0 \) for \( x \in (-\pi/2, 0) \),
\[
I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x) \, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x) \, dx.
\]
\[
I = \sqrt{2} \cos(x) \bigg|_{-\pi/2}^{0} - \sqrt{2} \cos(x) \bigg|_{0}^{\pi/2} = \sqrt{2}(1-0) - \sqrt{2}(0-1) = 2\sqrt{2}.
\]