Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac’s delta.
- Relation between deltas and steps.
- Dirac’s delta in Physics.
- The Laplace Transform of Dirac’s delta.
- Differential equations with Dirac’s delta sources.
The Dirac delta generalized function.

Definition
Consider the sequence of functions for $n \geq 1$,

$$
\delta_n(t) = \begin{cases} 
0, & t < 0 \\
n, & 0 \leq t \leq \frac{1}{n} \\
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\end{cases}
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Remarks:
(a) There exist infinitely many sequences $\delta_n$ that define the same generalized function $\delta$.

(b) For example, compare with the sequence $\delta_n$ in the textbook.
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Remarks:
(a) The Dirac $\delta$ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.
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\[ \begin{array}{c}
\text{d}_{1} \\
\text{d}_{2} \\
\text{d}_{3} \\
\hline
0 & 1 & \text{t}
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If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( t_0 \in \mathbb{R} \) and \( a > 0 \), then

\[
\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).
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Proof:

Introduce the change of variable \( \tau = t - t_0 \),

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I = \int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = \int_{-a}^{a} \delta(\tau) f(\tau + t_0) \, d\tau,
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Therefore,

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Remark:

▶ If we generalize the notion of derivative as $u_n'(t) = \lim_{n \to \infty} \delta_n(t)$, then holds $u_n'(t) = \delta(t)$.

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satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

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- Dirac’s delta is a generalized derivative of the step function.
Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac’s delta.
- Relation between deltas and steps.
- **Dirac’s delta in Physics.**
- The Laplace Transform of Dirac’s delta.
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That is, \( \Delta I = F_0. \)
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**Recall:** The Laplace Transform can be generalized from functions to $\delta$,
The Laplace Transform of Dirac’s delta.

Recall: The Laplace Transform can be generalized from functions to \( \delta \), as follows, \( \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)] \).
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$\mathcal{L}[\delta(t - c)] = e^{-cs}$. 
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\[ \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)], \quad \delta_n(t) = n \left[ u(t) - u(t - \frac{1}{n}) \right]. \]
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This is a singular limit, $0/0$. 

Use l'Hôpital rule.
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Proof: Recall: \( \mathcal{L}[\delta(t - c)] = e^{-cs} \lim_{n \to \infty} \left( 1 - \frac{e^{-s/n}}{s/n} \right). \)
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Proof: Recall: $\mathcal{L}[\delta(t - c)] = e^{-cs} \lim_{n \to \infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}$. 

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The Laplace Transform of Dirac’s delta.

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(a) This result is consistent with a previous result:
The Laplace Transform of Dirac’s delta.

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Generalized sources (Sect. 6.5).

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Example

Find the solution $y$ to the initial value problem

$$y'' - y = -20 \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 0.$$
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We arrive to the equation \( \mathcal{L}[y] = \frac{s}{(s^2 - 1)} - 20 e^{-3s} \frac{1}{(s^2 - 1)}, \)
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We conclude: $y(t) = \cosh(t) - 20 u(t - 3) \sinh(t - 3)$. \hfill \triangleleft
Differential equations with Dirac’s delta sources.

Example

Find the solution to the initial value problem

\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]
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that is, \[ \mathcal{L}[y] = \frac{e^{-\pi s}}{2} \frac{2}{(s^2 + 4)} - \frac{e^{-2\pi s}}{2} \frac{2}{(s^2 + 4)}. \]
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Recall: \( e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u(t - c) f(t - c)]. \)
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that is, \( \mathcal{L}[y] = \frac{e^{-\pi s}}{2} \frac{2}{(s^2 + 4)} - \frac{e^{-2\pi s}}{2} \frac{2}{(s^2 + 4)}. \)

Recall: \( e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u(t - c) f(t - c)]. \) Therefore,

\[ \mathcal{L}[y] = \frac{1}{2} \mathcal{L}[u(t-\pi) \sin[2(t-\pi)]] - \frac{1}{2} \mathcal{L}[u(t-2\pi) \sin[2(t-2\pi)]] . \]
Differential equations with Dirac’s delta sources.

Example

Find the solution to the initial value problem

\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Recall:

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We conclude:
\[ y(t) = \frac{1}{2} \left[ u(t - \pi) - u(t - 2\pi) \right] \sin(2t). \]
Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Definition

The convolution of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_{0}^{t} f(\tau)g(t - \tau) \, d\tau.$$
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Remarks:
- \( f \ast g \) is also called the generalized product of \( f \) and \( g \).
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**Remarks:**

- $f * g$ is also called the generalized product of $f$ and $g$.

- The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac’s delta.
Convolution of two functions.

Example

Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).
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We conclude:
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We conclude: 

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Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- **Properties of convolutions.**
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions $f$, $g$, and $h$, hold:

(i) **Commutativity:** $f \ast g = g \ast f$;

(ii) **Associativity:** $f \ast (g \ast h) = (f \ast g) \ast h$;

(iii) **Distributivity:** $f \ast (g + h) = f \ast g + f \ast h$;

(iv) **Neutral element:** $f \ast 0 = 0$;

(v) **Identity element:** $f \ast \delta = f$. 

Proof:

(v): $(f \ast \delta)(t) = \int_0^t f(\tau) \delta(t - \tau) d\tau = f(t)$. 

Properties of convolutions.

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We conclude: \( (f \ast g)(t) = (g \ast f)(t) \).
Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Laplace Transform of a convolution.

Theorem (Laplace Transform)

If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then

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\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].
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Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}. \)
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Laplace Transform of a convolution.

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Convolution solutions (Sect. 6.6).

- Convolution of two functions.
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Impulse response solution.

Definition
The *impulse response solution* is the function $y_\delta$ solution of the IVP

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \quad c \in \mathbb{R}.$$
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Example

Find the impulse response solution of the IVP

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Solution: $\mathcal{L}[y''_\delta] + 2 \mathcal{L}[y'_\delta] + 2 \mathcal{L}[y_\delta] = \mathcal{L}[\delta(t - c)].$
Impulse response solution.

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Example
Find the impulse response solution of the IVP

$$y''_\delta + 2 y'_\delta + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$

Solution: $\mathcal{L}[y''_\delta] + 2 \mathcal{L}[y'_\delta] + 2 \mathcal{L}[y_\delta] = \mathcal{L}[\delta(t - c)].$

$$(s^2 + 2s + 2) \mathcal{L}[y_\delta] = e^{-cs}$$
Impulse response solution.

Definition
The impulse response solution is the function $y_\delta$ solution of the IVP

$$y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \quad c \in \mathbb{R}.$$ 

Example
Find the impulse response solution of the IVP

$$y''_\delta + 2 y'_\delta + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$ 

Solution: $\mathcal{L}[y''_\delta] + 2 \mathcal{L}[y'_\delta] + 2 \mathcal{L}[y_\delta] = \mathcal{L}[\delta(t - c)].$

$$(s^2 + 2s + 2) \mathcal{L}[y_\delta] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)} \).
Impulse response solution.

Example

Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)} \).

Find the roots of the denominator,

\[ s^2 + 2s + 2 = 0 \]
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \)

Find the roots of the denominator,

\[ s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 - 8}\right] \]
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

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Complex roots.
Impulse response solution.

Example
Find the impulse response solution of the IVP

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Complex roots. We complete the square:
Impulse response solution.

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Find the impulse response solution of the IVP

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Complex roots. We complete the square:

\[ s^2 + 2s + 2 = \left[ s^2 + 2\left(\frac{2}{2}\right)s + 1 \right] - 1 + 2 \]
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_{\delta} + 2 y'_{\delta} + 2 y_{\delta} = \delta(t - c), \quad y_{\delta}(0) = 0, \quad y'_{\delta}(0) = 0. \]

Solution: Recall: \[ \mathcal{L}[y_{\delta}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \]

Find the roots of the denominator,

\[ s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 8} \right] \]

Complex roots. We complete the square:

\[ s^2 + 2s + 2 = \left[ s^2 + 2 \left( \frac{2}{2} \right) s + 1 \right] - 1 + 2 = (s + 1)^2 + 1. \]
Impulse response solution.

Example

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Complex roots. We complete the square:

\[ s^2 + 2s + 2 = \left[ s^2 + 2 \left( \frac{2}{2} \right) s + 1 \right] - 1 + 2 = (s + 1)^2 + 1. \]

Therefore, \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \)
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2 y'_\delta + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \).
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2 y'_\delta + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, . \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \), and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)] \).
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

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\[ \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \]
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\[ y''_\delta + 2 y'_\delta + 2 y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

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Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \), and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)] \).

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)].
\]
Impulse response solution.

Example

Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Solution: Recall:

\[ \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \]

Recall:

\[ \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}, \quad \text{and} \quad \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct}f(t)]. \]

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t}\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs} \mathcal{L}[e^{-t}\sin(t)].
\]

Since \( e^{-cs} \mathcal{L}[f](s) = \mathcal{L}[u(t - c)f(t - c)], \)
Impulse response solution.

Example
Find the impulse response solution of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \), and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)] \).

\[ \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)]. \]

Since \( e^{-cs} \mathcal{L}[f](s) = \mathcal{L}[u(t - c) f(t - c)] \),

we conclude \( y_\delta(t) = u(t - c) e^{-(t-c)} \sin(t - c) \).
Convolution solutions (Sect. 6.6).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Solution decomposition theorem.

Theorem (Solution decomposition)

The solution $y$ to the IVP

$$y'' + a_1 y' + a_0 y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta \ast g)(t),$$

where $y_h$ is the solution of the homogeneous IVP

$$y_h'' + a_1 y_h' + a_0 y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,$$

and $y_\delta$ is the impulse response solution, that is,

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]
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Example

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\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \[ \mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)], \]
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1), \]
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Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

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\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s (1) - (-1), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1. \]

\[ (s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)]. \]
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1.
\]

\[
(s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].
\]

\[
\mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)].
\]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

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But:

\[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} \]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

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But:

\[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} \]
Solution decomposition theorem.

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Use the Solution Decomposition Theorem to express the solution of

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But:

\[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \]
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Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

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But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} \)
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} \).
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

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and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \)
Example

Use the Solution Decomposition Theorem to express the solution of

\[y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.\]

Solution: Recall: \(\mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)].\)

But: \(\mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)],\)

and: \(\mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \) So,

\[\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)]\]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$ 

Solution: Recall:  

$$\mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)].$$

But:  

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$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[	ext{e}^{-t} \sin(t)].$$

So,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$
Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall:

\[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]

But:

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and:

\[ \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \]

So,

\[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta \ast g)(t), \]

So:

\[ y(t) = e^{-t} \cos(t) + \int_0^t e^{-\tau} \sin(\tau) \sin[a(t - \tau)] d\tau. \]
Solution decomposition theorem.

**Proof:** Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)], \)
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1,$$

and recall,

$$\mathcal{L}[y''] = s \mathcal{L}[y] - sy_0 - y_1.$$
Solution decomposition theorem.

**Proof:** Compute: \( L[y'''] + a_1 L[y'] + a_0 L[y] = L[g(t)] \), and recall,

\[
L[y'''] = s^2 L[y] - sy_0 - y_1, \quad L[y'] = s L[y] - y_0.
\]
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].
\]
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].
\]

\[
\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
\]
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1,
\mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
$$

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$
\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, 

Recall: $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$.
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1,$$
$$\mathcal{L}[y'] = s \mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$. 
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].
\]

\[
\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
\]

Recall: \( \mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} \), and \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)} \).

Since, \( \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \),
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y'''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y'''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[ (s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)]. \]

\[
\mathcal{L}[y] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
\]

Recall: \( \mathcal{L}[y_h] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} \), and \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)} \).

Since, \( \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \), so \( y(t) = y_h(t) + (y_\delta * g)(t) \).
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
$$

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$
\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta \ast g)(t)$.

Equivalently: $y(t) = y_h(t) + \int_0^t y_\delta(\tau)g(t - \tau) \, d\tau$. \qed
Systems of linear differential equations (Sect. 7.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.
$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example: Newton's law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

\[
\begin{bmatrix}
  x_1(t) \\
n x_2(t) \\
n x_3(t)
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  F_1(t, x(t)) \\
n F_2(t, x(t)) \\
n F_3(t, x(t))
\end{bmatrix}
\]

The equation of motion are:

\[
m \frac{d^2 x}{dt^2} = F(t, x(t)).
\]

These are three differential equations,

\[
m \frac{d^2 x_1}{dt^2} = F_1(t, x(t)),
\]

\[
m \frac{d^2 x_2}{dt^2} = F_2(t, x(t)),
\]

\[
m \frac{d^2 x_3}{dt^2} = F_3(t, x(t)).
\]
$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example
Newton’s law of motion for a particle of mass $m$ moving in space.
Remark: Many physical systems must be described with more than one differential equation.

Example
Newton’s law of motion for a particle of mass \( m \) moving in space. The unknown and the force are vector-valued functions,
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

$$
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},
$$
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

\[
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.
\]
n \times n systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

$$
\mathbf{x}(t) = \begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix}, \quad 
\mathbf{F}(t) = \begin{bmatrix}
    F_1(t, \mathbf{x}) \\
    F_2(t, \mathbf{x}) \\
    F_3(t, \mathbf{x})
\end{bmatrix}.
$$

The equation of motion are: $m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(t, \mathbf{x}(t))$. 
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass $m$ moving in space.

The unknown and the force are vector-valued functions,

$$
\mathbf{x}(t) = \begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix}, \quad 
\mathbf{F}(t) = \begin{bmatrix}
    F_1(t, \mathbf{x}) \\
    F_2(t, \mathbf{x}) \\
    F_3(t, \mathbf{x})
\end{bmatrix}.
$$

The equation of motion are:

$$
m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(t, \mathbf{x}(t)).
$$

These are three differential equations,

$$
m \frac{d^2 x_1}{dt^2} = F_1(t, \mathbf{x}(t)), \quad m \frac{d^2 x_2}{dt^2} = F_2(t, \mathbf{x}(t)), \quad m \frac{d^2 x_3}{dt^2} = F_3(t, \mathbf{x}(t)).
$$
Definition
An $n \times n$ system of linear first order differential equations is the following: Given the functions $a_{ij}, g_i : [a, b] \to \mathbb{R}$, where $i, j = 1, \cdots, n$, find $n$ functions $x_j : [a, b] \to \mathbb{R}$ solutions of the $n$ linear differential equations

\[ x_1' = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + g_1(t) \]
\[ \vdots \]
\[ x_n' = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + g_n(t). \]

The system is called homogeneous iff the source functions satisfy that $g_1 = \cdots = g_n = 0$. 
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x'_1 = a_{11}(t)x_1 + g_1(t).$$
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x'_1 = a_{11}(t) x_1 + g_1(t).$$

Example

$n = 2$: $2 \times 2$ linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x'_1 = a_{11}(t) x_1 + a_{12}(t) x_2 + g_1(t),$$
$$x'_2 = a_{21}(t) x_1 + a_{22}(t) x_2 + g_2(t).$$
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x'_1 = a_{11}(t)x_1 + g_1(t).$$

Example

$n = 2$: $2 \times 2$ linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + g_1(t),$$

$$x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + g_2(t).$$

Example

$n = 2$: $2 \times 2$ homogeneous linear system: Find $x_1(t)$ and $x_2(t)$,

$$x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2$$

$$x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2.$$
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
\begin{align*}
x_1' &= x_1 - x_2, \\
x_2' &= -x_1 + x_2.
\end{align*}
\]

\[
\begin{align*}
x_1 &= \frac{1}{2} (c_1 + c_2 e^{2t}), \\
x_2 &= \frac{1}{2} (c_1 - c_2 e^{2t}).
\end{align*}
\]
Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$
x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2.
$$

Solution: Add up the equations, and subtract the equations,
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0,$$
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$  

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$$
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$
x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2.
$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$,
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$x_1' = x_1 - x_2$,  \hspace{1cm}  $x_2' = -x_1 + x_2$.

Solution: Add up the equations, and subtract the equations,

$(x_1 + x_2)' = 0$,  \hspace{1cm}  $(x_1 - x_2)' = 2(x_1 - x_2)$.

Introduce the unknowns \( v = x_1 + x_2 \), \( w = x_1 - x_2 \),
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2,$ constant coefficients, homogeneous system
\[ x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2. \]

Solution: Add up the equations, and subtract the equations,
\[ (x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2). \]
Introduce the unknowns $v = x_1 + x_2, \ w = x_1 - x_2,$ then
\[ v' = 0. \]
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2.$$ 

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$
\( n \times n \) systems of linear differential equations.

Example

Find \( x_1(t) \), \( x_2(t) \) solutions of the \( 2 \times 2 \), constant coefficients, homogeneous system

\[
\begin{align*}
x'_1 &= x_1 - x_2, \\
x'_2 &= -x_1 + x_2.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
\begin{align*}
(x_1 + x_2)' &= 0, \\
(x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}
\]

Introduce the unknowns \( v = x_1 + x_2 \), \( w = x_1 - x_2 \), then

\[
\begin{align*}
v' &= 0 \quad \Rightarrow \quad v = c_1, \\
w' &= 2w
\end{align*}
\]
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$ 

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2, \ w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$
$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$
$n \times n$ systems of linear differential equations.

**Example**
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x'_1 = x_1 - x_2, \quad x'_2 = -x_1 + x_2.$$ 

**Solution:** Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$ 

Back to $x_1$ and $x_2$: 

$$x_1(t) = \frac{1}{2} (c_1 + c_2 e^{2t}), \quad x_2(t) = \frac{1}{2} (c_1 - c_2 e^{2t}).$$
Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$x_1' = x_1 - x_2,$

$x_2' = -x_1 + x_2.$

Solution: Add up the equations, and subtract the equations,

$(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).$

Introduce the unknowns $v = x_1 + x_2, \ w = x_1 - x_2$, then

$v' = 0 \Rightarrow v = c_1,$

$w' = 2w \Rightarrow w = c_2 e^{2t}.$

Back to $x_1$ and $x_2$: $x_1 = \frac{1}{2} (v + w)$,
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\begin{align*}
    x_1' &= x_1 - x_2, \\
    x_2' &= -x_1 + x_2.
\end{align*}

Solution: Add up the equations, and subtract the equations,

\begin{align*}
    (x_1 + x_2)' &= 0, \\
    (x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\begin{align*}
    v' &= 0 \quad \Rightarrow \quad v = c_1, \\
    w' &= 2w \quad \Rightarrow \quad w = c_2 e^{2t}.
\end{align*}

Back to $x_1$ and $x_2$: $x_1 = \frac{1}{2} (v + w)$, $x_2 = \frac{1}{2} (v - w)$. 

\[
\begin{align*}
    x_1 &= \frac{1}{2} (v + w), \\
    x_2 &= \frac{1}{2} (v - w).
\end{align*}
\]
Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
\begin{align*}
x_1' &= x_1 - x_2, \\
x_2' &= -x_1 + x_2.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
\begin{align*}
(x_1 + x_2)' &= 0, \\
(x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}
\]

Introduce the unknowns $v = x_1 + x_2, w = x_1 - x_2$, then

\[
\begin{align*}
v' &= 0 \quad \Rightarrow \quad v = c_1, \\
w' &= 2w \quad \Rightarrow \quad w = c_2 e^{2t}.
\end{align*}
\]

Back to $x_1$ and $x_2$:

\[
\begin{align*}
x_1 &= \frac{1}{2} (v + w), \\
x_2 &= \frac{1}{2} (v - w).
\end{align*}
\]

We conclude:

\[
\begin{align*}
x_1(t) &= \frac{1}{2} (c_1 + c_2 e^{2t}), \\
x_2(t) &= \frac{1}{2} (c_1 - c_2 e^{2t}).
\end{align*}
\]
Systems of linear differential equations (Sect. 7.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.
Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution $y$ to the second order linear equation

$$y'' + p(t) y' + q(t) y = g(t), \quad (1)$$

defines a solution $x_1 = y$ and $x_2 = y'$ of the $2 \times 2$ first order linear differential system

$$x_1' = x_2, \quad (2)$$
$$x_2' = -q(t) x_1 - p(t) x_2 + g(t). \quad (3)$$

Conversely, every solution $x_1, x_2$ of the $2 \times 2$ first order linear system in Eqs. (2)-(3) defines a solution $y = x_1$ of the second order differential equation in (1).
Second order equations and first order systems.

Proof:

\[(\Rightarrow)\] Given \(y\) solution of \(y'' + p(t)y' + q(t)y = g(t),\)

\[(\Leftarrow)\] Introduce \(x_2 = x_1'\) into \(x_2' = -q(t)x_1 - p(t)x_2 + g(t),\)

that is \(x_2'' + p(t)x_2' + q(t)x_1 = g(t).\)
Second order equations and first order systems.

Proof:
$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$,
iintroduce $x_1 = y$ and $x_2 = y'$,
Second order equations and first order systems.

Proof:
(⇒) Given $y$ solution of $y'' + p(t)y' + q(t)y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x'_1 = y' = x_2$, 

(⇐) Introduce $x_2 = x'_1$ into $x'_2 = -q(t)x_1 - p(t)x_2 + g(t)$. 

\[ x''_1 = -q(t)x_1 - p(t)x'_1 + g(t), \]
Second order equations and first order systems.

Proof:
\((\Rightarrow)\) Given \(y\) solution of \(y'' + p(t)y' + q(t)y = g(t)\), introduce \(x_1 = y\) and \(x_2 = y'\), hence \(x'_1 = y' = x_2\), that is,

\[ x'_1 = x_2. \]
Second order equations and first order systems.

Proof:

($\Rightarrow$) Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x'_1 = y' = x_2$, that is,

$$x'_1 = x_2.$$

Then, $x'_2 = y''$
Second order equations and first order systems.

Proof:

(⇒) Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$,
introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t) y - p(t) y' + g(t)$. 

Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x'_1 = y' = x_2$, that is,

$$x'_1 = x_2.$$

Then, $x'_2 = y'' = -q(t) y - p(t) y' + g(t)$. That is,

$$x'_2 = -q(t) x_1 - p(t) x_2 + g(t).$$
Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x'_1 = y' = x_2$, that is,

$$x'_1 = x_2.$$

Then, $x'_2 = y'' = -q(t) y - p(t) y' + g(t)$. That is,

$$x'_2 = -q(t) x_1 - p(t) x_2 + g(t).$$

$(\Leftarrow)$ Introduce $x_2 = x'_1$ into $x'_2 = -q(t) x_1 - p(t) x_2 + g(t)$. 
Second order equations and first order systems.

Proof:

(⇒) Given \( y \) solution of \( y'' + p(t) y' + q(t) y = g(t) \), introduce \( x_1 = y \) and \( x_2 = y' \), hence \( x_1' = y' = x_2 \), that is,

\[
x_1' = x_2.
\]

Then, \( x_2' = y'' = -q(t) y - p(t) y' + g(t) \). That is,

\[
x_2' = -q(t) x_1 - p(t) x_2 + g(t).
\]

(⇐) Introduce \( x_2 = x_1' \) into \( x_2' = -q(t) x_1 - p(t) x_2 + g(t) \).

\[
x_1'' = -q(t) x_1 - p(t) x_1' + g(t),
\]
Second order equations and first order systems.

Proof:

($\Rightarrow$) Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t) y - p(t) y' + g(t)$. That is,

$$x_2' = -q(t) x_1 - p(t) x_2 + g(t).$$

($\Leftarrow$) Introduce $x_2 = x_1'$ into $x_2' = -q(t) x_1 - p(t) x_2 + g(t)$.

$$x_1'' = -q(t) x_1 - p(t) x_1' + g(t),$$

that is

$$x_1'' + p(t) x_1' + q(t) x_1 = g(t).$$
Example
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]
Second order equations and first order systems.

Example
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]

Solution: Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \]
Second order equations and first order systems.

Example

Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]

Solution: Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2. \]
Second order equations and first order systems.

Example
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]

Solution: Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2. \]

Then, the differential equation can be written as

\[ x_2' + 2x_2 + 2x_1 = \sin(at). \]
Second order equations and first order systems.

Example
Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

We conclude that

$$x_1' = x_2.$$

$$x_2' = -2x_1 - 2x_2 + \sin(at).$$
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example: Express as a single second order equation the $2 \times 2$ system and solve it,

\[
\begin{align*}
    x_1' &= -x_1 + 3x_2, \\
    x_2' &= x_1 - x_2.
\end{align*}
\]

Solution: Compute $x_1$ from the second equation:

\[
x_1 = x_2' + x_2.
\]

Introduce this expression into the first equation,

\[
\left(x_2' + x_2\right)' = -\left(x_2' + x_2\right) + 3x_2,
\]

\[
x_2'' + 2x_2' - 2x_2 = 0.
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Express as a single second order equation the $2 \times 2$ system and solve it,

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\[ r^2 + 2r - 2 = 0 \]
Second order equations and first order systems.

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Express as a single second order equation the $2 \times 2$ system and solve it,

$$x_1' = -x_1 + 3x_2, \quad x_2' = x_1 - x_2.$$ 

Solution: Recall: 

$$x_2'' + 2x_2' - 2x_2 = 0.$$ 

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right]$$
Second order equations and first order systems.

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Therefore, \[ x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}. \]
Second order equations and first order systems.

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\[ x'_1 = -x_1 + 3x_2, \]
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Solution: Recall: $x''_2 + 2x'_2 - 2x_2 = 0$.

\[
\begin{align*}
  r^2 + 2r - 2 &= 0 \\
  r_\pm &= \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \\
  &= \frac{1}{2} \left[ -2 \pm \sqrt{12} \right] \\
  &= -1 \pm \sqrt{3}.
\end{align*}
\]

Therefore, $x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}$. Since $x_1 = x'_2 + x_2$,

\[
x_1 = \left( c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t} \right) + \left( c_1 e^{r_+ t} + c_2 e^{r_- t} \right),
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Second order equations and first order systems.

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Express as a single second order equation the $2 \times 2$ system and solve it,

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Solution: Recall:

\[
x_2'' + 2x_2' - 2x_2 = 0.
\]

\[
r^2 + 2r - 2 = 0 \implies r_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 + 8}\right] \implies r_{\pm} = -1 \pm \sqrt{3}.
\]

Therefore, \( x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t} \). Since \( x_1 = x_2' + x_2 \),

\[
x_1 = (c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t}) + (c_1 e^{r_+ t} + c_2 e^{r_- t}),
\]

We conclude:

\[
x_1 = c_1 (1 + r_+) e^{r_+ t} + c_2 (1 + r_-) e^{r_- t}.
\]

\( \triangleq \)
Systems of linear differential equations (Sect. 7.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.
Main concepts from Linear Algebra.

**Remark:** Ideas from Linear Algebra are useful to study systems of linear differential equations.
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We review:
- Matrices $m \times n$.
- Matrix operations.
- $n$-vectors, dot product.
- matrix-vector product.

Definition
An $m \times n$ matrix, $A$, is an array of numbers $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $m$ rows, $n$ columns. An $n \times n$ matrix is called a square matrix.
Main concepts from Linear Algebra.

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where $a_{ij} \in \mathbb{C}$ and $i = 1, \cdots, m$, and $j = 1, \cdots, n$. An $n \times n$ matrix is called a square matrix.
Main concepts from Linear Algebra.

Example

(a) 2 × 2 matrix:  \[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \].
Main concepts from Linear Algebra.

Example

(a) 2 × 2 matrix: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \).

(b) 2 × 3 matrix: \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \).
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(a) $2 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) $2 \times 3$ matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) $3 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) $2 \times 2$ complex-valued matrix: $A = \begin{bmatrix} 1+i & 2 - i \\ 3 & 4i \end{bmatrix}$.
Main concepts from Linear Algebra.

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(e) The coefficients of a linear system can be grouped in a matrix,

\begin{align*}
  x_1' &= -x_1 + 3x_2 \\
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\end{align*}
Main concepts from Linear Algebra.

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Remark: An $m \times 1$ matrix is called an $m$-vector.
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Example

The unknowns of a $2 \times 2$ linear system can be grouped in a $2$-vector, for example, 

$$
\begin{align*}
    x_1' &= -x_1 + 3x_2 \\
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$$

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. 

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The unknowns of a $2 \times 2$ linear system can be grouped in a 2-vector, for example,

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\begin{align*}
x_1' &= -x_1 + 3x_2 \\
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\end{align*}
\implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 + i & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 + i \\ -1 + 2i & 1 \end{bmatrix}.$$ 

Notice that:

$$(A^T)^T = A.$$ 

(b) $A$-conjugate: Conjugate every matrix coefficient:

$$A^* = \begin{bmatrix} 1 & 2 - i & -1 - 2i \\ -3i & 2 - i & -1 \end{bmatrix}.$$ 

Notice that:

$$(A^*)^* = A.$$ 

Matrix $A$ is real iff $A^* = A$.

Matrix $A$ is imaginary iff $A^* = -A$. 

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\]

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(b) Addition of two $m \times n$ matrices is performed component-wise:

\[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.
\]

The addition $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not defined.
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Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1 + 2) & (2 + 3) \\ (3 + 5) & (4 + 1) \end{bmatrix}.$$
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Main concepts from Linear Algebra.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.
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Also:

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Main concepts from Linear Algebra.

Example

(a) Matrix multiplication.
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

Matrix multiplication is defined when the number of columns in the first matrix is equal to the number of rows in the second matrix. The product of two matrices $A$ of size $m \times n$ and $B$ of size $n \times \ell$ results in a matrix $AB$ of size $m \times \ell$.

Example:

Let $A$ be a $2 \times 2$ matrix and $B$ be a $2 \times 3$ matrix. Then $AB$ is a $2 \times 3$ matrix.

The product $AB$ is calculated as follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

Notice that if $B$ is a $2 \times 3$ matrix and $A$ is a $2 \times 2$ matrix, then $BA$ is not defined, as the number of columns in $A$ does not match the number of rows in $B$.
Main concepts from Linear Algebra.

Example

(a) **Matrix multiplication.** The matrix sizes is important:

\[
A \quad \text{times} \quad B \quad \text{defines} \quad AB
\]

\[
m \times n \quad n \times \ell \quad \quad \quad \quad \quad \quad \quad \quad m \times \ell
\]

Example:

\[
A \text{ is } 2 \times 2, B \text{ is } 2 \times 3, \text{ so } AB \text{ is } 2 \times 3:
\]

\[
\begin{bmatrix}
4 & 3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
= 
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16 & 23 & 30 \\
6 & 9 & 12
\end{bmatrix}
\]

Notice \( B \) is \( 2 \times 3 \), \( A \) is \( 2 \times 2 \), so \( BA \) is not defined.

\[
BA =
\begin{bmatrix}
1 & 2 & 3 \\
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\end{bmatrix}
\begin{bmatrix}
4 & 3 \\
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\end{bmatrix}
\text{ not defined}
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Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

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A \text{ times } B \text{ defines } AB
\]

\[
\begin{aligned}
m \times n & \quad n \times \ell & \quad m \times \ell
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Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$. 

Example: Find $AB$ and $BA$ for

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$.

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Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$. 

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Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$. 

We have just shown that this statement is not true for matrices.
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AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1 - 1)(-1 + 1) & (-1 + 1)(1 - 1) \\ (-1 + 1)(1 - 1) & (1 - 1)(1 - 1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
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