Review for Exam 2.

▶ 5 problems.
▶ No multiple choice questions.
▶ No notes, no books, no calculators.
▶ Problems similar to homeworks.
▶ Exam covers:
  ▶ Regular-singular points (5.5).
  ▶ Euler differential equation (5.4).
  ▶ Power series solutions (5.2).
  ▶ Variation of parameters (3.6).
  ▶ Undetermined coefficients (3.5)
  ▶ Constant coefficients, homogeneous, (3.1)-(3.4).
Regular-singular points (5.5).

Summary:
Regular-singular points (5.5).

Summary:

- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(n+r)}$.
Regular-singular points (5.5).

Summary:

- Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(n+r)} \).
- Recall: Since \( r \neq 0 \), holds
  \[
y' = \sum_{n=0}^{\infty} (n+r)a_n(x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r)a_n(x - x_0)^{(n+r-1)},
  \]
Regular-singular points (5.5).

Summary:

► Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(n+r)}$.

► Recall: Since $r \neq 0$, holds

$$y' = \sum_{n=0}^{\infty} (n+r)a_n(x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r)a_n(x - x_0)^{(n+r-1)},$$

► Find the indicial equation for $r$, the recurrence relation for $a_n$. 

(a) If $(r + r - r - 1)$ is not an integer, then each $r + r$ and $r - r$ define linearly independent solutions.

(b) If $(r + r - r - 1)$ is an integer, then both $r + r$ and $r - r$ define proportional solutions.
Regular-singular points (5.5).

Summary:

- Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(n+r)} \).
- Recall: Since \( r \neq 0 \), holds
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  y' = \sum_{n=0}^{\infty} (n+r)a_n(x-x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r)a_n(x-x_0)^{(n+r-1)},
  \]
- Find the indicial equation for \( r \), the recurrence relation for \( a_n \).
- Introduce the larger root \( r_+ \) of the indicial polynomial into the recurrence relation and solve for \( a_n \).
Regular-singular points (5.5).

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► Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(n+r)} \).

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► Find the indicial equation for \( r \), the recurrence relation for \( a_n \).

► Introduce the larger root \( r_+ \) of the indicial polynomial into the recurrence relation and solve for \( a_n \).

(a) If \( (r_+ - r_-) \) is not an integer, then each \( r_+ \) and \( r_- \) define linearly independent solutions.
Summary:

- Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)} \).
- Recall: Since \( r \neq 0 \), holds
  \[
y' = \sum_{n=0}^{\infty} (n+r)a_n (x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r)a_n (x - x_0)^{(n+r-1)},
  \]
- Find the indicial equation for \( r \), the recurrence relation for \( a_n \).
- Introduce the larger root \( r_+ \) of the indicial polynomial into the recurrence relation and solve for \( a_n \).
  
  (a) If \((r_+ - r_-)\) is not an integer, then each \( r_+ \) and \( r_- \) define linearly independent solutions.
  
  (b) If \((r_+ - r_-)\) is an integer, then both \( r_+ \) and \( r_- \) define proportional solutions.
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( y = \sum_{n=0}^{\infty} a_n x^{(n+r)} \),
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right)y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r-2)}$, $x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r-2)}$, $\left(x^2 + \frac{1}{4}\right)y = \sum_{n=0}^{\infty} a_n x^{(n+r)+2} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$. 
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( y = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \ y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r-2)}, \ x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{(n+r)} \)
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 
\[
y = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \quad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r-2)},
\]

\[
x^2 y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)}
\]

We also need to compute 
\[
\left( x^2 + \frac{1}{4} \right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},
\]

Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( \left(x^2 + \frac{1}{4}\right)y = \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{n+r}. \)
Regular-singular points (5.5).

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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{n+r}$.

Re-label $m = n + 2$ in the first term and then switch back to $n,$
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \[
\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}.
\]

Re-label $m = n + 2$ in the first term and then switch back to $n$,

\[
\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},
\]
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$.

Re-label $m = n + 2$ in the first term and then switch back to $n$,

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

The equation is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$
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Consider the equation $x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:
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\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.
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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

$$\left[r(r-1) + \frac{1}{4}\right] a_0 x^r + \left[(r + 1)r + \frac{1}{4}\right] a_1 x^{(r+1)} +$$

$$\sum_{n=2}^{\infty} \left[(n + r)(n + r - 1) a_n + a_{(n-2)} + \frac{1}{4} a_n\right] x^{(n+r)} = 0.$$
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Consider the equation $x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 
\[
\begin{align*}
[r(r - 1) + \frac{1}{4}] a_0 &= 0,
[(r + 1)r + \frac{1}{4}] a_1 &= 0,
[(n + r)(n + r - 1) + \frac{1}{4}] a_n + a_{(n-2)} &= 0.
\end{align*}
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The indicial equation \( r^2 - r + \frac{1}{4} = 0 \) implies \( r_\pm = \frac{1}{2} \).
Regular-singular points (5.5).

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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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\begin{align*}
[r(r-1) + \frac{1}{4}] a_0 &= 0, \\
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\end{align*}
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The indicial equation \( r^2 - r + \frac{1}{4} = 0 \) implies \( r_\pm = \frac{1}{2} \).

The indicial equation \( r^2 + r + \frac{1}{4} = 0 \) implies \( r_\pm = -\frac{1}{2} \).
Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 
\[ r(r - 1) + \frac{1}{4} a_0 = 0, \quad (r + 1) r + \frac{1}{4} a_1 = 0, \]
\[ (n + r)(n + r - 1) + \frac{1}{4} a_n + a_{(n-2)} = 0. \]

The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r_\pm = \frac{1}{2}$.

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Choose $r = \frac{1}{2}$. 

Regular-singular points (5.5).
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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:
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\begin{align*}
\left[ r(r - 1) + \frac{1}{4} \right] a_0 &= 0, \\
\left[ (r + 1)r + \frac{1}{4} \right] a_1 &= 0, \\
\left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n + a_{n-2} &= 0.
\end{align*}
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The indicial equation \( r^2 - r + \frac{1}{4} = 0 \) implies \( r_{\pm} = \frac{1}{2} \).

The indicial equation \( r^2 + r + \frac{1}{4} = 0 \) implies \( r_{\pm} = -\frac{1}{2} \).

Choose \( r = \frac{1}{2} \). That implies \( a_0 \) arbitrary.
Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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[r(r-1) + \frac{1}{4}] a_0 &= 0, \\
[(r+1)r + \frac{1}{4}] a_1 &= 0, \\
[(n+r)(n+r-1) + \frac{1}{4}] a_n + a_{(n-2)} &= 0.
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The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r_\pm = \frac{1}{2}$.

The indicial equation $r^2 + r + \frac{1}{4} = 0$ implies $r_\pm = -\frac{1}{2}$.

Choose $r = \frac{1}{2}$. That implies $a_0$ arbitrary and $a_1 = 0$. 

Regular-singular points (5.5).
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Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ (n + r)(n + r - 1) + \frac{1}{4} a_n = -a_{(n-2)}. \)
Regular-singular points (5.5).

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Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)} \).

\[
\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \quad a_1 = 0, \quad \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)}. \)

\[
\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \Rightarrow \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)}. \)

\[
\left[ \left( n+\frac{1}{2} \right) \left( n-\frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \quad \Rightarrow \quad \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\[ n^2 a_n = -a_{(n-2)} \]
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)}. \)

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\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \Rightarrow \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\[
n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2}
\]
Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ \left[ (n + r)(n + r - 1) + \frac{1}{4} \right] a_n = -a_{(n-2)} \).

\[
\left[ \left( n + \frac{1}{2} \right) \left( n - \frac{1}{2} \right) + \frac{1}{4} \right] a_n = -a_{(n-2)} \Rightarrow \left[ n^2 - \frac{1}{4} + \frac{1}{4} \right] a_n = -a_{(n-2)}
\]

\[ n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2} \Rightarrow \begin{cases} a_2 = -\frac{a_0}{4}, \\ a_4 = -\frac{a_2}{16} \end{cases} \]
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \frac{1}{2}, \quad a_1 = 0, \quad \left[(n + r)(n + r - 1) + \frac{1}{4}\right]a_n = -a_{(n-2)}$.

\[
\left[\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) + \frac{1}{4}\right]a_n = -a_{(n-2)} \Rightarrow \left[n^2 - \frac{1}{4} + \frac{1}{4}\right]a_n = -a_{(n-2)}
\]

\[
n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2} \Rightarrow \begin{cases} a_2 = -\frac{a_0}{4}, \\ a_4 = -\frac{a_2}{16} = \frac{a_0}{64}. \end{cases}
\]
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ a_2 = -\frac{a_0}{4}, \ \text{and} \ a_4 = \frac{a_0}{64} \).
Regular-singular points (5.5).

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2}, \ a_1 = 0, \ a_2 = -\frac{a_0}{4}, \) and \( a_4 = \frac{a_0}{64} \). Then,

\[
y(x) = x^r \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).
\]
Regular-singular points (5.5).

Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \frac{1}{2}$, $a_1 = 0$, $a_2 = -\frac{a_0}{4}$, and $a_4 = \frac{a_0}{64}$. Then,

$$y(x) = x^r \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for $n$ odd.
Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2} \), \( a_1 = 0 \), \( a_2 = -\frac{a_0}{4} \), and \( a_4 = \frac{a_0}{64} \). Then,

\[
y(x) = x^r \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right).
\]

Recall: \( a_1 = 0 \) and the recurrence relation imply \( a_n = 0 \) for \( n \) odd. Therefore,

\[
y(x) = a_0 x^{1/2} \left( 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 + \cdots \right). \quad \triangleleft
\]
Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - **Euler differential equation (5.4).**
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).
Euler differential equation (5.4).

Summary:
Euler differential equation (5.4).

Summary:

\[ (x - x_0)^2 y'' + (x - x_0) p_0 y' + q_0 y = 0. \]
Euler differential equation (5.4).

Summary:

- $(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0$.
- Find $r_{\pm}$ solutions of $r(r - 1) + p_0 r + q_0 = 0$. 
Euler differential equation (5.4).

Summary:
- $(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0$.
- Find $r_{\pm}$ solutions of $r(r - 1) + p_0 r + q_0 = 0$.
- If $r_+ \neq r_-$ and both are real, then fundamental solutions are
  \[ y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}. \]
Euler differential equation (5.4).

Summary:

- \((x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0\).
- Find \(r_\pm\) solutions of \(r(r - 1) + p_0 r + q_0 = 0\).
- If \(r_+ \neq r_-\) and both are real, then fundamental solutions are
  \[ y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}. \]
- If \(r_\pm = \alpha \pm i\beta\), then real-valued fundamental solutions are
  \[ y_+ = |x - x_0|^\alpha \cos(\beta \ln |x - x_0|), \quad y_- = |x - x_0|^\alpha \sin(\beta \ln |x - x_0|). \]
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Example
Using a power series centered at $x_0 = 0$ find the three first terms of the general solution of $(4 - x^2) y'' + 2y = 0$. 

Solution:
We look for solutions $y = \sum_{n=0}^{\infty} a_n x^n$. Therefore, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$. The differential equation is then given by $(4 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$,
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Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]
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u'_1 = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}}
\]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2 \),

\[
 u'_1 = -\frac{y_2 g}{W} = -\frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = -\frac{1}{x}.
\]
Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \ y_2 = x e^{-2x}, \ g = x^{-2} e^{-2x}, \ W = e^{-4x}. \)

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\[ u'_1 = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \quad \Rightarrow \quad u_1 = - \ln |x|. \]
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Use the variation of parameters to find the general solution of

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\[
\begin{align*}
  u'_1 &= -\frac{y_2 g}{W} = -\frac{x \ e^{-2x} \ x^{-2} \ e^{-2} \ x^{-2} \ e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|. \\
  u'_2 &= \frac{y_1 g}{W}
\end{align*}
\]
Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

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 u_1' = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \quad \Rightarrow \quad u_1 = - \ln |x|.
\]

\[
 u_2' = \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}}
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Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

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\[
\begin{align*}
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    u'_2 &= \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = - \frac{1}{x}.
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Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

\[ y'' + 4y' + 4y = x^{-2} e^{-2x}. \]

Solution: \( y_1 = e^{-2x}, \quad y_2 = xe^{-2x}, \quad g = x^{-2} e^{-2x}, \quad W = e^{-4x}. \)

Now we find the functions \( u_1 \) and \( u_2, \)

\[
u_1' = -\frac{y_2 g}{W} = -\frac{xe^{-2x}x^{-2}e^{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.
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u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}x^{-2}e^{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}.
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\[ y_p = u_1 y_1 + u_2 y_2 \]
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\[
y_p = u_1 y_1 + u_2 y_2 = -\ln |x| e^{-2x} - \frac{1}{x} x e^{-2x}
\]
Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2} e^{-2x}.$$ 

Solution: $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$, $g = x^{-2} e^{-2x}$, $W = e^{-4x}$.

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$$u_1' = - \frac{y_2 g}{W} = - \frac{x e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = - \frac{1}{x} \quad \Rightarrow \quad u_1 = - \ln |x|.$$ 

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$$y_p = u_1 y_1 + u_2 y_2 = - \ln |x| e^{-2x} - \frac{1}{x} x e^{-2x} = -(1 + \ln |x|) e^{-2x}.$$
Variation of parameters (3.6).

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Use the variation of parameters to find the general solution of

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Now we find the functions \( u_1 \) and \( u_2, \)

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\[
u'_2 = \frac{y_1 g}{W} = \frac{e^{-2x} x^{-2} e^{-2x}}{e^{-4x}} = x^{-2} \Rightarrow \ u_2 = -\frac{1}{x}.\]

\[
y_p = u_1 y_1 + u_2 y_2 = -\ln|x| e^{-2x} - \frac{1}{x} x e^{-2x} = -(1 + \ln|x|) e^{-2x}.\]

Since \( \tilde{y}_p = -\ln|x| e^{-2x} \) is solution,
Variation of parameters (3.6).

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Use the variation of parameters to find the general solution of

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\[ y_p = u_1 y_1 + u_2 y_2 = -\ln |x| e^{-2x} - \frac{1}{x} xe^{-2x} = -(1 + \ln |x|) e^{-2x}. \]

Since \( \tilde{y}_p = -\ln |x| e^{-2x} \) is solution, \( y = (c_1 + c_2 x - \ln |x|) e^{-2x}. \) \( \triangle \)
Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x). \]
Undetermined coefficients (3.5)

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Use the undetermined coefficients to find the general solution of

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Solution: Find the solutions of the homogeneous problem,
Undetermined coefficients (3.5)

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Use the undetermined coefficients to find the general solution of

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\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]
Undetermined coefficients (3.5)

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Use the undetermined coefficients to find the general solution of
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Solution: Find the solutions of the homogeneous problem,
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i. \]
\[ y_1 = \cos(2x), \quad y_2 = \sin(2x). \]
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The function \( \tilde{y}_p = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess,
Undetermined coefficients (3.5)

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The function \( \tilde{y}_p = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess, since
it is solution of the homogeneous equation.
Undetermined coefficients (3.5)

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The function \( \tilde{y}_p = k_1 \sin(2x) + k_2 \cos(2x) \) is the wrong guess, since it is solution of the homogeneous equation. We guess:
\[ y_p = x[k_1 \sin(2x) + k_2 \cos(2x)]. \]
Undetermined coefficients (3.5)

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\[ y_p = x[k_1 \sin(2x) + k_2 \cos(2x)]. \]

\[ y_p' = [k_1 \sin(2x) + k_2 \cos(2x)] + 2x[k_1 \cos(2x) - k_2 \sin(2x)]. \]
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Use the undetermined coefficients to find the general solution of

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\[ y'_p = [k_1 \sin(2x) + k_2 \cos(2x)] + 2x[k_1 \cos(2x) - k_2 \sin(2x)]. \]

\[ y''_p = 4[k_1 \cos(2x) - k_2 \sin(2x)] + 4x[-k_1 \sin(2x) - k_2 \cos(2x)]. \]
Undetermined coefficients (3.5)

Example
Use the undetermined coefficients to find the general solution of
\[ y'' + 4y = 3 \sin(2x). \]

Solution: Recall: \( y_1 = \sin(2x), \) and \( y_2 = \cos(2x). \)
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

\[ y'' + 4y = 3 \sin(2x). \]

Solution: Recall: \( y_1 = \sin(2x), \) and \( y_2 = \cos(2x). \)

\[
4 \left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x \left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] + 4x \left[ k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x),
\]
Example

Use the undetermined coefficients to find the general solution of

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Evaluating at \( x = 0 \) and \( x = \pi/4 \)
Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

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Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

$$4\left[ k_1 \cos(2x) - k_2 \sin(2x) \right] + 4x \left[ -k_1 \sin(2x) - k_2 \cos(2x) \right] +$$

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Therefore, $4\left[ k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x)$.

Evaluating at $x = 0$ and $x = \pi/4$ we get

$$4k_1 = 0, \quad -4k_2 = 3$$
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4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.
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Use the undetermined coefficients to find the general solution of

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\[
4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.
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Therefore,
\[
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Evaluating at \( x = 0 \) and \( x = \pi/4 \) we get
\[
4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.
\]

Therefore, \( y_p = -\frac{3}{4}x \cos(2x). \) The general solution is
\[
y(x) = c_1 \sin(2x) + \left( c_2 - \frac{3}{4}x \right) \cos(2x).
\]
The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.
Overview and notation.

**Overview:** The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions.*
Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with discontinuous source functions.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$. 

Remark: One can show that for a particular type of functions $f$, that includes all functions we work with in this Section, the notation above is well-defined.
Overview and notation.

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Example
From the Laplace Transform table we know that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.
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Example
From the Laplace Transform table we know that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.
Then also holds that $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$
The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.
The definition of a step function.

Definition
A function $u$ is called a \textit{step function} at $t = 0$ iff holds

$$u(t) = \begin{cases} 
0 & \text{for } t < 0, \\
1 & \text{for } t \geq 0.
\end{cases}$$
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\end{cases}$$

Example
Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$. 
The definition of a step function.

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Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

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The definition of a step function.

**Remark:** Given any function values \( f(t) \) and \( c > 0 \), then \( f(t - c) \) is a right translation of \( f \) and \( f(t + c) \) is a left translation of \( f \).
The definition of a step function.

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\[ f(t) = e^{at} \]
The definition of a step function.

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**Example**

\[ f(t) = e^{at} \]

\[ f(t) = u(t) e^{at} \]
The definition of a step function.

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Example
The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- **Piecewise discontinuous functions.**
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Example
Graph of the function \( b(t) = u(t - a) - u(t - b) \), with \( 0 < a < b \).
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Graph of the function \( f(t) = e^{at} \left[ u(t - 1) - u(t - 2) \right] \).
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Notation: The function values \( u(t - c) \) are denoted in the textbook as \( u_c(t) \).
The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.
The Laplace Transform of discontinuous functions.

**Theorem**

*Given any real number c, the following equation holds,*

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We conclude that $\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}$. \qed
The Laplace Transform of discontinuous functions.

Example
Compute $\mathcal{L}[3u(t - 2)]$. 

Solution:

$\mathcal{L}[3u(t - 2)] = 3 \mathcal{L}[u(t - 2)]$

We conclude:

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The Laplace Transform of discontinuous functions.

**Example**
Compute $\mathcal{L}[3u(t-2)]$.

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The Laplace Transform of discontinuous functions.

Example
Compute $\mathcal{L}[3u(t − 2)]$.

Solution: $\mathcal{L}[3u(t − 2)] = 3 \mathcal{L}[u(t − 2)] = 3 \frac{e^{-2s}}{s}$. 
The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t - 2)]$.

Solution: 

$$\mathcal{L}[3u(t - 2)] = 3 \mathcal{L}[u(t - 2)] = 3 \frac{e^{-2s}}{s}.$$ 

We conclude: 

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The Laplace Transform of discontinuous functions.

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Example
Compute $\mathcal{L}^{-1}\left[\frac{e^{3s}}{s}\right]$. 

\[ \triangleq \]
The Laplace Transform of discontinuous functions.

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\( \triangle \)

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The Laplace Transform of discontinuous functions.

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We conclude: $\mathcal{L}^{-1}\left[\frac{e^{3s}}{s}\right] = u(t + 3)$.
The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
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Properties of the Laplace Transform.

Theorem (Translations)

If \( F(s) = \mathcal{L}[f(t)] \) exists for \( s > a \geq 0 \) and \( c > 0 \), then holds

\[
\mathcal{L}[u(t - c)f(t - c)] = e^{-cs} F(s), \quad s > a.
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Furthermore,

\[
\mathcal{L}[e^{ct}f(t)] = F(s - c), \quad s > a + c.
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Properties of the Laplace Transform.

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Properties of the Laplace Transform.

Example

Compute \( \mathcal{L}[u(t - 2) \sin(a(t - 2))] \).

\[
\begin{align*}
\mathcal{L}[\sin(at)] &= as^2 + a^2, \\
\mathcal{L}[u(t - c)f(t - c)] &= e^{-cs}\mathcal{L}[f(t)]
\end{align*}
\]

\[
\mathcal{L}[u(t - 2) \sin(a(t - 2))] = e^{-2s}a\left(s^2 + a^2\right)
\]

We conclude:

\[
\mathcal{L}[u(t - 2) \sin(a(t - 2))] = e^{-2s}a\left(s^2 + a^2\right).
\]
Properties of the Laplace Transform.

Example

Compute \( \mathcal{L}[u(t-2) \sin(a(t-2))] \).

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Properties of the Laplace Transform.

Example
Compute $\mathcal{L}[u(t - 2) \sin(a(t - 2))].$

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Properties of the Laplace Transform.

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Compute $L[u(t - 2) \sin(a(t - 2))]$.

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Compute $\mathcal{L}[e^{3t} \sin(at)]$. 

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Properties of the Laplace Transform.

Example
Compute \( \mathcal{L}[u(t - 2) \sin(a(t - 2))] \).

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Properties of the Laplace Transform.

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Properties of the Laplace Transform.

Example

Find the Laplace transform of \( f(t) = \begin{cases} 
0, & t < 1, \\
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Properties of the Laplace Transform.

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Find the Laplace transform of \( f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases} \)

Solution: Recall: \( f(t) = u(t - 1) [(t - 1)^2 + 1] \).

This is equivalent to

\[ f(t) = u(t - 1)(t - 1)^2 + u(t - 1). \]
Properties of the Laplace Transform.

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Properties of the Laplace Transform.

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\[
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\]

We conclude: \( \mathcal{L}[f(t)] = \frac{e^{-s}}{s^3} (2 + s^2). \) \( \triangle \)
Properties of the Laplace Transform.

**Remark:** The inverse of the formulas in the Theorem above are:

\[ \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c), \]

Example

Find \( \mathcal{L}^{-1}[e^{-4s} s^2 + 9] \).

Solution:

\[ \mathcal{L}^{-1}[e^{-4s} s^2 + 9] = \frac{1}{3} \mathcal{L}^{-1}[e^{-4s} s^2 + 9]. \]

Recall:

\[ \mathcal{L}^{-1}[a s^2 + a^2] = \sin(at). \]

Then, we conclude that

\[ \mathcal{L}^{-1}[e^{-4s} s^2 + 9] = \frac{1}{3} u(t - 4) \sin(3(t - 4)). \]
Properties of the Laplace Transform.

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Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] \).
Properties of the Laplace Transform.

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Find \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{3}{s^2 + 9} \right] \).
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Example

Find \(\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 + 9}\right]\).

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Recall: \(\mathcal{L}^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin(at).\)
Properties of the Laplace Transform.

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\[ \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t). \]

**Example**

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] \).

**Solution:**

\[ \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{3}{s^2 + 9} \right]. \]

Recall: \( \mathcal{L}^{-1}\left[ \frac{a}{s^2 + a^2} \right] = \sin(at) \). Then, we conclude that

\[ \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} u(t - 4) \sin(3(t - 4)). \]
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos(at)$,
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos(at)$, $\mathcal{L}^{-1}\left[F(s - c)\right] = e^{ct} f(t)$.
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{s}{s^2 + a^2} \right] = \cos(at) \), \( \mathcal{L}^{-1}\left[ F(s - c) \right] = e^{ct} f(t) \).

We conclude: \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] = e^{2t} \cos(3t) \). \( \triangle \)
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1} \left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1} \left[ \frac{s}{s^2 + a^2} \right] = \cos(at) \), \( \mathcal{L}^{-1} [F(s - c)] = e^{ct} f(t) \).

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Example
Find \( \mathcal{L}^{-1} \left[ \frac{2e^{-3s}}{s^2 - 4} \right] \).
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).

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Example

Find \( \mathcal{L}^{-1} \left[ \frac{2e^{-3s}}{s^2 - 4} \right] \).

Solution: Recall: \( \mathcal{L}^{-1} \left[ \frac{a}{s^2 - a^2} \right] = \sinh(at) \)
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right] \).

Solution: \( \mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos(at), \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t) \).

We conclude: \( \mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right] = e^{2t} \cos(3t) \).

Example
Find \( \mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] \).

Solution: Recall: \( \mathcal{L}^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at) \)

and \( \mathcal{L}^{-1}\left[e^{-cs} F(s)\right] = u(t - c) f(t - c) \).
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right]$.

Solution: Recall:

$$\mathcal{L}^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at), \quad \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c).$$
Properties of the Laplace Transform.

Example
Find $\mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right]$.

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\]

\[
\mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] = \mathcal{L}^{-1}\left[ e^{-3s} \frac{2}{s^2 - 4} \right].
\]

\[
\text{We conclude: } \mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] = u(t - 3) \sinh(2(t - 3)).
\]
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right]$.

Solution: Recall:

$$\mathcal{L}^{-1}\left[ \frac{a}{s^2 - a^2} \right] = \sinh(at), \quad \mathcal{L}^{-1}\left[ e^{-cs} F(s) \right] = u(t - c) f(t - c).$$

$$\mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] = \mathcal{L}^{-1}\left[ e^{-3s} \frac{2}{s^2 - 4} \right].$$

We conclude: $$\mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] = u(t - 3) \sinh(2(t - 3)).$$
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right]$. 

Solution:
Find the roots of the denominator:

$s \pm \sqrt{1 + 8} = 1, -2$.

Therefore,

$s^2 + s - 2 = (s - 1)(s + 2)$.

Use partial fractions to simplify the rational function:

$\frac{1}{s^2 + s - 2} = \frac{a}{s - 1} + \frac{b}{s + 2}$.

$1 = a(s + 2) + b(s - 1)$.
Properties of the Laplace Transform.

Example
Find $\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 + 8} \right]$$
Example
Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Find the roots of the denominator:

\[
s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 + 8} \right] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}
\]
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right]$.

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Therefore, $s^2 + s - 2 = (s - 1)(s + 2)$. 
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

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s_+ = 1, \\
s_- = -2.
\end{cases}
\]

Therefore, \( s^2 + s - 2 = (s - 1)(s + 2) \).

Use partial fractions to simplify the rational function:

\[
\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)}
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Find the roots of the denominator:

\[
s_{\pm} = \frac{1}{2} \left[-1 \pm \sqrt{1 + 8}\right] \quad \Rightarrow \quad \begin{cases} 
s_+ = 1, \\
s_- = -2. \end{cases}
\]

Therefore, \( s^2 + s - 2 = (s - 1)(s + 2) \).

Use partial fractions to simplify the rational function:

\[
\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Find the roots of the denominator:

\[
s_{\pm} = \frac{1}{2} [-1 \pm \sqrt{1 + 8}] \quad \Rightarrow \quad \begin{cases} 
 s_+ = 1, \\
 s_- = -2.
\end{cases}
\]

Therefore, \( s^2 + s - 2 = (s - 1)(s + 2) \).

Use partial fractions to simplify the rational function:

\[
\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},
\]

\[
\frac{1}{s^2 + s - 2} = a(s + 2) + b(s - 1)
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Find the roots of the denominator:

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s_\pm = \frac{1}{2} \left[ -1 \pm \sqrt{1 + 8} \right] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}
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Therefore, \( s^2 + s - 2 = (s - 1)(s + 2) \).

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\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},
\]

\[
\frac{1}{s^2 + s - 2} = a(s + 2) + b(s - 1) = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}.
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:

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\frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}
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Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:

\[
\frac{1}{s^2 + s - 2} = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}
\]

\( a + b = 0, \)
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall: \( \frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)} \)

\[
a + b = 0, \quad 2a - b = 1,
\]
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:

\[
\frac{1}{s^2 + s - 2} = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}
\]

\[\begin{align*}
a + b &= 0, \\
2a - b &= 1,
\end{align*}\]

\( \Rightarrow \)

\[a = \frac{1}{3}, \quad b = -\frac{1}{3}.\]
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right]$.

Solution: Recall: \[
\frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}
\]

\[
a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.
\]

\[
\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s - 1}\right] - \frac{1}{3} \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s + 2}\right].
\]
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:
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\frac{1}{s^2 + s - 2} = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}
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a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.
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\[
\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} \mathcal{L}^{-1}\left[ e^{-2s} \frac{1}{s - 1} \right] - \frac{1}{3} \mathcal{L}^{-1}\left[ e^{-2s} \frac{1}{s + 2} \right].
\]

Recall: \( \mathcal{L}^{-1}\left[ \frac{1}{s - a} \right] = e^{at} \),
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:

\[
\frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}
\]

\( a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}. \)

\[
\mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} \mathcal{L}^{-1} \left[ e^{-2s} \frac{1}{s - 1} \right] - \frac{1}{3} \mathcal{L}^{-1} \left[ e^{-2s} \frac{1}{s + 2} \right].
\]

Recall:

\( \mathcal{L}^{-1} \left[ \frac{1}{s - a} \right] = e^{at}, \quad \mathcal{L}^{-1} \left[ e^{-cs} F(s) \right] = u(t - c) f(t - c), \)
Properties of the Laplace Transform.

Example

Find \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] \).

Solution: Recall:

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\frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}
\]

\[
a + b = 0, \quad 2a - b = 1,
\]

\[
\Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.
\]

\[
\mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} \mathcal{L}^{-1} \left[ e^{-2s} \frac{1}{s - 1} \right] - \frac{1}{3} \mathcal{L}^{-1} \left[ e^{-2s} \frac{1}{s + 2} \right].
\]

Recall: \( \mathcal{L}^{-1} \left[ \frac{1}{s - a} \right] = e^{at} \), \( \mathcal{L}^{-1} \left[ e^{-cs} F(s) \right] = u(t - c) f(t - c) \),

\[
\mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2 + s - 2} \right] = \frac{1}{3} u(t - 2) e^{(t-2)} - \frac{1}{3} u(t - 2) e^{-2(t-2)}. 
\]
Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] \).

Solution: Recall:
\[
\frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}
\]
\[
a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.
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\[
\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s - 1}\right] - \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s + 2}\right].
\]

Recall:
\[
\mathcal{L}^{-1}\left[\frac{1}{s - a}\right] = e^{at}, \quad \mathcal{L}^{-1}\left[e^{-cs} F(s)\right] = u(t - c) f(t - c),
\]

\[
\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t - 2) e^{(t-2)} - \frac{1}{3} u(t - 2) e^{-2(t-2)}.
\]

Hence:
\[
\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t - 2) \left[e^{(t-2)} - e^{-2(t-2)}\right]. \quad \triangleq
\]
Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:
  
  \[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

  (b) Example 2:
  
  \[
  y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
  1, & t \in [0, \pi) \\
  0, & t \in [\pi, \infty) 
  \end{cases}.
  \]

  (c) Example 3:
  
  \[
  y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad g(t) = \begin{cases} 
  \sin(t), & t \in [0, \pi) \\
  0, & t \in [\pi, \infty) 
  \end{cases}.
  \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP
\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Compute the Laplace transform of the whole equation,
\[ \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] \]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Compute the Laplace transform of the whole equation,

\[ \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}. \]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Compute the Laplace transform of the whole equation,

\[ \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}. \]

From the previous Section we know that

\[ [s \mathcal{L}[y] - y(0)] + 2 \mathcal{L}[y] = \frac{e^{-4s}}{s} \]

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$ 

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}.$$ 

From the previous Section we know that

$$[s \mathcal{L}[y] - y(0)] + 2 \mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Compute the Laplace transform of the whole equation,

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\[ [s \mathcal{L}[y] - y(0)] + 2 \mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}. \]

Introduce the initial condition,
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$  

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}.  

From the previous Section we know that

$$[s \mathcal{L}[y] - y(0)] + 2 \mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.  

Introduce the initial condition,  

$$\mathcal{L}[y] = \frac{3}{s + 2} + e^{-4s} \frac{1}{s(s + 2)},$$
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Compute the Laplace transform of the whole equation,

\[ \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}. \]

From the previous Section we know that

\[ [s \mathcal{L}[y] - y(0)] + 2 \mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}. \]

Introduce the initial condition, \( \mathcal{L}[y] = \frac{3}{s + 2} + e^{-4s} \frac{1}{s(s + 2)} \),

Use the table: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term.
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term. Partial fractions:
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term. Partial fractions:

\[ \frac{1}{s(s + 2)} = \frac{a}{s} + \frac{b}{s + 2}. \]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( L[y] = 3L[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term.

Partial fractions:

\[
\frac{1}{s(s + 2)} = \frac{a}{s} + \frac{b}{s + 2} = \frac{a(s + 2) + bs}{s(s + 2)}
\]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term. Partial fractions:

\[
\frac{1}{s(s + 2)} = \frac{a}{s} + \frac{b}{(s + 2)} = \frac{a(s + 2) + bs}{s(s + 2)} = \frac{(a + b)s + (2a)}{s(s + 2)}
\]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP
\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \]

We need to invert the Laplace transform on the last term. Partial fractions:
\[ \frac{1}{s(s + 2)} = \frac{a}{s} + \frac{b}{s + 2} = \frac{a(s + 2) + bs}{s(s + 2)} = \frac{(a + b)s + (2a)}{s(s + 2)}. \]

We get, \( a + b = 0, \ 2a = 1. \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall: \( \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s + 2)}. \)

We need to invert the Laplace transform on the last term.
Partial fractions:

\[
\frac{1}{s(s + 2)} = \frac{a}{s} + \frac{b}{s + 2} = \frac{a(s + 2) + bs}{s(s + 2)} = \frac{(a + b)s + (2a)}{s(s + 2)}
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We get, \( a + b = 0, \ 2a = 1. \) We obtain: \( a = \frac{1}{2}, \ b = -\frac{1}{2}. \)
Differential equations with discontinuous sources.

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We get, \( a + b = 0, \ 2a = 1. \) We obtain: \( a = \frac{1}{2}, \ b = -\frac{1}{2}. \) Hence,

\[ \frac{1}{s(s + 2)} = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s + 2} \right]. \]
Differential equations with discontinuous sources.

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The algebraic equation for \( \mathcal{L}[y] \) has the form,

\[
\mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + \frac{1}{2} \left[ e^{-4s} \frac{1}{s} - e^{-4s} \frac{1}{s + 2} \right].
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\[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] \]
Differential equations with discontinuous sources.

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\[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + \frac{1}{2} \left[ e^{-4s} \frac{1}{s} - e^{-4s} \frac{1}{(s + 2)} \right]. \]

\[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + \frac{1}{2} \left( \mathcal{L}[u(t - 4)] \right) \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall:

\[ \frac{1}{s(s + 2)} = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s + 2} \right]. \]

The algebraic equation for \( \mathcal{L}[y] \) has the form,

\[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + \frac{1}{2} \left[ e^{-4s} \frac{1}{s} - e^{-4s} \frac{1}{s + 2} \right]. \]

\[ \mathcal{L}[y] = 3 \mathcal{L}[e^{-2t}] + \frac{1}{2} \left( \mathcal{L}[u(t - 4)] - \mathcal{L}[u(t - 4) e^{-2(t - 4)}] \right). \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

Solution: Recall:

\[ \frac{1}{s(s + 2)} = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s + 2} \right]. \]

The algebraic equation for \( L[y] \) has the form,

\[ L[y] = 3 \cdot L[e^{-2t}] + \frac{1}{2} \left[ e^{-4s} \cdot \frac{1}{s} - e^{-4s} \cdot \frac{1}{s + 2} \right]. \]

\[ L[y] = 3 \cdot L[e^{-2t}] + \frac{1}{2} \left( L[u(t - 4)] - L[u(t - 4) \cdot e^{-2(t-4)}] \right). \]

We conclude that

\[ y(t) = 3e^{-2t} + \frac{1}{2} u(t - 4) \left[ 1 - e^{-2(t-4)} \right]. \]
Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:
  \[ y' + 2y = u(t - 4), \quad y(0) = 3. \]

(b) Example 2:
\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
0, & t \in [\pi, \infty). 
\end{cases} \]

(c) Example 3:
\[ y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 
\sin(t), & t \in [0, \pi) \\
0, & t \in [\pi, \infty). 
\end{cases} \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]

where \( b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
0, & t \in [\pi, \infty). 
\end{cases} \)
Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution:

Rewrite the source function using step functions.
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]

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1, & t \in [0, \pi) \\
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Rewrite the source function using step functions.
Differential equations with discontinuous sources.

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Solution: The graphs imply: \( b(t) = u(t) - u(t - \pi) \)
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

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Now is simple to find \( \mathcal{L}[b] \),
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]
\[ b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
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Now is simple to find \( \mathcal{L}[b] \), since

\[ \mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

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\[ \mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}. \]
Differential equations with discontinuous sources.

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So, the source is \( \mathcal{L}[b(t)] = (1 - e^{-\pi s}) \frac{1}{s} \),
Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: The graphs imply: \( b(t) = u(t) - u(t - \pi) \)

Now is simple to find \( \mathcal{L}[b] \), since

\[ \mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}. \]

So, the source is \( \mathcal{L}[b(t)] = (1 - e^{-\pi s}) \frac{1}{s} \), and the equation is

\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: So:

\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]
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The initial conditions imply:
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Solution: So:\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]

The initial conditions imply:\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: So:

\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]

The initial conditions imply: \( \mathcal{L}[y''] = s^2 \mathcal{L}[y] \) and \( \mathcal{L}[y'] = s \mathcal{L}[y] \).
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}, \quad y'(0) = 0, \]

Solution: So:

\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]

The initial conditions imply: \( \mathcal{L}[y''] = s^2 \mathcal{L}[y] \) and \( \mathcal{L}[y'] = s \mathcal{L}[y] \).

Therefore,

\[ \left( s^2 + s + \frac{5}{4} \right) \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]
\[ b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: So: \[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]

The initial conditions imply: \[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] \text{ and } \mathcal{L}[y'] = s \mathcal{L}[y]. \]

Therefore, \[ \left( s^2 + s + \frac{5}{4} \right) \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}. \]

We arrive at the expression: \[ \mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s \left(s^2 + s + \frac{5}{4}\right)}. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s \left(s^2 + s + \frac{5}{4}\right)}. \)
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Solution: Recall: \( \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \).

Denoting: \( H(s) = \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \).
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( \mathcal{L}[y] = (1 - e^{-\pi s}) \cdot \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \).

Denoting: \( H(s) = \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \),

we obtain, \( \mathcal{L}[y] = (1 - e^{-\pi s}) \cdot H(s) \).
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \).

Denoting: \( H(s) = \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \),

we obtain, \( \mathcal{L}[y] = (1 - e^{-\pi s}) H(s) \).

In other words: \( y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)] \).
Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]. \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \[ y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]. \]

Denoting: \[ h(t) = \mathcal{L}^{-1}[H(s)], \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases} \]

Solution: Recall: \( y(t) = \mathcal{L}^{-1} \left[ H(s) \right] - \mathcal{L}^{-1} \left[ e^{-\pi s} H(s) \right] \).

Denoting: \( h(t) = \mathcal{L}^{-1} \left[ H(s) \right] \), the \( \mathcal{L}[ \cdot ] \) properties imply

\[ \mathcal{L}^{-1} \left[ e^{-\pi s} H(s) \right] = u(t - \pi) h(t - \pi). \]
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall:

\[ y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]. \]

Denoting: \( h(t) = \mathcal{L}^{-1}[H(s)] \), the \( \mathcal{L}[\cdot] \) properties imply

\[ \mathcal{L}^{-1}[e^{-\pi s} H(s)] = u(t - \pi) h(t - \pi). \]

Therefore, the solution has the form

\[ y(t) = h(t) - u(t - \pi) h(t - \pi). \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]. \)

Denoting: \( h(t) = \mathcal{L}^{-1}[H(s)] \), the \( \mathcal{L}[\ ] \) properties imply

\[ \mathcal{L}^{-1}[e^{-\pi s} H(s)] = u(t - \pi) h(t - \pi). \]

Therefore, the solution has the form

\[ y(t) = h(t) - u(t - \pi) h(t - \pi). \]

We only need to find \( h(t) = \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\right]. \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \[ h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \right]. \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]

\[ b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases} \]

Solution: Recall: \( h(t) = \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\right]. \)

Partial fractions:
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( h(t) = \mathcal{L}^{-1}\left[ \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \right] \).

Partial fractions: Find the zeros of the denominator,
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
0, & t \in [\pi, \infty). 
\end{cases} \]

\[ y'(0) = 0, \quad b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
0, & t \in [\pi, \infty). 
\end{cases} \]

Solution: Recall: \( h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \right] \).

Partial fractions: Find the zeros of the denominator,

\[ s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 - 5} \right] \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( h(t) = \mathcal{L}^{-1}\left[ \frac{1}{s \left( s^2 + s + \frac{5}{4} \right)} \right]. \)

Partial fractions: Find the zeros of the denominator,

\[ s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 - 5} \right] \quad \Rightarrow \quad \text{Complex roots}. \]
Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

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Differential equations with discontinuous sources.

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Solution: Recall:

\[ H(s) = \frac{1}{s^2 + s + \frac{5}{4}} \frac{1}{s} = \frac{a}{s} + \frac{bs + c}{s^2 + s + \frac{5}{4}}. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( H(s) = \frac{1}{(s^2 + s + \frac{5}{4}) s} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + s + \frac{5}{4})}. \)

The partial fraction decomposition is:

\[ 1 = a \left( s^2 + s + \frac{5}{4} \right) + s (bs + c) \]
Differential equations with discontinuous sources.

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The partial fraction decomposition is:

\[ 1 = a \left( s^2 + s + \frac{5}{4} \right) + s (bs + c) = (a + b) s^2 + (a + c) s + \frac{5}{4}a. \]
Differential equations with discontinuous sources.

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Solution: Recall: $H(s) = \frac{1}{s^2 + s + \frac{5}{4}} = \frac{a}{s} + \frac{b s + c}{s^2 + s + \frac{5}{4}}$.

The partial fraction decomposition is:

$$ 1 = a \left( s^2 + s + \frac{5}{4} \right) + s (b s + c) = (a + b) s^2 + (a + c) s + \frac{5}{4} a. $$

This equation implies that $a$, $b$, and $c$, are solutions of

$$ a + b = 0, \quad a + c = 0, \quad \frac{5}{4} a = 1. $$
Differential equations with discontinuous sources.

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Solution: So: \( a = \frac{4}{5}, \quad b = -\frac{4}{5}, \quad c = -\frac{4}{5}. \)
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Solution: So:

\[ a = \frac{4}{5}, \quad b = -\frac{4}{5}, \quad c = -\frac{4}{5}. \]

Hence, we have found that,

\[ H(s) = \frac{1}{s^2 + s + \frac{5}{4}} \]
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We have to compute the inverse Laplace Transform.
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \]

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We have to compute the inverse Laplace Transform

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s + 1)}{\left( s^2 + s + \frac{5}{4} \right)} \right] \]
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Solution: Recall:

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s + 1)}{(s^2 + s + \frac{5}{4})} \right]. \]

In this case we complete the square in the denominator,
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
1, & t \in [0, \pi) \\
0, & t \in [\pi, \infty).
\end{cases} \]

Solution: Recall: \( h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s + 1)}{(s^2 + s + \frac{5}{4})} \right]. \)

In this case we complete the square in the denominator,

\[ s^2 + s + \frac{5}{4} = \left[ s^2 + 2 \left( \frac{1}{2} \right) s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

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\[ s^2 + s + \frac{5}{4} = \left[ s^2 + 2 \left( \frac{1}{2} \right) s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} = \left( s + \frac{1}{2} \right)^2 + 1. \]
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\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
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So: \( h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s + 1)}{\left( (s + \frac{1}{2})^2 + 1 \right)} \right]. \)
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

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In this case we complete the square in the denominator,

\[ s^2 + s + \frac{5}{4} = \left[ s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} = \left( s + \frac{1}{2} \right)^2 + 1. \]

So:

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{1}{s} - \frac{(s + 1)}{\left( (s + \frac{1}{2})^2 + 1 \right)} \right]. \]

That is,

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{\left( s + \frac{1}{2} \right) + \frac{1}{2}}{\left( (s + \frac{1}{2})^2 + 1 \right)} \right]. \]
Differential equations with discontinuous sources.

Example

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\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall:

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5} \mathcal{L}^{-1}\left[\frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1}\right]. \]
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Solution: Recall:

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{(s + \frac{1}{2}) + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} \right]. \]

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1}\left[ \frac{(s + \frac{1}{2})}{(s + \frac{1}{2})^2 + 1} \right] - \frac{2}{5} \mathcal{L}^{-1}\left[ \frac{1}{(s + \frac{1}{2})^2 + 1} \right]. \]
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Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
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0, & t \in [\pi, \infty). 
\end{cases} \]

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\[ h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5} \mathcal{L}^{-1}\left[\frac{(s + \frac{1}{2}) + \frac{1}{2}}{((s + \frac{1}{2})^2 + 1)}\right]. \]

Recall:

\[ \mathcal{L}^{-1}\left[F(s - c)\right] = e^{ct} f(t). \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall:

\[
    h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5} \mathcal{L}^{-1}\left[\frac{(s + \frac{1}{2}) + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1}\right].
\]

Recall: \( \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t) \). Hence,

\[
    h(t) = \frac{4}{5} \left[1 - e^{-t/2} \cos(t) - \frac{1}{2} e^{-t/2} \sin(t)\right].
\]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall:

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{\left( s + \frac{1}{2} \right) + \frac{1}{2}}{\left( s + \frac{1}{2} \right)^2 + 1} \right]. \]

\[ h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{\left( s + \frac{1}{2} \right)}{\left( s + \frac{1}{2} \right)^2 + 1} \right] - \frac{2}{5} \mathcal{L}^{-1} \left[ \frac{1}{\left( s + \frac{1}{2} \right)^2 + 1} \right]. \]

Recall: \( \mathcal{L}^{-1} [F(s - c)] = e^{ct} f(t) \). Hence,

\[ h(t) = \frac{4}{5} \left[ 1 - e^{-t/2} \cos(t) - \frac{1}{2} e^{-t/2} \sin(t) \right]. \]

We conclude:

\[ y(t) = h(t) + u(t - \pi) h(t - \pi). \]
Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  
  (a) Example 1:

  \[
  y' + 2y = u(t - 4), \quad y(0) = 3.
  \]

  (b) Example 2:

  \[
  y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 
  1, & t \in [0, \pi) \\
  0, & t \in [\pi, \infty) 
  \end{cases}
  \]

  (c) Example 3:

  \[
  y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 
  \sin(t), & t \in [0, \pi) \\
  0, & t \in [\pi, \infty) 
  \end{cases}
  \]
Example
Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases} \]
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Solution:

Rewrite the source function using step functions.
Differential equations with discontinuous sources.

Example

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\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases} \]

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Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 
\sin(t) & t \in [0, \pi) \\
0 & t \in [\pi, \infty).
\end{cases}
\]

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Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

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Example
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\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: The graphs imply: \( g(t) = [u(t) - u(t - \pi)] \sin(t). \)
Differential equations with discontinuous sources.

Example
Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: The graphs imply: \( g(t) = [u(t) - u(t - \pi)] \sin(t). \)

Recall the identity: \( \sin(t) = -\sin(t - \pi). \)
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: The graphs imply: \( g(t) = [u(t) - u(t - \pi)] \sin(t). \)

Recall the identity: \( \sin(t) = -\sin(t - \pi). \) Then,

\[ g(t) = u(t) \sin(t) - u(t - \pi) \sin(t), \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases} \]

\[ y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: The graphs imply: \( g(t) = [u(t) - u(t - \pi)] \sin(t) \).

Recall the identity: \( \sin(t) = -\sin(t - \pi) \). Then,

\[ g(t) = u(t) \sin(t) - u(t - \pi) \sin(t), \]

\[ g(t) = u(t) \sin(t) + u(t - \pi) \sin(t - \pi). \]
Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}, \quad y'(0) = 0, \]

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Now is simple to find \( \mathcal{L}[g] \),
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

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Now is simple to find \( \mathcal{L}[g] \), since

\[ \mathcal{L}[g(t)] = \mathcal{L}[u(t) \sin(t)] + \mathcal{L}[u(t - \pi) \sin(t - \pi)]. \]
Differential equations with discontinuous sources.

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\[ \mathcal{L}[g(t)] = \mathcal{L}[u(t) \sin(t)] + \mathcal{L}[u(t - \pi) \sin(t - \pi)]. \]

\[ \mathcal{L}[g(t)] = \frac{1}{s^2 + 1} + e^{-\pi s} \frac{1}{s^2 + 1}. \]
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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \]

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Recall the Laplace transform of the differential equation

\[ \mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = \mathcal{L}[g]. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

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The initial conditions imply:
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 
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\[ \mathcal{L}[g(t)] = \frac{1}{(s^2 + 1)} + e^{-\pi s} \frac{1}{(s^2 + 1)}. \]

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The initial conditions imply: \( \mathcal{L}[y''] = s^2 \mathcal{L}[y] \)
Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] \text{ and } \mathcal{L}[y'] = s \mathcal{L}[y]. \]

Therefore,

\[ \left( s^2 + s + \frac{5}{4} \right) \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}. \]
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 
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Solution: Recall:

\[ \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}. \]

\[ \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}. \]
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\[
\mathcal{L}[y] = \left( s^2 + s + \frac{5}{4} \right) \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}.
\]

Introduce the function

\[
H(s) = \frac{1}{\left( s^2 + s + \frac{5}{4} \right) (s^2 + 1)}.
\]
Differential equations with discontinuous sources.

Example

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Solution: Recall:

\[ (s^2 + s + \frac{5}{4}) \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}. \]

Introduce the function \( H(s) = \frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)} \).

Then, \( y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s} H(s)]. \)
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: Recall: \( y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s} H(s)], \) and

\[ H(s) = \frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)}. \]
Differential equations with discontinuous sources.

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\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \]
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Partial fractions:
Differential equations with discontinuous sources.

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Partial fractions: Find the zeros of the denominator,
Differential equations with discontinuous sources.

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Partial fractions: Find the zeros of the denominator,

\[ s_{\pm} = \frac{1}{2} [-1 \pm \sqrt{1 - 5}] \]
Differential equations with discontinuous sources.

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\[ s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 - 5} \right] \quad \Rightarrow \quad \text{Complex roots}. \]
Differential equations with discontinuous sources.

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The partial fraction decomposition is:
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\[ s_{\pm} = \frac{1}{2} \left[ -1 \pm \sqrt{1 - 5} \right] \Rightarrow \text{Complex roots.} \]

The partial fraction decomposition is:

\[ \frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: So:

\[ \frac{1}{(s^2 + s + \frac{5}{4}) (s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}. \]
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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: So:

\[
\frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}.\]

Therefore, we get

\[ 1 = (as + b)(s^2 + 1) + (cs + d)\left(s^2 + s + \frac{5}{4}\right), \]
Differential equations with discontinuous sources.

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\[
1 = (as + b)(s^2 + 1) + (cs + d)\left(s^2 + s + \frac{5}{4}\right),
\]

\[
1 = (a + c) s^3 + (b + c + d) s^2 + \left(a + \frac{5}{4} c + d\right) s + \left(b + \frac{5}{4} d\right).
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Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Solution: So:

\[ \frac{1}{(s^2 + s + \frac{5}{4}) (s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}. \]

Therefore, we get

\[ 1 = (as + b)(s^2 + 1) + (cs + d)\left(s^2 + s + \frac{5}{4}\right), \]

\[ 1 = (a + c) s^3 + (b + c + d) s^2 + \left(a + \frac{5}{4} c + d\right) s + \left(b + \frac{5}{4} d\right). \]

This equation implies that \(a, b, c,\) and \(d,\) are solutions of

\[ a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4} c + d = 0, \quad b + \frac{5}{4} d = 1. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty) \end{cases}. \]

Solution: So:

\[ a = \frac{16}{17}, \quad b = \frac{12}{17}, \quad c = -\frac{16}{17}, \quad d = \frac{4}{17}. \]
Example

Use the Laplace transform to find the solution of the IVP

\[ y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \]

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Solution: So:

\[ a = \frac{16}{17}, \quad b = \frac{12}{17}, \quad c = -\frac{16}{17}, \quad d = \frac{4}{17}. \]

We have found:

\[ H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{(s^2 + s + \frac{5}{4})} + \frac{(-4s + 1)}{(s^2 + 1)} \right]. \]
Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Complete the square in the denominator,
Differential equations with discontinuous sources.

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\[ s^2 + s + \frac{5}{4} = \left[ s^2 + 2\left(\frac{1}{2}\right) s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} \]
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4} y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

$$g(t) = \begin{cases} 
\sin(t) & t \in [0, \pi) \\
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\end{cases}.$$

Solution: So:

$$a = \frac{16}{17}, \quad b = \frac{12}{17}, \quad c = -\frac{16}{17}, \quad d = \frac{4}{17}.$$

We have found:

$$H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{(s^2 + s + \frac{5}{4})} + \frac{(-4s + 1)}{(s^2 + 1)} \right].$$

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We have found: \( H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{(s^2 + s + \frac{5}{4})} + \frac{(-4s + 1)}{(s^2 + 1)} \right]. \)

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\[ H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{\left( s + \frac{1}{2} \right)^2 + 1} + \frac{(-4s + 1)}{(s^2 + 1)} \right]. \]
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Solution: So: \[ H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{(s + \frac{1}{2})^2 + 1} + \frac{(-4s + 1)}{(s^2 + 1)} \right]. \]
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\[ H(s) = \frac{4}{17} \left[ \frac{(4s + 3)}{(s + \frac{1}{2})^2 + 1} \right] + \frac{(-4s + 1)}{(s^2 + 1)}. \]

Rewrite the polynomial in the numerator,
Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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We conclude: \( y(t) = h(t) + u(t - \pi)h(t - \pi). \) \( \triangle \)