Review Exam 3.

- Sections 6.1-6.6, 7.1-7.6, 7.8.
- 5 problems.
- 50 minutes.
- Laplace Transform table included.

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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Solution: Eigenvalues of A:

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$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix}$$

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$$\lambda^2 + 5\lambda + 4 = 0$$

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$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}\big[-5 \pm \sqrt{25-16}\big]$$

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Hence $\lambda_+ = -1$,

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Hence $\lambda_+ = -1$, $\lambda_- = -4$.

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Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for λ_+.

$$(A+I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$$

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$2v_1 = \sqrt{2}\,v_2$.

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$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

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Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

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Eigenvector for λ_-.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

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$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}$$

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Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$,

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Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. ◁

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Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

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Solution:

We start plotting the vectors

$$\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix},$$

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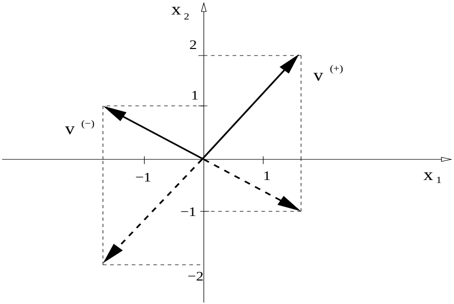

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Solution:

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$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

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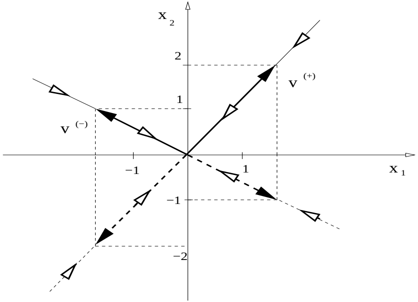

Exam: November 12, 2008. Problem 4.

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Solution:

Recall: $\lambda_- < \lambda_+ < 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)}\, e^{-t} + \mathbf{v}^{(-)}\, e^{-4t}.$$

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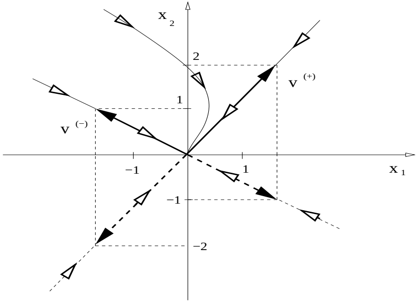

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Solution:

We plot the solutions

$$\mathbf{x} = c_1 \, \mathbf{x}^{(+)} + c_2 \, \mathbf{x}^{(-)},$$

for different values of c_1 and c_2.

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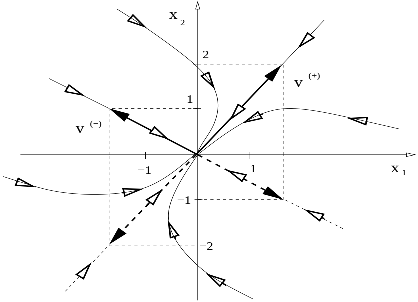
for different values of c_1 and c_2.



Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

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Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

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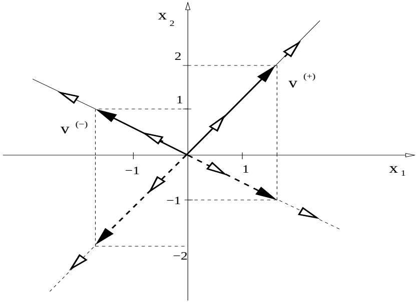

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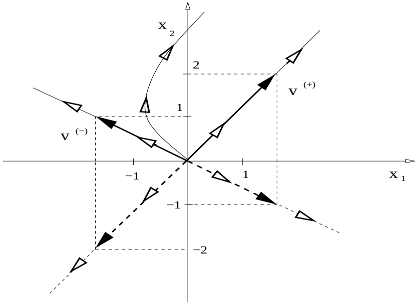

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$$\mathbf{x} = c_1 \mathbf{x}^{(+)} + c_2 \mathbf{x}^{(-)},$$

for different values of c_1 and c_2.

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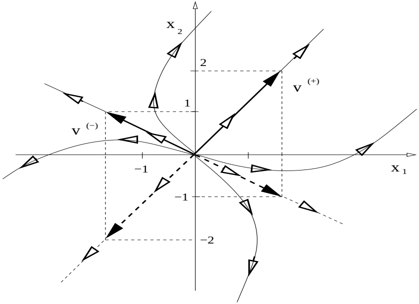

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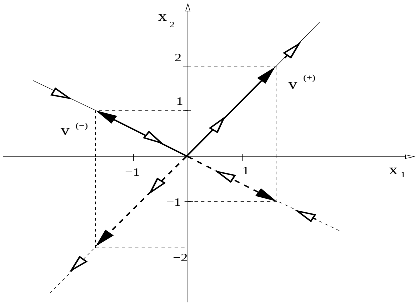

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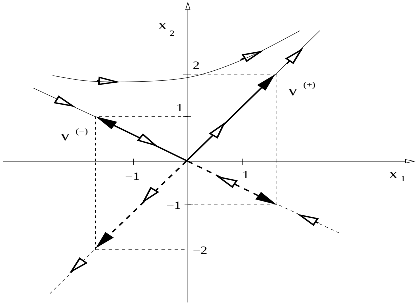

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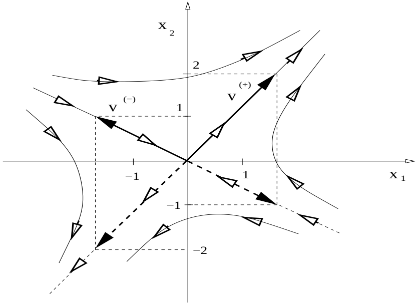

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Solution: Eigenvalues of A:

Extra problem.

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Extra problem.

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$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

Extra problem.

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$v_1 = 2\,v_2$.

Extra problem.

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$,

Extra problem.

Example

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

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Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

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Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Find $\mathbf{x}$ solution of the IVP

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Hence $w_1 = 2w_2 - 1$,

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

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Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

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General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

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The solution is $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\triangleleft$

Extra problem.

Example

Let $\lambda = -1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{-t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{-t}$.

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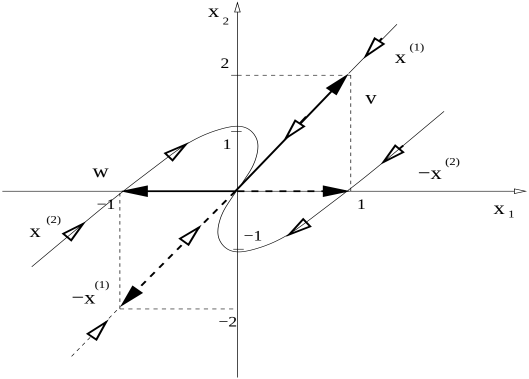

Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm \mathbf{x}^{(1)} = \pm \mathbf{v}\, e^{t}$ and $\pm \mathbf{x}^{(2)} = \pm \big(\mathbf{v}\, t + \mathbf{w} \big)\, e^{t}$.

Extra problem.

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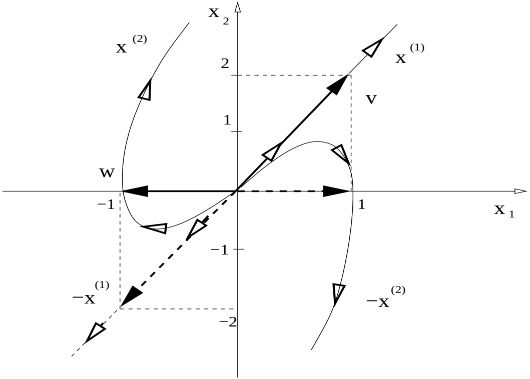

Extra problem.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

for the cases $\alpha = 0$, and $\alpha > 0$, where $\beta > 0$.

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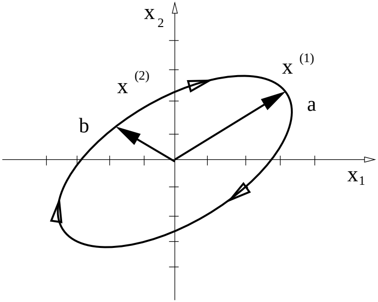

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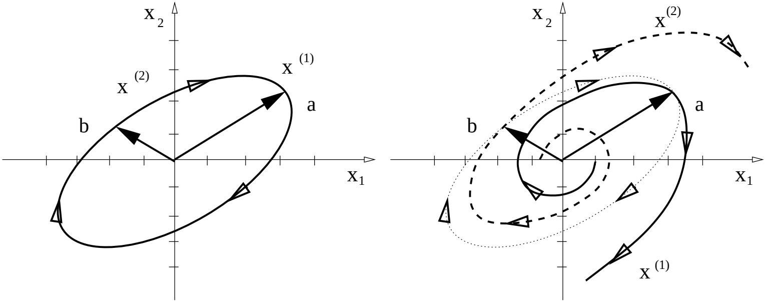

◁

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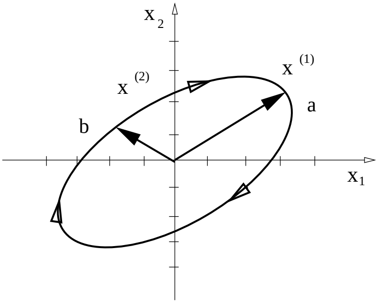

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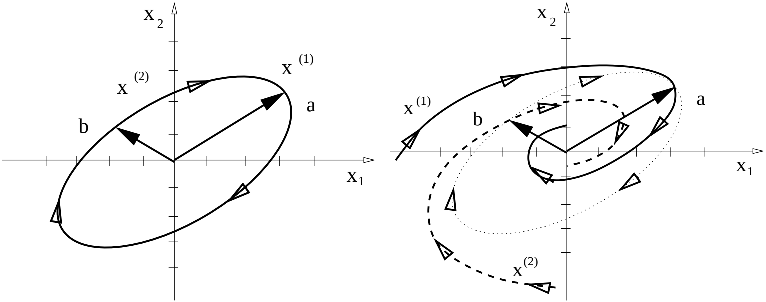

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Overview of Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Origins of the Fourier Series.

Summary:

Daniel Bernoulli ($\sim$ 1750) found solutions to the equation that describes waves propagating on a vibrating string.

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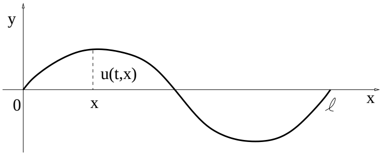

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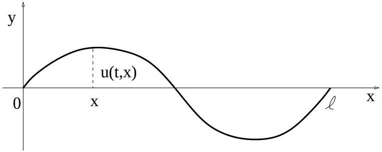

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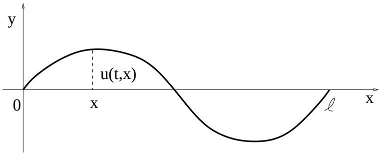

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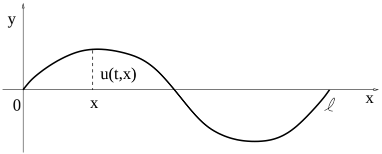

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$$\partial_t^2 u(t,x) = v^2\, \partial_x^2 u(t,x), \quad v \in \mathbb{R}, \quad x \in [0, L], \quad t \in [0, \infty),$$

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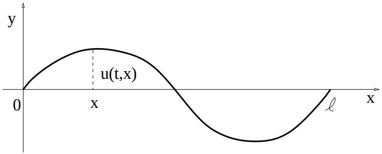

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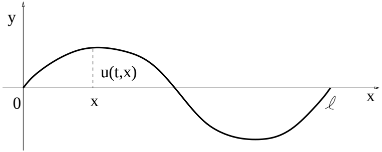

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Origins of the Fourier Series.

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Bernoulli found particular solutions to the wave equation.

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Remark: The wave equation and its solutions provide a mathematical description of music.

Origins of the Fourier Series.

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$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \quad k > 0, \quad x \in [0,L], \quad t \in [0,\infty),$$

$$\text{I.C.} \quad u(0,x) = f(x),$$

Origins of the Fourier Series.

Remarks:

- Bernoulli claimed he had obtained all solutions to the problem above for the wave equation.
- However, he did not prove that claim.
- A proof is: Given a function F with $F(0) = F(L) = 0$, but otherwise arbitrary, find N and the coefficients a_n such that F is approximated by an expansion F_N given in the previous slide.
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Fourier also realized that

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Remark: The heat equation and its solutions provide a mathematical description of heat transport in a solid material.

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- More precisely: Every continuous, τ-periodic function F, there exist constants a_0, a_n, b_n, for $n = 1, 2, \cdots$ such that

$$F_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right],$$

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Notation: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$

Origins of the Fourier Series.

The main problem in our class:

Given a continuous, τ-periodic function f, find the formulas for a_n and b_n such that

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Remarks: We need to review two main concepts:

- The notion of periodic functions.
- The notion of orthogonal functions, in particular the orthogonality of Sines and Cosines.

Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- **Periodic functions.**
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Periodic functions.

Definition

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

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Notation:

A periodic function with period T is also called T-periodic.

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Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

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Show that the function below is periodic, and find its period,

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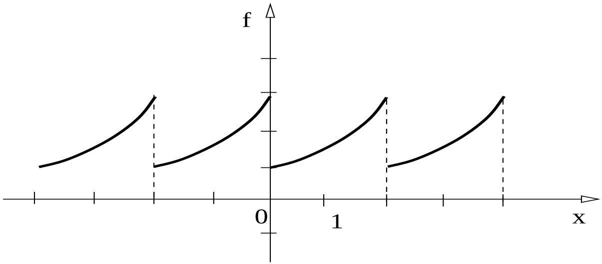

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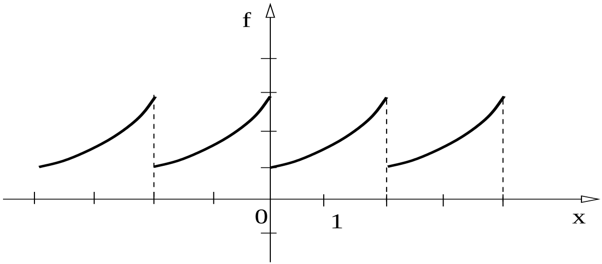

So the function is periodic with period $T = 2$. ◁

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Proof: If $f(x+T) = f(x)$ and $g(x+T) = g(x)$, then

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so $(af + bg)$ is also T-periodic. □

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Since f and g are invariant under translations by T/n, they are also invariant under translations by T.

Periodic functions.

Corollary

Any function f given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big]$$

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A function f is T-periodic iff holds

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Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- Main result on Fourier Series.

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Remark:
From now on we work on the following domain: $[-L, L]$.

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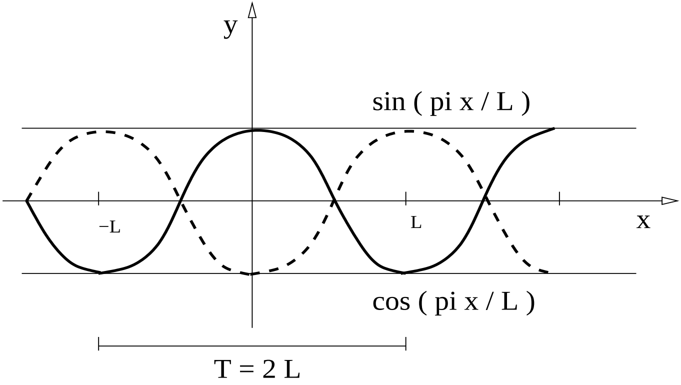

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

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- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\, g(x)\, dx$ is an inner product in the vector space of functions.

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The following relations hold for all $n,\, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = 0.$$

Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right];$$

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right];$$

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Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

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In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx$$
$$+\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

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Orthogonality of Sines and Cosines.

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$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

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We obtain that

$$\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

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If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

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This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. □

Overview of Fourier Series (Sect. 10.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- **Main result on Fourier Series.**

Main result on Fourier Series.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then f can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \tag{1}$$

with the constants a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$-periodic extension of f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.

Examples of the Fourier Theorem (Sect. 10.3).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

The Fourier Theorem: Continuous case.

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The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

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- Express f_N as a convolution of Sine, Cosine, functions and the original function f.
- Use the convolution properties to show that

$$\lim_{N\to\infty} f_N(x) = f(x), \qquad x \in [-L, L].$$

□

Examples of the Fourier Theorem (Sect. 10.3).

- The Fourier Theorem: Continuous case.
- **Example: Using the Fourier Theorem.**
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Example: Using the Fourier Theorem.

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$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (1 + x)\, dx + \int_{0}^{1} (1 - x)\, dx.$$

Example: Using the Fourier Theorem.

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$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

$$a_0 = \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1}$$

Example: Using the Fourier Theorem.

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Example: Using the Fourier Theorem.

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We obtain: $a_0 = 1$.

Example: Using the Fourier Theorem.

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$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

Example: Using the Fourier Theorem.

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Example: Using the Fourier Theorem.

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x),$

Example: Using the Fourier Theorem.

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)$, and

$$\int x \cos(n\pi x)\, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{0}^{1}$$

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We then conclude that $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big]$.

Example: Using the Fourier Theorem.

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$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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Finally, we must find the coefficients b_n.

Example: Using the Fourier Theorem.

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Finally, we must find the coefficients b_n.

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Then, the Fourier series of f is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big] \cos(n\pi x).$$

$\triangleleft$

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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Example: Using the Fourier Theorem.

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We conclude: $f(x) = \dfrac{1}{2} + \sum_{k=1}^{\infty} \dfrac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x)$. $\triangleleft$

Examples of the Fourier Theorem (Sect. 10.3).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- **The Fourier Theorem: Piecewise continuous case.**
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The Fourier Theorem: Piecewise continuous case.

Recall:

Definition

A function $f : [a, b] \to \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that f is continuous on the interior of these sub-intervals.

(b) f has finite limits at the endpoints of all sub-intervals.

The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is piecewise continuous, then the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

where a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

satisfies that:

(a) $f_F(x) = f(x)$ *for all x where f is continuous;*

(b) $f_F(x_0) = \frac{1}{2}\Big[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \Big]$ *for all x_0 where f is discontinuous.*

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0}(-1)\,\cos\big(n\pi x\big)\,dx + \int_{0}^{1}(1)\,\cos\big(n\pi x\big)\,dx,$$

$$a_n = \frac{(-1)}{n\pi}\Big[\sin(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi}\Big[\sin(n\pi x)\Big|_{0}^{1}\Big],$$

Example: Using the Fourier Theorem.

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and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, $b_{2k} = \dfrac{4}{(2k-1)\pi}$, and $a_n = 0$.

Therefore, we conclude that

$$f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\big((2k-1)\pi\, x\big).$$

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