Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Repeated roots as a limit case.
- Main result for repeated roots.
- Reduction of the order method:
  - Constant coefficients equations.
  - Variable coefficients equations.
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation
\[
y'' + a_1 y' + a_0 y = 0
\]
with characteristic polynomial having roots
\[
r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]
Summary: Given constants $a_1, a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.$$ 

(1) If $a_1^2 - 4a_0 > 0$, 

(2) If $a_1^2 - 4a_0 < 0$, then introducing $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2} \sqrt{a_1^2 - 4a_0}$, 

$$y_1(t) = e^{\alpha t} \cos(\beta t),$$

$$y_2(t) = e^{\alpha t} \sin(\beta t).$$

(3) If $a_1^2 - 4a_0 = 0$, then 

$$y_1(t) = e^{-\frac{a_1}{2} t}.$$
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

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(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).
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y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
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Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation
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Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

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Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation

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y'' + a_1 y' + a_0 y = 0
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(3) If \( a_1^2 - 4a_0 = 0 \), then \( y_1(t) = e^{-\frac{a_1}{2} t} \).
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Question:
Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \frac{a_1^2}{4}$. 

▶ Does the equation $y'' + a_1 y' + a_{12}^4 y = 0$ have two linearly independent solutions?

▶ Or, every solution to the equation above is proportional to $y_1(t) = e^{-\frac{a_1}{2} t}$. 

▶
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

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Consider the case (3), with \( a_1^2 - 4a_0 = 0 \), that is, \( a_0 = \frac{a_1^2}{4} \).

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Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- **Repeated roots as a limit case.**
- Main result for repeated roots.
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Repeated roots as a limit case.

Remark:

- Case (3), where \( 4a_0 - a_1^2 = 0 \) can be obtained as the limit \( \beta \to 0 \) in case (2).
Repeated roots as a limit case.

Remark:
- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).
- Let us study the solutions of the differential equation in the case (2) as $\beta \to 0$ for fixed $t$. 

Since $\cos(\beta t) \to 1$ as $\beta \to 0$, we conclude that $y_{1\beta}(t) = e^{-a_1^2 t} \cos(\beta t) \to e^{-a_1^2 t} = y_1(t)$.

Since $\sin(\beta t) / \beta t \to 1$ as $\beta \to 0$, that is, $\sin(\beta t) \to \beta t$, we have $y_{2\beta}(t) = e^{-a_1^2 t} \sin(\beta t) \to 0$.

Is $y_2(t) = ty_1(t)$ solution of the differential equation? Introducing $y_2$ in the differential equation one obtains: Yes.

Since $y_2$ is not proportional to $y_1$, the functions $y_1$, $y_2$ are a fundamental set for the differential equation in case (3).
Repeated roots as a limit case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).
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y_{1\beta}(t) = e^{-\frac{a_1}{2} t} \cos(\beta t)
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Repeated roots as a limit case.

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Repeated roots as a limit case.

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Repeated roots as a limit case.

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Repeated roots as a limit case.

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  \[ y_{2\beta}(t) = e^{-\frac{a_1}{2} t} \sin(\beta t) \to \beta t e^{-\frac{a_1}{2} t} \to 0. \]
- Is \( y_2(t) = t \, y_1(t) \) solution of the differential equation?
Repeated roots as a limit case.

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► Is $y_2(t) = t \cdot y_1(t)$ solution of the differential equation? Introducing $y_2$ in the differential equation one obtains: Yes.
Repeated roots as a limit case.

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- Let us study the solutions of the differential equation in the case (2) as \(\beta \to 0\) for fixed \(t\).
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- Is \(y_2(t) = t y_1(t)\) solution of the differential equation? Introducing \(y_2\) in the differential equation one obtains: Yes.
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Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
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- **Main result for repeated roots.**
- Reduction of the order method:
  - Constant coefficients equations.
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Main result for repeated roots.

Theorem
If \( a_1, a_0 \in R \) satisfy that \( a_1^2 = 4a_0 \), then the functions
\[
y_1(t) = e^{-\frac{a_1}{2} t}, \quad y_2(t) = t e^{-\frac{a_1}{2} t},
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**Example**

Find the general solution of \( 9y'' + 6y' + y = 0 \).
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**Example**
Find the general solution of $9y'' + 6y' + y = 0$.

**Solution:** The characteristic equation is $9r^2 + 6r + 1 = 0$, so

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r_{\pm} = \frac{1}{(2)(9)} \left[ -6 \pm \sqrt{36 - 36} \right]
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Example
Find the general solution of \( 9y'' + 6y' + y = 0. \)

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\[
r_{\pm} = \frac{1}{(2)(9)} \left[ -6 \pm \sqrt{36 - 36} \right] \Rightarrow r_{\pm} = -\frac{1}{3}.
\]
The Theorem above implies that the general solution is
\[
y(t) = (c_1 + c_2 t) e^{-t/3}.
\]
Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
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Reduction of the order method: Constant coefficients.

Proof: Recall: The characteristic equation is \( r^2 + a_1 r + a_0 = 0 \),
Reduction of the order method: Constant coefficients.

Proof: Recall: The characteristic equation is \( r^2 + a_1 r + a_0 = 0 \), and its solutions are \( r_\pm = (1/2)\left[-a_1 \pm \sqrt{a_1^2 - 4a_0}\right] \).
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The hypothesis \( a_1^2 = 4a_0 \) implies \( r_+ = r_- = -a_1/2 \).
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The hypothesis \( a_1^2 = 4a_0 \) implies \( r_+ = r_- = -a_1/2 \).

So, the solution \( r_+ \) of the characteristic equation satisfies both

\[
r_+^2 + a_1 r_+ + a_0 = 0, \quad 2r_+ + a_1 = 0.
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Reduction of the order method: Constant coefficients.

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It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.
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It is clear that \( y_1(t) = e^{r_+ t} \) is solutions of the differential equation.

A second solution \( y_2 \) not proportional to \( y_1 \) can be found as follows: (D’Alembert \( \sim 1750 \).)
Reduction of the order method: Constant coefficients.

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The hypothesis \( a_1^2 = 4a_0 \) implies \( r_+ = r_- = -a_1/2 \).

So, the solution \( r_+ \) of the characteristic equation satisfies both

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It is clear that \( y_1(t) = e^{r_+ t} \) is solutions of the differential equation.

A second solution \( y_2 \) not proportional to \( y_1 \) can be found as follows: (D’Alembert \( \sim 1750 \).)

Express: \( y_2(t) = v(t) y_1(t) \), and find the equation that function \( v \) satisfies from the condition \( y''_2 + a_1 y'_2 + a_0 y_2 = 0 \).
Reduction of the order method: Constant coefficients.

Recall: $y_2 = vy_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. 
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1y_2' + a_0y_2 = 0 \). So, \( y_2 = ve^{r_+t} \) and

\[
y_2' = v' e^{r_+t} + r_+ ve^{r_+t}, \quad y_2'' = v'' e^{r_+t} + 2r_+ v' e^{r_+t} + r_+^2 ve^{r_+t}.
\]
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+t} \) and

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y_2' = v'e^{r_+t} + r_+ ve^{r_+t}, \quad y_2'' = v''e^{r_+t} + 2r_+ v'e^{r_+t} + r_+^2 ve^{r_+t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+t} + a_1 \left[ v' + r_+ v \right] e^{r_+t} + a_0 v e^{r_+t} = 0.
\]
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y''_2 + a_1 y'_2 + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y'_2 = v' e^{r_+ t} + r_+ ve^{r_+ t}, \quad y''_2 = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 ve^{r_+ t}.
\]

Introducing this information into the differential equation

\[
[v'' + 2r_+ v' + r_+^2 v] e^{r_+ t} + a_1 [v' + r_+ v] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
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\[
[v'' + 2r_+ v' + r_+^2 v] + a_1 [v' + r_+ v] + a_0 v = 0.
\]
Reduction of the order method: Constant coefficients.

Recall: $y_2 = vy_1$ and $y_2'' + a_1y_2' + a_0y_2 = 0$. So, $y_2 = ve^{r_+t}$ and

$$y_2' = v'e^{r_+t} + r_+ve^{r_+t}, \quad y_2'' = v''e^{r_+t} + 2r_+v'e^{r_+t} + r_+^2ve^{r_+t}.$$ Introducing this information into the differential equation

$$[v'' + 2r_+v' + r_+^2 v] e^{r_+t} + a_1[v' + r_+v] e^{r_+t} + a_0v e^{r_+t} = 0.$$ \[v'' + 2r_+v' + r_+^2 v] + a_1[v' + r_+v] + a_0v = 0$$

$$v'' + (2r_+ + a_1)v' + (r_+^2 + a_1 r_+ + a_0)v = 0$$
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1y'_2 + a_0y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y'_2 = v' e^{r_+ t} + r_+ ve^{r_+ t}, \quad y''_2 = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 ve^{r_+ t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
\]

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] + a_1 \left[ v' + r_+ v \right] + a_0 v = 0
\]

\[
v'' + (2r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
\]

Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2r_+ + a_1 = 0 \).
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = v y_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y_2' = v' e^{r_+ t} + r_+ v e^{r_+ t}, \quad y_2'' = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
\]

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] + a_1 \left[ v' + r_+ v \right] + a_0 v = 0 \]

\[
v'' + (2r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
\]

Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2r_+ + a_1 = 0 \).

\[
v'' = 0
\]
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y_2' = v' e^{r_+ t} + r_+ ve^{r_+ t}, \quad y_2'' = v'' e^{r_+ t} + 2 r_+ v' e^{r_+ t} + r_+^2 ve^{r_+ t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2 r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
\]

\[
\left[ v'' + 2 r_+ v' + r_+^2 v \right] + a_1 \left[ v' + r_+ v \right] + a_0 v = 0
\]

\[
v'' + (2 r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
\]

Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2 r_+ + a_1 = 0 \).

\[
v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t)
\]
Reduction of the order method: Constant coefficients.

Recall: $y_2 = vy_1$ and $y_2'' + a_1y_2' + a_0y_2 = 0$. So, $y_2 = ve^{r_+t}$ and

$$y_2' = v' e^{r_+t} + r_+ ve^{r_+t}, \quad y_2'' = v'' e^{r_+t} + 2r_+ v' e^{r_+t} + r_+^2 ve^{r_+t}.$$

Introducing this information into the differential equation

$$[v'' + 2r_+ v' + r_+^2 v] e^{r_+t} + a_1 [v' + r_+ v] e^{r_+t} + a_0 v e^{r_+t} = 0.$$

Recall that $r_+$ satisfies: $r_+^2 + a_1 r_+ + a_0 = 0$ and $2r_+ + a_1 = 0$.

$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t) e^{r_+t}.$$
Reduction of the order method: Constant coefficients.

Recall: We have obtained that \( y_2(t) = (c_1 + c_2 t) e^{r+t} \).
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r+t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r+t}$ and $y_1 = e^{r+t}$ are linearly dependent functions.
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r+t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r+t}$ and $y_1 = e^{r+t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t) e^{r+t}$ and $y_1 = e^{r+t}$ are linearly independent functions.
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2t) e^{r+t}$.

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Simplest choice: $c_1 = 0$ and $c_2 = 1$. 
Reduction of the order method: Constant coefficients.

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\[
    y_1(t) = e^{r+t}, \quad y_2(t) = t e^{r+t}
\]
Reduction of the order method: Constant coefficients.

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$$y_1(t) = e^{r+t}, \quad y_2(t) = t e^{r+t} \quad \Box$$

The general solution to the differential equation is

$$y(t) = (c_1 + c_2 t) e^{r+t}.$$
Reduction of the order method: Constant coefficients.

Example
Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \quad y(0) = 1, \quad y'(0) = \frac{5}{3}.$$
Reduction of the order method: Constant coefficients.

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Solution: The solutions of \( 9r^2 + 6r + 1 = 0 \), are \( r_+ = r_- = -\frac{1}{3} \).
Reduction of the order method: Constant coefficients.

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The Theorem above says that the general solution is

\[ y(t) = c_1 e^{-t/3} + c_2 te^{-t/3} \]
Reduction of the order method: Constant coefficients.

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The initial conditions imply that

\[ 1 = y(0) \]
Reduction of the order method: Constant coefficients.

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\[ 1 = y(0) = c_1, \]
Reduction of the order method: Constant coefficients.

Example
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\[ 1 = y(0) = c_1, \]
\[ \frac{5}{3} = y'(0). \]
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\[ 1 = y(0) = c_1, \]

\[ \frac{5}{3} = y'(0) = -\frac{c_1}{3} + c_2 \]
Example
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The initial conditions imply that

\[
\begin{align*}
1 &= y(0) = c_1, \\
\frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2
\end{align*}
\]

\[ \Rightarrow \quad c_1 = 1, \quad c_2 = 2. \]
Reduction of the order method: Constant coefficients.

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Find the solution to the initial value problem

\[ 9y'' + 6y' + y = 0, \quad y(0) = 1, \quad y'(0) = \frac{5}{3}. \]

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The Theorem above says that the general solution is

\[ y(t) = c_1 e^{-t/3} + c_2 te^{-t/3} \Rightarrow y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \left(1 - \frac{t}{3}\right) e^{-t/3}. \]

The initial conditions imply that

\[ \begin{cases} 1 = y(0) = c_1, \\ \frac{5}{3} = y'(0) = -\frac{c_1}{3} + c_2 \end{cases} \Rightarrow c_1 = 1, \quad c_2 = 2. \]

We conclude that \( y(t) = (1 + 2t) e^{-t/3} \).
Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
  - Variable coefficients equations.
Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove constant coefficients Theorem above can be used in variable coefficients equations.
Reduction of the order method: Variable coefficients.

**Remark:** The same idea used to prove constant coefficients Theorem above can be used in variable coefficients equations.

**Theorem**

*Given continuous functions* $p, q : (t_1, t_2) \to \mathbb{R}$, let $y_1 : (t_1, t_2) \to \mathbb{R}$ *be a solution of*

$$y'' + p(t) y' + q(t) y = 0,$$

*If the function* $v : (t_1, t_2) \to \mathbb{R}$ *is solution of*

$$y_1(t) v'' + [2y'(t) + p(t)y_1(t)] v' = 0.$$  \hspace{1cm} (1)

*then the functions* $y_1$ *and* $y_2 = v y_1$ *are fundamental solutions to the differential equation above.*
Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove constant coefficients Theorem above can be used in variable coefficients equations.

Theorem

Given continuous functions $p, q : (t_1, t_2) \to \mathbb{R}$, let $y_1 : (t_1, t_2) \to \mathbb{R}$ be a solution of

$$y'' + p(t) y' + q(t) y = 0,$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ is solution of

$$y_1(t) v'' + [2y'(t) + p(t)y_1(t)] v' = 0. \quad (1)$$

then the functions $y_1$ and $y_2 = v y_1$ are fundamental solutions to the differential equation above.

Remark: The reason for the name Reduction of order method is that the function $v$ does not appear in Eq. (1). This is a first order equation in $v'$. 
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.
Example
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\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t)y_1(t) \).
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = \nu(t)y_1(t) \). The equation for \( \nu \) comes from \( t^2y_2'' + 2ty_2' - 2y_2 = 0 \).
Example

Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = \nu(t)y_1(t) \). The equation for \( \nu \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute

\[ y_2 = \nu \, t, \]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = \nu(t)y_1(t) \). The equation for \( \nu \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute

\[ y_2 = \nu t, \quad y_2' = t \nu' + \nu, \]
Example

Find a fundamental set of solutions to

\[ t^2y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t)y_1(t) \). The equation for \( v \) comes from \( t^2y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute

\[
\begin{align*}
y_2 &= v t, \\
y_2' &= t v' + v, \\
y_2'' &= t v'' + 2v'.
\end{align*}
\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2ty'_2 - 2y_2 = 0 \). We need to compute

\[
\begin{align*}
y_2 &= vt, \\
y_2' &= t v' + v, \\
y_2'' &= t v'' + 2v'.
\end{align*}
\]

So, the equation for \( v \) is given by

\[
t^2 (t v'' + 2v') + 2t (t v' + v) - 2t v = 0
\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t)y_1(t) \). The equation for \( v \) comes from \( t^2y''_2 + 2ty'_2 - 2y_2 = 0 \). We need to compute

\[
\begin{align*}
y_2 &= vt, \\
y'_2 &= tv' + v, \\
y''_2 &= tv'' + 2v'.
\end{align*}
\]
So, the equation for \( v \) is given by

\[
\begin{align*}
t^2(tv'' + 2v') + 2t(tv' + v) - 2tv &= 0 \\
t^3v'' + (2t^2 + 2t^2)v' + (2t - 2t)v &= 0
\end{align*}
\]
Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t) y_1(t)$. The equation for $v$ comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

$$y_2 = v t, \quad y_2' = t v' + v, \quad y_2'' = t v'' + 2v'.$$

So, the equation for $v$ is given by

$$t^2 (t v'' + 2v') + 2t (t v' + v) - 2t v = 0$$

$$t^3 v'' + (2t^2 + 2t^2) v' + (2t - 2t) v = 0$$

$$t^3 v'' + (4t^2) v' = 0$$
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2t y' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2t y_2' - 2y_2 = 0 \). We need to compute
\[
\begin{align*}
y_2 &= v t, \\
y_2' &= t v' + v, \\
y_2'' &= t v'' + 2v'.
\end{align*}
\]
So, the equation for \( v \) is given by
\[
\begin{align*}
t^2 (t v'' + 2v') + 2t (t v' + v) - 2t v &= 0 \\
t^3 v'' + (2t^2 + 2t^2) v' + (2t - 2t) v &= 0 \\
t^3 v'' + (4t^2) v' &= 0 \quad \Rightarrow \quad v'' + \frac{4}{t} v' = 0.
\end{align*}
\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t}v' = 0$. 

\[\text{\Halmos}\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Recall: \( v'' + \frac{4}{t} v' = 0. \)
This is a first order equation for \( w = v' \),
\[ w'' + 4w' = 0. \]
Reduction of the order method: Variable coefficients.

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Find a fundamental set of solutions to
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knowing that \( y_1(t) = t \) is a solution.

Solution: Recall: \( v'' + \frac{4}{t} v' = 0. \)

This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t} w = 0, \)
Reduction of the order method: Variable coefficients.

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This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \), so
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\frac{w'}{w} = -\frac{4}{t}
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\[
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\[
\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad \ln(w) = -4\ln(t) + c_0 \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}.
\]
Reduction of the order method: Variable coefficients.

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\frac{w'}{w} = -\frac{4}{t} \Rightarrow \ln(w) = -4\ln(t) + c_0 \Rightarrow w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}.
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Integrating \( w \) we obtain \( v \),
Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Recall: \( v'' + \frac{4}{t}v' = 0. \)

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Integrating \( w \) we obtain \( v \), that is, \( v = c_2 t^{-3} + c_3 \), with \( c_2, c_3 \in \mathbb{R} \).
Recalling that \( y_2 = t v \) we then conclude that \( y_2 = c_2 t^{-2} + c_3 t \).
Choosing \( c_2 = 1 \) and \( c_3 = 0 \) we obtain the fundamental solutions \( y_1(t) = t \) and \( y_2(t) = \frac{1}{t^2} \).  \( \triangle \)
Reduction of the order method: Variable coefficients.

Proof of the Theorem: The choice of $y_2 = vy_1$ implies

$$y'_2 = v' y_1 + v y'_1, \quad y''_2 = v'' y_1 + 2v' y'_1 + v y''_1.$$
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This information introduced into the differential equation says that

$$(v'' y_1 + 2v' y'_1 + v y''_1) + p(v' y_1 + v y'_1) + qv y_1 = 0$$
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$$y_2' = v' y_1 + vy_1', \quad y_2'' = v'' y_1 + 2v' y_1' + vy_1''.$$

This information introduced into the differential equation says that

$$(v'' y_1 + 2v' y_1' + vy_1'') + p (v' y_1 + vy_1') + qv y_1 = 0$$

$$y_1 v'' + (2y_1' + p y_1) v' + (y_1'' + p y_1' + q y_1) v = 0.$$
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The function \( y_1 \) is solution of \( y''_1 + p y'_1 + q y_1 = 0 \).
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y_2' = v' y_1 + v y_1', \quad y_2'' = v'' y_1 + 2v' y_1' + v y_1''.
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This information introduced into the differential equation says that

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The function \( y_1 \) is solution of \( y_1'' + p y_1' + q y_1 = 0 \).

Then, the equation for \( v \) is given by Eq. (1), that is,

\[
y_1 v'' + (2y_1' + p y_1) v' = 0.
\]
Reduction of the order method: Variable coefficients.

Proof: Recall $y_1 v'' + (2y_1' + p y_1) v' = 0$. 

We now need to show that $y_1$ and $y_2 = v y_1$ are linearly independent.

We obtain $W y_1 y_2 = v' y_2$.

We need to find $v'$. Denote $w = v'$, so $y_1 w' + (2y_1' + p y_1) w = 0 \Rightarrow w w' = -2 y_1' y_1 - p$.

Let $P$ be a primitive of $p$, that is, $P'(t) = p(t)$, then

$$\ln(w) = -2 \ln(y_1) - P \Rightarrow w = e^{\ln(y_1^{-2} - P)} \Rightarrow w = y_1^{-2} e^{-P}.$$ 

We obtain $v' y_2 = e^{-P}$, hence $W y_1 y_2 = e^{-P}$, which is non-zero.

We conclude that $y_1$ and $y_2 = v y_1$ are linearly independent.
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Proof: Recall $y_1 v'' + (2y_1' + p y_1) v' = 0$. We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent.
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$W_{y_1y_2}$
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Proof: Recall $y_1 v'' + (2y'_1 + p y_1) v' = 0$. We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent.

$$W_{y_1y_2} = \begin{vmatrix} y_1 & vy_1 \\ y'_1 & (v'y_1 + vy'_1) \end{vmatrix}$$
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Proof: Recall $y_1 \ v'' + (2y_1' + p y_1) \ v' = 0$. We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent.

$$W_{y_1y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v'y_1 + vy_1') \end{vmatrix} = y_1(v'y_1 + vy_1') - vy_1y_1'.$$
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Proof: Recall \( y_1 \, v'' + (2y_1' + p \, y_1) \, v' = 0 \). We now need to show that \( y_1 \) and \( y_2 = v \, y_1 \) are linearly independent.

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W_{y_1y_2} = \begin{vmatrix}
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We obtain \( W_{y_1y_2} = v' y_1^2 \).
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We conclude that $y_1$ and $y_2 = vy_1$ are linearly independent. \qed
Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1 y' + a_0 y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.
Operator notation and preliminary results.

**Notation:** Given functions $p$, $q$, denote

$$L(y) = y'' + p(t)y' + q(t)y.$$
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The function $L$ acting on a function $y$ is called an **operator**.
Operator notation and preliminary results.

**Remark:** The operator $L$ is a linear function of $y$. 

**Theorem:** For every continuously differentiable functions $y_1$, $y_2$: $(t_1, t_2) \rightarrow \mathbb{R}$ and every $c_1, c_2 \in \mathbb{R}$ holds that

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

**Proof:**

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)' + p(t) (c_1 y_1 + c_2 y_2)' + q(t) (c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$
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$$= (c_1 y_1'' + p(t) c_1 y_1' + q(t) c_1 y_1)$$

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$$L(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)'+ q(t)(c_1y_1 + c_2y_2)$$

$$L(c_1y_1 + c_2y_2) = (c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1)$$
$$+ (c_2y_2'' + p(t)c_2y_2' + q(t)c_2y_2)$$

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$
Operator notation and preliminary results.

Theorem
Given functions $p$, $q$, $f$, let $L(y) = y'' + p(t)y' + q(t)y$.
If the functions $y_1$ and $y_2$ are fundamental solutions of the homogeneous equation

$$L(y) = 0,$$

and $y_p$ is any solution of the non-homogeneous equation

$$L(y_p) = f,$$  \hspace{1cm} (2)

then any other solution $y$ of the non-homogeneous equation above is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t),$$  \hspace{1cm} (3)

where $c_1, c_2 \in \mathbb{R}$. 

Operator notation and preliminary results.

Theorem

Given functions \( p, q, f \), let \( L(y) = y'' + p(t) y' + q(t) y \).

If the functions \( y_1 \) and \( y_2 \) are fundamental solutions of the homogeneous equation

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\]

where \( c_1, c_2 \in \mathbb{R} \).

Notation: The expression for \( y \) in Eq. (3) is called the general solution of the non-homogeneous Eq. (2).
Operator notation and preliminary results.

Theorem

Given functions $p$, $q$, let $L(y) = y'' + p(t)y' + q(t)y$.

If the function $f$ can be written as $f(t) = f_1(t) + \cdots + f_n(t)$, with $n \geq 1$, and if there exists functions $y_{p_1}, \ldots, y_{p_n}$ such that

$$L(y_{p_i}) = f_i, \quad i = 1, \ldots, n,$$

then the function $y_p = y_{p_1} + \cdots + y_{p_n}$ satisfies the non-homogeneous equation

$$L(y_p) = f.$$
Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1 y' + a_0 y = b(t)$.
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- **Summary of the undetermined coefficients method.**
- Using the method in few examples.
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Summary of the undetermined coefficients method.

**Problem:** Given a constant coefficients linear operator 
\[ L(y) = y'' + a_1 y' + a_0 y, \] with \( a_1, a_2 \in \mathbb{R} \), find every solution of the non-homogeneous differential equation

\[ L(y) = f. \]
Summary of the undetermined coefficients method.

**Problem:** Given a constant coefficients linear operator

\[ L(y) = y'' + a_1 y' + a_0 y, \]  

with \( a_1, a_2 \in \mathbb{R} \), find every solution of the non-homogeneous differential equation

\[ L(y) = f. \]

**Remarks:**

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
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Remarks:

▶ The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
▶ It consists in guessing the solution \( y_p \) of the non-homogeneous equation\[ L(y_p) = f, \] for particularly simple source functions \( f. \)
Summary of the undetermined coefficients method.

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Summary:

(1) Find the general solution of the homogeneous equation $L(y_h) = 0$. 

(2) If $f$ has the form $f = f_1 + \cdots + f_n$, with $n \geq 1$, then look for solutions $y_{p_i}$, with $i = 1, \ldots, n$, to the equations $L(y_{p_i}) = f_i$.

(3) Given the source functions $f_i$, guess the solutions functions $y_{p_i}$ following the Table below.
Summary of the undetermined coefficients method.

Summary:

(1) Find the general solution of the homogeneous equation

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\[ L(y_{p_i}) = f_i. \]

Once the functions \( y_{p_i} \) are found, then construct

\[ y_p = y_{p_1} + \cdots + y_{p_n}. \]
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Summary:

1. Find the general solution of the homogeneous equation $L(y_h) = 0$.

2. If $f$ has the form $f = f_1 + \cdots + f_n$, with $n \geq 1$, then look for solutions $y_{p_i}$, with $i = 1, \cdots, n$ to the equations

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3. Given the source functions $f_i$, guess the solutions functions $y_{p_i}$ following the Table below.
Summary of the undetermined coefficients method.

Summary (cont.):

<table>
<thead>
<tr>
<th>$f_i(t)$ (given)</th>
<th>$y_{pi}(t)$ (Guess)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ke^{at}$</td>
<td>$ke^{at}$</td>
</tr>
<tr>
<td>$Kt^m$</td>
<td>$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$</td>
</tr>
<tr>
<td>$K \cos(bt)$</td>
<td>$k_1 \cos(bt) + k_2 \sin(bt)$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$Kt^m e^{at}$</td>
<td>$e^{at} (k_m t^m + \cdots + k_0)$</td>
</tr>
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Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function $y_{p_i}$ satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function $t^s y_{p_i}$, with $s \geq 1$, and $s$ sufficiently large such that $L(t^s y_{p_i}) \neq 0$.
Summary of the undetermined coefficients method.

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(4) If any guessed function $y_{pi}$ satisfies the homogeneous equation $L(y_{pi}) = 0$, then change the guess to the function $t^s y_{pi}$, with $s \geq 1$, and $s$ sufficiently large such that $L(t^s y_{pi}) \neq 0$.

(5) Impose the equation $L(y_{pi}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate $m$, given in the table above.
Summary of the undetermined coefficients method.

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(6) The general solution to the original differential equation \( L(y) = f \) is then given by

\[
y(t) = y_h(t) + y_{p_1} + \cdots + y_{p_n}.
\]
Non-homogeneous equations (Sect. 3.5).

- We study: \( y'' + a_1 y' + a_0 y = b(t) \).
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.
Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$
Using the method in few examples.

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Find all solutions to the non-homogeneous equation
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Solution: Notice: \( L(y) = y'' - 3y' - 4y \) and \( f(t) = 3e^{2t} \).
Using the method in few examples.

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Using the method in few examples.

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\[ r^2 - 3r - 4 = 0 \quad \Rightarrow \quad \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases} \]
Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}. \tag{1}$$

Solution: Notice: $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

(1) Find all solutions $y_h$ to the homogeneous equation $L(y_h) = 0$. The characteristic equation is

$$r^2 - 3r - 4 = 0 \implies \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases}$$

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t}. \tag{2}$$

Trivial in our case. The source function $f(t) = 3e^{2t}$ cannot be simplified into a sum of simpler functions.

Table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = ke^{2t}$. 
Using the method in few examples.

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\[
(2^2 - 6 - 4)ke^{2t} = 3e^{2t}
\]

\( k = -\frac{1}{2} \).

We have obtained that \( y_p(t) = -\frac{1}{2} e^{2t} \).

(6) The general solution to the inhomogeneous equation is

\[
y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}.
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\[(2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3\]
Using the method in few examples.

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Using the method in few examples.

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Using the method in a few examples.

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Using the method in few examples.

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Using the method in few examples.

**Example**

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\[ y'' - 3y' - 4y = 3e^{4t}. \]

**Solution:** We know that the general solution to homogeneous equation is \( y(t) = c_1 e^{4t} + c_2 e^{-t} \).

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However, this guess satisfies \( L(y_p) = 0 \).

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Introduce the guess into \( L(y_p) = f \). We need to compute

\[ y'_p = k e^{4t} + 4kt e^{4t}, \quad y''_p = 8k e^{4t} + 16kt e^{4t}. \]
Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

\[ y'' - 3y' - 4y = 3e^{4t}. \]

Solution: Recall:

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[(8 + 16t) - 3(1 + 4t) - 4t] k &= 3
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We obtain that \( k = \frac{3}{5}. \)
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We obtain that \(k = \frac{3}{5}\). Therefore, \(y_p(t) = \frac{3}{5} t e^{4t}\), and
\[
y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5} t e^{4t}. \]
\[\triangleq\]
Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation

\[ y'' - 3y' - 4y = 2 \sin(t). \]
Using the method in few examples.

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Following the table: Since \( f = 2 \sin(t) \),

\[ \text{This guess satisfies } L(y_p) \neq 0. \]
\[ L(y_p) = \left[ -k_1 \sin(t) - k_2 \cos(t) \right] - 3 \left[ k_1 \cos(t) - k_2 \sin(t) \right] - 4 \left[ k_1 \sin(t) + k_2 \cos(t) \right] = 2 \sin(t), \]
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\[ y_p = k_1 \cos(t) + k_2 \sin(t). \]

This guess satisfies \( \mathcal{L}(y_p) \neq 0. \)

Compute: \( y'_p = k_1 \cos(t) - k_2 \sin(t), \) \( y''_p = -k_1 \sin(t) - k_2 \cos(t). \)
Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2 \sin(t). \]

Solution: We know that the general solution to homogeneous equation is \( y(t) = c_1 e^{4t} + c_2 e^{-t}. \)

Following the table: Since \( f = 2 \sin(t), \) then we guess
\[ y_p = k_1 \cos(t) + k_2 \sin(t). \]

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L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]
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\[ (-5k_1 + 3k_2) \sin(t) + (-3k_1 - 5k_2) \cos(t) = 2 \sin(t). \]
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Example
Find all the solutions to the inhomogeneous equation
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L(y_p) = \left[ -k_1 \sin(t) - k_2 \cos(t) \right] - 3 \left[ k_1 \cos(t) - k_2 \sin(t) \right] \\
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This equation holds for all \( t \in \mathbb{R} \). In particular, at \( t = \frac{\pi}{2}, t = 0 \).

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\begin{align*}
-5k_1 + 3k_2 &= 2, \\
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\[
\Rightarrow \begin{cases} \\
k_1 = -\frac{5}{17}, \\
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\end{cases}
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Using the method in few examples.

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So the particular solution to the inhomogeneous equation is

\[ y_p(t) = \frac{1}{17} \left[ -5 \sin(t) + 3 \cos(t) \right]. \]
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Find all the solutions to the inhomogeneous equation

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We have just found out that

\[ y_{p1}(t) = -\frac{1}{2} e^{2t}, \quad y_{p2}(t) = \frac{1}{17} \left[ -5\sin(t) + 3\cos(t) \right]. \]
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\[ y_{p1}(t) = -\frac{1}{2} e^{2t}, \quad y_{p2}(t) = \frac{1}{17} \left[ -5\sin(t) + 3\cos(t) \right]. \]

We conclude that

\[ y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2} e^{2t} + \frac{1}{17} \left[ -5\sin(t) + 3\cos(t) \right]. \]
Using the method in few examples.

Example

▶ For $y'' - 3y' - 4y = 3e^{2t} \sin(t)$,
Using the method in few examples.

Example

- For \( y'' - 3y' - 4y = 3e^{2t} \sin(t) \), guess

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y_p(t) = [k_1 \sin(t) + k_2 \cos(t)] e^{2t}.
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Example

- For $y'' - 3y' - 4y = 3e^{2t} \sin(t)$, guess

  $$y_p(t) = \left[ k_1 \sin(t) + k_2 \cos(t) \right] e^{2t}.$$  

- For $y'' - 3y' - 4y = 2t^2 e^{3t}$, guess

  $$y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{3t}.$$  

- For $y'' - 3y' - 4y = 3t \sin(t)$, guess

  $$y_p(t) = (1 + k_1 t) \left[ k_2 \sin(t) + k_3 \cos(t) \right].$$
Non-homogeneous equations (Sect. 3.6).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.
Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

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Method of variation of parameters.

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- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

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- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.

- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.
Method of variation of parameters.

Theorem (Variation of parameters)

Let $p, q, f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, let $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

and let $W_{y_1y_2}$ be the Wronskian of $y_1$ and $y_2$. If the functions $u_1$ and $u_2$ are defined by

$$u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} dt,$$

then the function $y_p = u_1y_1 + u_2y_2$ is a particular solution to the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = f(t).$$
Non-homogeneous equations (Sect. 3.6).

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Using the method in an example.

Example
Find the general solution of the inhomogeneous equation
\[ y'' - 5y' + 6y = 2e^t. \]
Using the method in an example.

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Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$ 

Solution:
First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is

$$r^2 - 5r + 6 = 0$$
Using the method in an example.

Example
Find the general solution of the inhomogeneous equation
\[ y'' - 5y' + 6y = 2e^t. \]

Solution:
First: Find fundamental solutions to the homogeneous equation. The characteristic equation is
\[ r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = \frac{1}{2} (5 \pm \sqrt{25 - 24}) \]
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Hence, \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{2t} \). Compute their Wronskian,

\[ W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \]
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Hence, \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{2t} \). Compute their Wronskian,
\[ W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1y_2}(t) = -e^{5t}. \]
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Second: We compute the functions \( u_1 \) and \( u_2. \)
Using the method in an example.

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Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Compute their Wronskian,

$$W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \Rightarrow W_{y_1y_2}(t) = -e^{5t}.$$  

Second: We compute the functions $u_1$ and $u_2$. By definition,

$$u'_1 = -\frac{y_2f}{W_{y_1y_2}}, \quad u'_2 = \frac{y_1f}{W_{y_1y_2}}.$$
Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

\[ y'' - 5y' + 6y = 2e^t. \]

Solution: Recall: \( y_1(t) = e^{3t}, \ y_2(t) = e^{2t}, \ W_{y_1y_2}(t) = -e^{5t} \), and

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    u'_1 = -e^{2t}(2e^t)(-e^{-5t})
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\[ y(t) = c_1e^{3t} + c_2e^{2t} + e^t, \quad c_1, c_2 \in \mathbb{R}. \]
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Find the general solution of the inhomogeneous equation

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Third: The particular solution is

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\[ y'' - 5y' + 6y = 2e^t. \]

**Solution:** Recall: \( y_1(t) = e^{3t}, \ y_2(t) = e^{2t}, \ W_{y_1y_2}(t) = -e^{5t}, \) and

\[ u'_1 = -\frac{y_2f}{W_{y_1y_2}}, \quad u'_2 = \frac{y_1f}{W_{y_1y_2}}. \]

\[ u'_1 = -e^{2t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u'_1 = 2e^{-2t} \quad \Rightarrow \quad u_1 = -e^{-2t}, \]

\[ u'_2 = e^{3t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u'_2 = -2e^{-t} \quad \Rightarrow \quad u_2 = 2e^{-t}. \]

**Third:** The particular solution is

\[ y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t. \]

The general solution is \( y(t) = c_1e^{3t} + c_2e^{2t} + e^t, \ c_1, c_2 \in \mathbb{R}. \) \( \triangleq \)
Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t) y' + q(t) y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.
The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)y' + q(t)y$. 
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Idea: The reduction of order method:
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Idea: The reduction of order method: Find \( y_2 \) proposing \( y_2 = uy_1 \).

First idea: Propose that \( y_p \) is given by \( y_p = u_1 y_1 + u_2 y_2 \).
The proof of the variation of parameter method.

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The proof of the variation of parameter method.

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\[
y'_p = u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2,
\]
The proof of the variation of parameter method.

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We need to find $y_p$ solution of $L(y_p) = f$.

We know $y_1$ and $y_2$ solutions of $L(y_1) = 0$ and $L(y_2) = 0$.

Idea: The reduction of order method: Find $y_2$ proposing $y_2 = uy_1$.

First idea: Propose that $y_p$ is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for $u_1$ and $u_2$ will be simpler than the original equation for $y_p$, since $y_1$ and $y_2$ are solutions to the homogeneous equation. Compute:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$
The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\left[ u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2 \right]$$

$$p(t)\left[ u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2 \right] + q(t)\left[ u_1 y_1 + u_2 y_2 \right] = f(t).$$
The proof of the variation of parameter method.

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$$
\left[u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2\right]
$$

$$
p(t)\left[u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2\right] + q(t)\left[u_1 y_1 + u_2 y_2\right] = f(t).
$$

$$
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p\left(u'_1 y_1 + u'_2 y_2\right)
$$

$$
+ u_1 (y''_1 + p y'_1 + q y_1) + u_2 (y''_2 + p y'_2 + q y_2) = f
$$
The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\left[ u''_1 y_1 + 2u'_1 y'_1 + u_1 y''' + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''' \right]$$

$$p(t)\left[ u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2 \right] + q(t)\left[ u_1 y_1 + u_2 y_2 \right] = f(t).$$

$$u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p \left( u_1 y_1 + u_2 y_2 \right)$$

$$+ u_1 \left( y'''_1 + p y'_1 + q y_1 \right) + u_2 \left( y'''_2 + p y'_2 + q y_2 \right) = f$$

Recall: $y'''_1 + p y'_1 + q y_1 = 0$ and $y'''_2 + p y'_2 + q y_2 = 0$. 
The proof of the variation of parameter method.

Proof: Then \( L(y_p) = f \) is given by

\[
[u'' y_1 + 2u'_1 y'_1 + u_1 y''' + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''']
\]

\[
p(t)[u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2] + q(t)[u_1 y_1 + u_2 y_2] = f(t).
\]

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2)
\]

\[
+ u_1 (y''_1 + p y'_1 + q y_1) + u_2 (y''_2 + p y'_2 + q y_2) = f
\]

Recall: \( y''_1 + p y'_1 + q y_1 = 0 \) and \( y''_2 + p y'_2 + q y_2 = 0 \). Hence,

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) = f
\]
The proof of the variation of parameter method.

Proof: Then \( L(y_p) = f \) is given by

\[
\begin{align*}
\left[ u''_1 y_1 + 2u'_1 y'_1 + u_1 y'' + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2 \right] \\
p(t)\left[ u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2 \right] + q(t)\left[ u_1 y_1 + u_2 y_2 \right] &= f(t).
\end{align*}
\]

Recall: \( y''_1 + p y'_1 + q y_1 = 0 \) and \( y''_2 + p y'_2 + q y_2 = 0 \). Hence,

\[
\begin{align*}
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) \\
+ u_1 (y''_1 + p y'_1 + q y_1) + u_2 (y''_2 + p y'_2 + q y_2) &= f
\end{align*}
\]

Second idea: Look for \( u_1 \) and \( u_2 \) that satisfy the extra equation

\[
u'_1 y_1 + u'_2 y_2 = 0.
\]
The proof of the variation of parameter method.

Proof: Recall: \( u_1'y_1 + u_2'y_2 = 0 \) and

\[
u_1''y_1 + u_2''y_2 + 2(u_1'y_1' + u_2'y_2') + p(u_1'y_1 + u_2'y_2) = f.
\]
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Proof: Recall: \( u'_1 y_1 + u'_2 y_2 = 0 \) and

\[
    u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) = f. 
\]

These two equations imply that \( L(y_p) = f \) is

\[
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From \( u'_1 y_1 + u'_2 y_2 = 0 \) we get \( [u'_1 y_1 + u'_2 y_2]' = 0, \)
The proof of the variation of parameter method.

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\[
u''_1 y_1 + u''_2 y_2 + (u'_1 y'_1 + u'_2 y'_2) = 0.
\]
The proof of the variation of parameter method.

Proof: Recall: \(u_1'y_1 + u_2'y_2 = 0\) and

\[u_1''y_1 + u_2''y_2 + 2(u_1'y_1 + u_2'y_2) + p(u_1'y_1 + u_2'y_2) = f.\]

These two equations imply that \(L(y_p) = f\) is

\[u_1''y_1 + u_2''y_2 + 2(u_1'y_1 + u_2'y_2) = f.\]

From \(u_1'y_1 + u_2'y_2 = 0\) we get \([u_1'y_1 + u_2'y_2]' = 0\), that is

\[u_1''y_1 + u_2''y_2 + (u_1'y_1 + u_2'y_2) = 0.\]

This information in \(L(y_p) = f\) implies

\[u_1'y_1 + u_2'y_2 = f.\]
The proof of the variation of parameter method.

Proof: Recall: \( u'_1 y_1 + u'_2 y_2 = 0 \) and

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) = f.
\]

These two equations imply that \( L(y_p) = f \) is

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) = f.
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From \( u'_1 y_1 + u'_2 y_2 = 0 \) we get \( [u'_1 y_1 + u'_2 y_2]' = 0 \), that is

\[
u''_1 y_1 + u''_2 y_2 + (u'_1 y'_1 + u'_2 y'_2) = 0.
\]

This information in \( L(y_p) = f \) implies

\[
u'_1 y'_1 + u'_2 y'_2 = f.
\]

Summary: If \( u_1 \) and \( u_2 \) satisfy \( u'_1 y_1 + u'_2 y_2 = 0 \) and \( u'_1 y'_1 + u'_2 y'_2 = f \), then \( y_p = u_1 y_1 + u_2 y_2 \) satisfies \( L(y_p) = f \).
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Proof: Summary: If \( u_1 \) and \( u_2 \) satisfy \( u'_1 y_1 + u'_2 y_2 = 0 \) and \( u'_1 y'_1 + u'_2 y'_2 = f \), then \( y_p = u_1 y_1 + u_2 y_2 \) satisfies \( L(y_p) = f \).
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Proof: Summary: If \( u_1 \) and \( u_2 \) satisfy \( u_1'y_1 + u_2'y_2 = 0 \) and \( u_1'y_1' + u_2'y_2' = f \), then \( y_p = u_1y_1 + u_2y_2 \) satisfies \( L(y_p) = f \).

The equations above are simple to solve for \( u_1 \) and \( u_2 \),
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**Proof:** Summary: If $u_1$ and $u_2$ satisfy $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u_2' = -\frac{y_1}{y_2} u_1'$$
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**Proof:** Summary: If \( u_1 \) and \( u_2 \) satisfy \( u'_1 y_1 + u'_2 y_2 = 0 \) and \( u'_1 y'_1 + u'_2 y'_2 = f \), then \( y_p = u_1 y_1 + u_2 y_2 \) satisfies \( L(y_p) = f \).

The equations above are simple to solve for \( u_1 \) and \( u_2 \),

\[
u'_2 = -\frac{y_1}{y_2} u'_1 \quad \Rightarrow \quad u'_1 y'_1 - \frac{y_1 y'_2}{y_2} u'_1 = f
\]
The proof of the variation of parameter method.

Proof: Summary: If \( u_1 \) and \( u_2 \) satisfy \( u_1'y_1 + u_2'y_2 = 0 \) and \( u_1'y_1' + u_2'y_2' = f \), then \( y_p = u_1y_1 + u_2y_2 \) satisfies \( L(y_p) = f \).

The equations above are simple to solve for \( u_1 \) and \( u_2 \),

\[
    u_2' = -\frac{y_1}{y_2} \quad u_1' \quad \Rightarrow \quad u_1'y_1' - \frac{y_1y_2'}{y_2} \quad u_1' = f \quad \Rightarrow \quad u_1' \left( \frac{y_1'y_2 - y_1y_2'}{y_2} \right) = f.
\]
The proof of the variation of parameter method.

**Proof:** Summary: If $u_1$ and $u_2$ satisfy $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1 + u_2'y_2 = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u_2' = -\frac{y_1}{y_2} u_1' \quad \Rightarrow \quad u_1'y_1 - \frac{y_1y_2'}{y_2} u_1' = f \quad \Rightarrow \quad u_1'\left(\frac{y_1'y_2 - y_1y_2'}{y_2}\right) = f.$$

Since $W_{y_1y_2} = y_1y_2' - y_1'y_2$, ...
The proof of the variation of parameter method.

Proof: Summary: If $u_1$ and $u_2$ satisfy $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1 + u_2'y_2' = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u'_2 = -\frac{y_1}{y_2}u'_1 \implies u'_1y'_1 - \frac{y_1y'_2}{y_2}u'_1 = f \implies u'_1\left(\frac{y'_1y_2 - y_1y'_2}{y_2}\right) = f.$$

Since $W_{y_1y_2} = y_1y'_2 - y'_1y_2$,

$$u'_1 = -\frac{y_2f}{W_{y_1y_2}}$$
The proof of the variation of parameter method.

Proof: Summary: If $u_1$ and $u_2$ satisfy $u'_1y_1 + u'_2y_2 = 0$ and $u'_1y'_1 + u'_2y'_2 = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u'_2 = -\frac{y_1}{y_2} u'_1 \Rightarrow u'_1 y'_1 - \frac{y_1 y'_2}{y_2} u'_1 = f \Rightarrow u'_1 \left( \frac{y'_1 y_2 - y_1 y'_2}{y_2} \right) = f.$$ 

Since $W_{y_1y_2} = y_1 y'_2 - y'_1 y_2$,

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The proof of the variation of parameter method.

Proof: Summary: If $u_1$ and $u_2$ satisfy $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f$, then $y_p = u_1y_1 + u_2y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for $u_1$ and $u_2$,

$$u_2' = -\frac{y_1}{y_2} u_1' \quad \Rightarrow \quad u_1'y_1 - \frac{y_1y_2'}{y_2} u_1' = f \quad \Rightarrow \quad u_1'\left(\frac{y_1y_2 - y_1y_2'}{y_2}\right) = f.$$ 

Since $W_{y_1y_2} = y_1y_2' - y_1'y_2$,

$$u_1' = -\frac{y_2f}{W_{y_1y_2}} \quad \Rightarrow \quad u_2' = \frac{y_1f}{W_{y_1y_2}}.$$ 

Integrating in the variable $t$ we obtain

$$u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt,$$

This establishes the Theorem. □
Non-homogeneous equations (Sect. 3.6).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
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Using the method in another example.

Example
Find a particular solution to the differential equation

\[ t^2y'' - 2y = 3t^2 - 1, \]

knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2y'' - 2y = 0 \).
Using the method in another example.

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$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$  

We know that $y_1 = t^2$ and $y_2 = 1/t$. 

Using the method in another example.

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Find a particular solution to the differential equation

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Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by \( t^2 \),

\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \Rightarrow f(t) = 3 - \frac{1}{t^2}. \]

We know that \( y_1 = t^2 \) and \( y_2 = 1/t \). Their Wronskian is

\[ W_{y_1y_2}(t) = (t^2)\left(\frac{-1}{t^2}\right) - (2t)\left(\frac{1}{t}\right) \]
Using the method in another example.

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knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by \( t^2 \),

\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}. \]

We know that \( y_1 = t^2 \) and \( y_2 = 1/t \). Their Wronskian is

\[ W_{y_1y_2}(t) = (t^2)\left(\frac{-1}{t^2}\right) - (2t)\left(\frac{1}{t}\right) \quad \Rightarrow \quad W_{y_1y_2}(t) = -3. \]
Using the method in another example.

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Find a particular solution to the differential equation
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We now compute \( y_1 \) and \( u_2 \),

\[ u_1' = -\frac{1}{t} \left( 3 - \frac{1}{t^2} \right) \frac{1}{-3} \]
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\[ \tilde{y}_p = \left[ \ln(t) + \frac{1}{6} t^{-2} \right] (t^2) + \frac{1}{3} (-t^3 + t)(t^{-1}) \]
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A simpler expression is \( y_p = t^2 \ln(t) + \frac{1}{2} \). \( \triangle \)