Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).

Regular-singular points (5.5).

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- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.

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- Recall: Since $r \neq 0$, holds
  $$y' = \sum_{n=0}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)},$$

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- Find the indicial equation for r, the recurrence relation for a_n.
- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.

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- Find the indicial equation for r, the recurrence relation for a_n.
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  (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.

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- Find the indicial equation for r, the recurrence relation for a_n.
- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.

  (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.

  (b) If $(r_+ - r_-)$ is an integer, then both r_+ and r_- define proportional solutions.

Regular-singular points (5.5).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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Solution: $y = \displaystyle\sum_{n=0}^{\infty} a_n x^{(n+r)}$,

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Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$,

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We also need to compute

$$\left(x^2 + \frac{1}{4}\right)y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

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Re-label $m = n + 2$ in the first term and then switch back to n,

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Re-label $m = n + 2$ in the first term and then switch back to n,

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

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The equation is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

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$$\left[r(r-1) + \frac{1}{4}\right] a_0\, x^r + \left[(r+1)r + \frac{1}{4}\right] a_1\, x^{(r+1)} +$$

$$\sum_{n=2}^{\infty}\left[(n+r)(n+r-1)a_n + a_{(n-2)} + \frac{1}{4} a_n\right] x^{(n+r)} = 0.$$

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The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r_\pm = \frac{1}{2}$.

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$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2}$$

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$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2} \;\Rightarrow\; \begin{cases} a_2 = -\dfrac{a_0}{4}, \\ a_4 = -\dfrac{a_2}{16} \end{cases}$$

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$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2} \;\Rightarrow\; \begin{cases} a_2 = -\dfrac{a_0}{4}, \\[2mm] a_4 = -\dfrac{a_2}{16} = \dfrac{a_0}{64}. \end{cases}$$

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Consider the equation $x^2\,y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$.

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Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$. Then,

$$y(x) = x^r \big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\big).$$

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$$y(x) = x^r\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd.

Regular-singular points (5.5).

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Consider the equation $x^2\,y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$. Then,

$$y(x) = x^r\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd. Therefore,

$$y(x) = a_0 x^{1/2}\left(1 - \frac{1}{4}\,x^2 + \frac{1}{64}\,x^4 + \cdots\right). \qquad \triangleleft$$

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  - **Euler differential equation (5.4).**
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).

Euler differential equation (5.4).

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- $(x - x_0)^2\, y'' + (x - x_0)p_0\, y' + q_0\, y = 0.$

Euler differential equation (5.4).

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- $(x - x_0)^2\, y'' + (x - x_0) p_0\, y' + q_0\, y = 0.$
- Find $r_\pm$ solutions of $r(r-1) + p_0 r + q_0 = 0.$

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- $(x - x_0)^2\, y'' + (x - x_0) p_0\, y' + q_0\, y = 0.$
- Find $r_\pm$ solutions of $r(r-1) + p_0 r + q_0 = 0$.
- If $r_+ \neq r_-$ and both are real, then fundamental solutions are

$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}.$$

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- If $r_\pm = \alpha \pm i\beta$, then real-valued fundamental solutions are

$$y_+ = |x - x_0|^{\alpha} \cos\big(\beta \ln |x - x_0|\big), \; y_- = |x - x_0|^{\alpha} \sin\big(\beta \ln |x - x_0|\big).$$

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$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_+} \ln|x - x_0|.$$

Euler differential equation (5.4).

Example

Find real-valued fundamental solutions of

$$(x-2)^2\, y'' + 5(x-2)\, y' + 8\, y = 0.$$

Euler differential equation (5.4).

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Solution: $y_1 = e^{-2x}$, $y_2 = x\, e^{-2x}$, $g = x^{-2}\, e^{-2x}$, $W = e^{-4x}$.

Now we find the functions u_1 and u_2,

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Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - **Undetermined coefficients (3.5)**
  - Constant coefficients, homogeneous, (3.1)-(3.4).

Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

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Therefore, $y_p = -\frac{3}{4}\,x\cos(2x)$. The general solution is

$$y(x) = c_1\sin(2x) + \Big(c_2 - \frac{3}{4}\,x\Big)\cos(2x).$$

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

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Definition

A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

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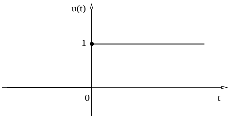

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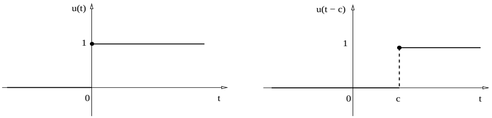

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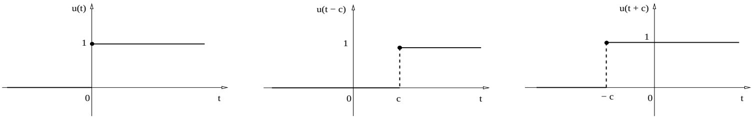

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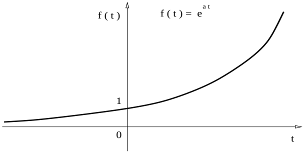

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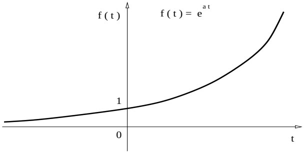

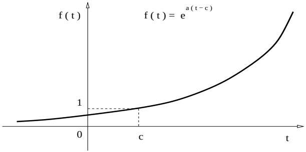

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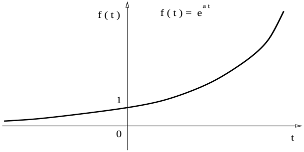

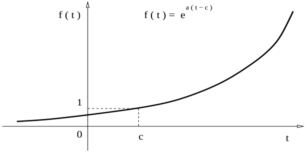

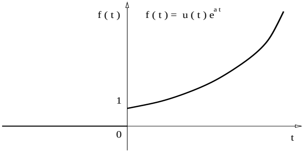

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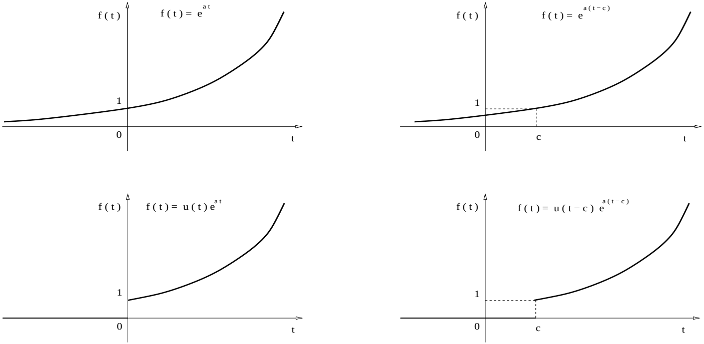

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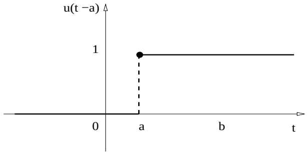

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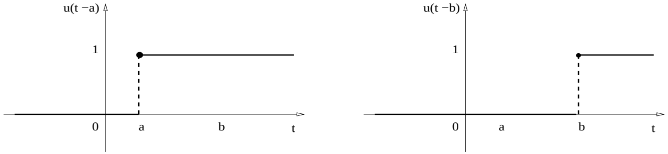

Piecewise discontinuous functions.

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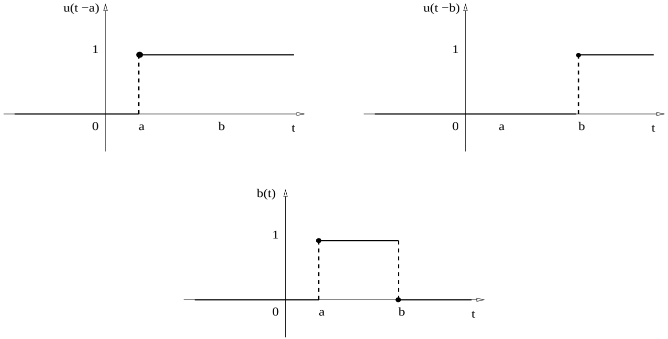

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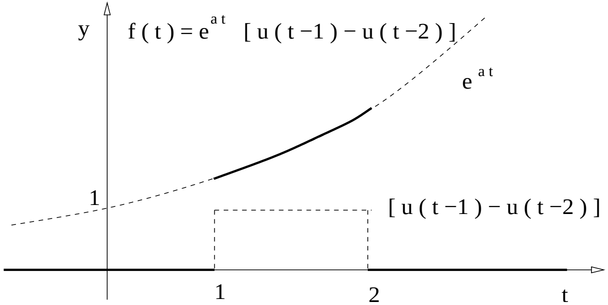

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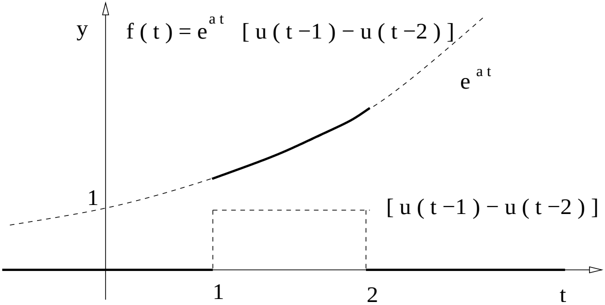

Notation: The function values $u(t-c)$ are denoted in the textbook as $u_c(t)$.

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- **The Laplace Transform of discontinuous functions.**
- Properties of the Laplace Transform.

The Laplace Transform of discontinuous functions.

Theorem

Given any real number c, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

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We conclude that $\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}$. □

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

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- Overview and notation.
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Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c > 0$, *then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}F(s), \qquad s > a.$$

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Example

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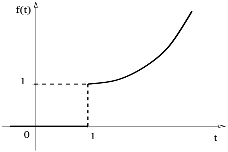

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Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

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$$\mathcal{L}^{-1}\Big[\frac{e^{-4s}}{s^2+9}\Big] = \frac{1}{3}\,u(t-4)\,\sin\big(3(t-4)\big). \qquad \triangleleft$$

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Properties of the Laplace Transform.

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Find $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{s}{s^2+a^2}\Big] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\, f(t)$.

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Solution: Recall: $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh(at)$

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Example

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Properties of the Laplace Transform.

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$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1+8}\right]$$

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Hence: $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big] = \dfrac{1}{3}\,u(t-2)\Big[e^{(t-2)} - e^{-2(t-2)}\Big]$. $\triangleleft$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

  (c) Example 3:

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Differential equations with discontinuous sources.

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$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s}$$

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Introduce the initial condition,

Differential equations with discontinuous sources.

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Use the table: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

Differential equations with discontinuous sources.

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We need to invert the Laplace transform on the last term.

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We get, $a+b=0$, $2a=1$. We obtain: $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. Hence,

$$\frac{1}{s(s+2)} = \frac{1}{2}\Big[\frac{1}{s} - \frac{1}{(s+2)}\Big].$$

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The algebraic equation for $\mathcal{L}[y]$ has the form,

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$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{1}{(s+2)}\right].$

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\left[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\right].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\big[u(t-4)\,e^{-2(t-4)}\big]\Big).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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$$\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\left[u(t-4)\,e^{-2(t-4)}\right]\Big).$$

We conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big].$$

◁

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) **Example 2:**

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

  (c) Example 3:

$$y'' + y' + \frac{5}{4}\, y = g(t), \quad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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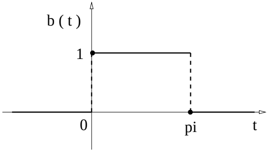

Differential equations with discontinuous sources.

Example

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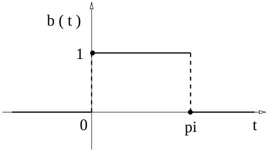

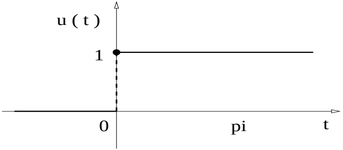

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{l} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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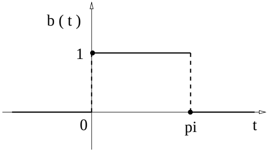

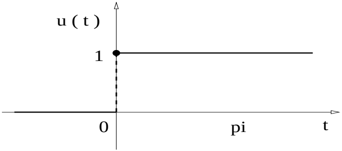

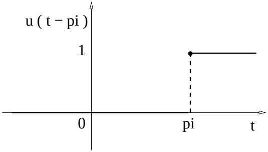

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Now is simple to find $\mathcal{L}[b]$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is $\mathcal{L}[b(t)] = \left(1 - e^{-\pi s}\right) \frac{1}{s}$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t-\pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is $\mathcal{L}[b(t)] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$, and the equation is

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\, \mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}.$

The initial conditions imply: $\mathcal{L}[y''] = s^2\, \mathcal{L}[y]$ and $\mathcal{L}[y'] = s\, \mathcal{L}[y]$.

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$.

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$.

We arrive at the expression: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)},$

we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s).$

Differential equations with discontinuous sources.

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Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s)$.

In other words: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\ ]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t-\pi)\, h(t-\pi).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$\mathcal{L}^{-1}\big[e^{-\pi s}H(s)\big] = u(t-\pi)\,h(t-\pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t-\pi)\,h(t-\pi).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{l} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\;]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t - \pi)\, h(t - \pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t - \pi)\, h(t - \pi).$$

We only need to find $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\right].$

Differential equations with discontinuous sources.

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Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1 - 5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c)$$

Differential equations with discontinuous sources.

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This equation implies that a, b, and c, are solutions of

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}\,a = 1.$$

Differential equations with discontinuous sources.

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Solution: So: $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$.

Differential equations with discontinuous sources.

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We have to compute the inverse Laplace Transform

Differential equations with discontinuous sources.

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$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\right]$$

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Solution: Recall: $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s} - \dfrac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big].$

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In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4}$$

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big].$

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So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}\Big].$

That is, $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\left(s+\frac{1}{2}\right) + \frac{1}{2}}{\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}\Big].$

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$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$

Differential equations with discontinuous sources.

Example

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$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\, f(t)$. Hence,

$$h(t) = \frac{4}{5}\,\Big[1 - e^{-t/2}\,\cos(t) - \frac{1}{2}\, e^{-t/2}\,\sin(t)\Big].$$

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$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\lhd$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

  (c) **Example 3:**

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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Example

Use the Laplace transform to find the solution of the IVP

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Solution:

Rewrite the source function using step functions.

Differential equations with discontinuous sources.

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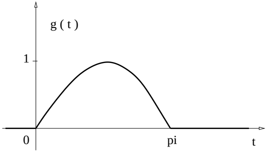

Differential equations with discontinuous sources.

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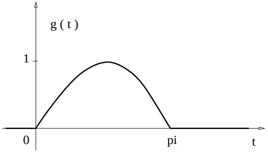

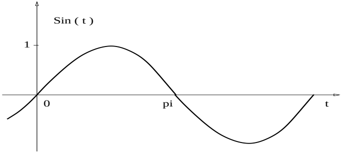

Differential equations with discontinuous sources.

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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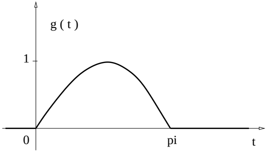

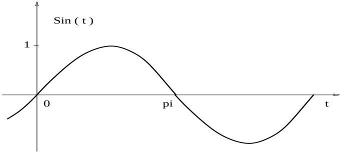

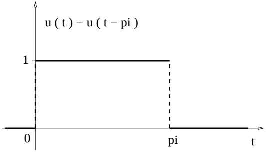

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Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

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Recall the identity: $\sin(t) = -\sin(t-\pi)$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\, \sin(t)$.

Recall the identity: $\sin(t) = -\sin(t - \pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

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Now is simple to find $\mathcal{L}[g]$,

Differential equations with discontinuous sources.

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Now is simple to find $\mathcal{L}[g]$, since

$$\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)].$$

Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Differential equations with discontinuous sources.

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Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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Differential equations with discontinuous sources.

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The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)}$.

Differential equations with discontinuous sources.

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$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

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Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}$.

Differential equations with discontinuous sources.

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Then, $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\, H(s)].$

Differential equations with discontinuous sources.

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Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

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Differential equations with discontinuous sources.

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Partial fractions: Find the zeros of the denominator,

Differential equations with discontinuous sources.

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$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1-5}\right]$$

Differential equations with discontinuous sources.

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$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 5}\right] \quad \Rightarrow \quad \text{Complex roots.}$$

Differential equations with discontinuous sources.

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The partial fraction decomposition is:

Differential equations with discontinuous sources.

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$$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Differential equations with discontinuous sources.

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Solution: So: $$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Differential equations with discontinuous sources.

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Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\Big(s^2 + s + \frac{5}{4}\Big),$$

Differential equations with discontinuous sources.

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$$1 = (a + c)\,s^3 + (b + c + d)\,s^2 + \left(a + \frac{5}{4}\,c + d\right)s + \left(b + \frac{5}{4}\,d\right).$$

Differential equations with discontinuous sources.

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$$1 = (a + c)\,s^3 + (b + c + d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0,\pi) \\ 0 & t \in [\pi,\infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

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We have found: $H(s) = \dfrac{4}{17}\Big[\dfrac{(4s+3)}{\big(s^2+s+\frac{5}{4}\big)} + \dfrac{(-4s+1)}{(s^2+1)}\Big]$.

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Complete the square in the denominator,

Differential equations with discontinuous sources.

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$$s^2 + s + \frac{5}{4} = \left[s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}\right] - \frac{1}{4} + \frac{5}{4}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

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$$s^2 + s + \frac{5}{4} = \left[s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}\right] - \frac{1}{4} + \frac{5}{4} = \left(s + \frac{1}{2}\right)^2 + 1.$$

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$$s^2 + s + \frac{5}{4} = \left[s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}\right] - \frac{1}{4} + \frac{5}{4} = \left(s + \frac{1}{2}\right)^2 + 1.$$

$$H(s) = \frac{4}{17}\left[\frac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2 + 1\right]} + \frac{(-4s+1)}{(s^2+1)}\right].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \dfrac{4}{17}\left[\dfrac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \dfrac{(-4s+1)}{(s^2+1)}\right].$

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Rewrite the polynomial in the numerator,

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Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3$$

Differential equations with discontinuous sources.

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Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\left[4\,\mathcal{L}\left[e^{-t/2}\cos(t)\right] + \mathcal{L}\left[e^{-t/2}\sin(t)\right] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\right].$$

Differential equations with discontinuous sources.

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We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\triangleleft$