Definition as an integral

Recall:
(a) The derivative of \( y = x^n \) is \( y' = nx^{n-1} \), for \( n \) integer.
(b) The integral of \( y = x^n \) is \( \int x^n \, dx = \frac{x^{n+1}}{n+1} \), for \( n \neq -1 \).
(c) Case \( n = -1 \): \( \int \frac{dx}{x} \) is neither rational nor trigonometric function. This is a new function.

Definition
The natural logarithm is the function
\[
\ln(x) = \int_1^x \frac{dt}{t}, \quad x \in (0, \infty).
\]
In particular: \( \ln(1) = 0 \).
Definition as an integral

Definition
The natural logarithm is the function
\[ \ln(x) = \int_1^x \frac{dt}{t}, \quad x \in (0, \infty). \]

In particular: \( \ln(1) = 0 \).

Definition
The number \( e \) is the number satisfying \( \ln(e) = 1 \), that is,
\[ \int_1^e \frac{dt}{t} = 1. \]

\((e = 2.718281...).\)

Natural Logarithms (Sect. 7.2)

- Definition as an integral.
- The derivative and properties.
- The graph of the natural logarithm.
- Integrals involving logarithms.
- Logarithmic differentiation.
The derivative and properties

**Theorem (Derivative of \( \ln \))**

*The Fundamental Theorem of Calculus implies \( \ln'(x) = \frac{1}{x} \).*

Proof:
\[
\ln(x) = \int_1^x \frac{dt}{t} \Rightarrow \ln'(x) = \frac{1}{x}.
\]

**Theorem (Chain rule)**

*For every differentiable function \( u \) holds \( \left[ \ln(u) \right]' = \frac{u'}{u} \).*

Proof:
\[
\frac{d \ln(u)}{dx} = \frac{d \ln}{du}(u) \frac{du}{dx} = \frac{1}{u} u' \Rightarrow \frac{d \ln(u)}{dx}(x) = \frac{u'(x)}{u(x)}.
\]

---

**Example**

Find the derivative of \( y(x) = \ln(3x) \), and \( z(x) = \ln(2x^2 + \cos(x)) \).

**Solution:** We use the chain rule.
\[
y'(x) = \frac{1}{(3x)} (3) = \frac{1}{x} \Rightarrow y'(x) = \frac{1}{x}.
\]

We also use chain rule,
\[
z'(x) = \frac{1}{(2x^2 + \cos(x))} (4x - \sin(x)) \]
\[
z'(x) = \frac{4x - \sin(x)}{2x^2 + \cos(x)}.
\]

**Remark:** \( y(x) = \ln(3x) \), satisfies \( y'(x) = \ln'(x) \).
The derivative and properties

Theorem (Algebraic properties)
For every positive real numbers $a$ and $b$ holds,
(a) $\ln(ab) = \ln(a) + \ln(b)$,  \ (product rule);
(b) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$,  \ (quotient rule);
(c) $\ln\left(\frac{1}{a}\right) = - \ln(a)$,  \ (reciprocal rule);
(d) $\ln(a^b) = b \ln(a)$,  \ (power rule).

Proof of (a): (only)
The function $y(x) = \ln(ax)$ satisfies $y'(x) = \frac{1}{ax} a = \frac{1}{x} = \ln'(x)$
Therefore $\ln(ax) = \ln(x) + c$. Evaluating at $x = 1$ we obtain $c$.

$$\ln(a) = \ln(1) + c \Rightarrow c = \ln(a) \Rightarrow \ln(ax) = \ln(x) + \ln(a).$$

The derivative and properties

Example
Compute the derivative of $y(x) = \ln\left[\frac{(x + 1)^2}{3(x + 2)}\right]$.

Solution: Before computing the derivative of $y$, we simplify it,

$$y = \ln[(x + 1)^2] - \ln[3(x + 2)],$$

$$y = 2 \ln(x + 1) - [\ln(3) + \ln(x + 2)].$$

The derivative of function $y$ is: $y' = 2 \frac{1}{(x + 1)} - \frac{1}{(x + 2)}$.

$$y' = \frac{2(x + 2) - (x + 1)}{(x + 1)(x + 2)} \Rightarrow y' = \frac{(x + 3)}{(x + 1)(x + 2)}. \triangleleft$$
Natural Logarithms (Sect. 7.2)

▶ Definition as an integral.
▶ The derivative and properties.
▶ The graph of the natural logarithm.
▶ Integrals involving logarithms.
▶ Logarithmic differentiation.

The graph of the natural logarithm

Remarks:
The graph of \( \ln \) function has:
(a) A vertical asymptote at \( x = 0 \).
(b) No horizontal asymptote.

Proof: Recall \( e = 2.718281... > 1 \) and \( \ln(e) = 1 \).
(a): If \( x = e^n \), then \( \ln(e^n) = n \ln(e) = n \). Hence
\[
\lim_{x \to \infty} \ln(x) = \infty.
\]
(b): If \( x = \frac{1}{e^n} \), then \( \ln\left(\frac{1}{e^n}\right) = -\ln(e^n) - n \ln(e) = -n \). Hence
\[
\lim_{x \to 0^+} \ln(x) = -\infty.
\]
Integrals involving logarithms.

Remark: It holds \( \int \frac{dx}{x} = \ln(|x|) + c \) for \( x \neq 0 \) and \( c \in \mathbb{R} \).

Indeed, for \( x > 0 \) this is the definition of logarithm. And for \( x < 0 \), we have that \( -x > 0 \), then,

\[
\int \frac{dx}{x} = \int \frac{(-dx)}{(-x)} = \ln(-x) + c, \quad -x > 0.
\]

We conclude,

\[
\int \frac{dx}{x} = \begin{cases} 
\ln(-x) + c & \text{if } x < 0, \\
\ln(x) + c & \text{if } x > 0.
\end{cases}
\]

Remark: It also holds \( \int \frac{f'(x)}{f(x)} \, dx = \ln(|f(x)|) + c \), for \( f(x) \neq 0 \).
Integrals involving logarithms.

Remarks:
(a) \[ \int \tan(x) \, dx = -\ln(|\cos(x)|) + c. \] Indeed,
\[ \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx \quad u = \cos(x), \quad du = -\sin(x) \, dx. \]
\[ \int \tan(x) \, dx = -\int \frac{du}{u} = -\ln(|u|) + c = -\ln(|\cos(x)|) + c. \]

(b) \[ \int \cot(x) \, dx = \ln(|\sin(x)|) + c. \] Indeed,
\[ \int \cot(x) \, dx = \int \frac{\cos(x)}{\sin(x)} \, dx \quad u = \sin(x), \quad du = \cos(x) \, dx. \]
\[ \int \cot(x) \, dx = \int \frac{du}{u} = \ln(|u|) + c = \ln(|\sin(x)|) + c. \]

Example
Find \[ y(t) = \int \frac{3 \sin(t)}{(2 + \cos(t))} \, dt. \]

Solution:
\[ y(t) = \int \frac{3 \sin(t)}{(2 + \cos(t))} \, dt, \quad u = 2 + \cos(t), \quad du = -\sin(t) \, dt. \]
\[ y(t) = \int \frac{3(-du)}{u} = -3 \int \frac{du}{u} = -3 \ln(|u|) + c \]

We conclude that \[ y(t) = -3 \ln(|2 + \cos(t)|) + c. \]
Logarithmic differentiation

Remark: Logarithms can be used to simplify the derivative of complicated functions.

Example
Find the derivative of \( y(x) = \frac{x^3(x + 2)^2}{\cos^3(x)} \).

Solution: First compute \( \ln[y(x)] = \ln \left[ \frac{x^3(x + 2)^2}{\cos^3(x)} \right] \),

\[
\ln[y(x)] = \ln[x^3(x + 2)^2] - \ln[\cos^3(x)],
\]

\[
\ln[y(x)] = \ln[x^3] + \ln[(x + 2)^2] - \ln[\cos^3(x)],
\]

\[
\ln[y(x)] = 3 \ln(x) + 2 \ln(x + 2) - 3 \ln[\cos(x)].
\]
Logarithmic differentiation

Example
Find the derivative of \( y(x) = \frac{x^3(x + 2)^2}{\cos^3(x)} \).

Solution: Recall: \( \ln[y(x)] = 3 \ln(x) + 2 \ln(x + 2) - 3 \ln[\cos(x)] \).

\[
\frac{y'(x)}{y(x)} = 3 \frac{x}{x} + 2 \frac{1}{(x + 2)} + 3 \frac{\sin(x)}{\cos(x)}.
\]

\[
y'(x) = \left[ 3 \frac{x}{x} + 2 \frac{1}{(x + 2)} + 3 \frac{\sin(x)}{\cos(x)} \right] y(x).
\]

We conclude that

\[
y'(x) = \left[ 3 \frac{x}{x} + 2 \frac{1}{(x + 2)} + 3 \frac{\sin(x)}{\cos(x)} \right] \frac{\cos^3(x)}{x^3(x + 2)^2}.
\]