Review 2 for Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks, webwork.
- Exam covers:
  - Linear equations (2.1).
  - Separable equations (2.2).
  - Homogeneous equations (2.2).
  - Modeling (2.3).
  - Non-linear equations (2.4).
  - Bernoulli equation (2.4).
  - Exact equations (2.6).
  - Exact equations with integrating factors (2.6).

Review 2 Exam 1.

Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

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Solution: We first verify if the equation is not exact.

$$N = \left(x^3 e^y + \frac{x}{y}\right)$$

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$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)}$$

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Review 2 Exam 1.

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Find every solution y of the equation

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Review 2 Exam 1.

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Review 2 Exam 1.

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Review 2 Exam 1.

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Review 2 Exam 1.

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Review 2 Exam 1.

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The solution y satisfies $x^2 e^{y(x)} + \ln(y(x)) + \ln(x) = c$. $\triangleleft$

Review 2 Exam 1.

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Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.

$$2xe^y + x^2 e^y y' + \frac{1}{y} y' + \frac{1}{x} = 0.$$

Review 2 for Exam 1.

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Find every solution of the initial value problem

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Review 2 for Exam 1.

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Review 2 for Exam 1.

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It is separable: $\dfrac{y'}{y + \sqrt{y}} = 4x.$

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Review 2 for Exam 1.

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Review 2 for Exam 1.

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$$\int \frac{y'}{y + \sqrt{y}}\,dt = \int 4x\,dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c.$$

The integral on the left-hand side requires an integration table.

Review 2 for Exam 1.

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Review 2 for Exam 1.

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The coefficient function is $a(x) = -2x$, so $A(x) = -x^2$, and the integrating factor is $\mu(x) = e^{-x^2}$.

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We conclude that $v = c\, e^{x^2} - 1$.

Review 2 for Exam 1.

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Review 2 for Exam 1.

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The points $\pm\sqrt{3}$ do not belong to the domain of y, since y' and the differential equation are not defined there. $\lhd$

Review 2 for Exam 1.

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Find the domain of the function y solution of the IVP

$$y' = -\frac{2t}{y}, \qquad y(t_0) = y_0.$$

Review 2 for Exam 1.

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The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2}$$

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Review 2 for Exam 1.

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The domain of the solution depends on the initial condition t_0, y_0:

$$D = \left(-\sqrt{t_0^2 + \frac{y_0^2}{2}}, +\sqrt{t_0^2 + \frac{y_0^2}{2}}\right). \qquad \triangleleft$$

Review 2 for Exam 1.

Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Review 2 for Exam 1.

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Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear,

Review 2 for Exam 1.

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Solution: The equation is not linear, not Bernoulli,

Review 2 for Exam 1.

Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable.

Review 2 for Exam 1.

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We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$,

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We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$, and $\psi(x, y(x)) = c$. $\triangleleft$

Review 2 for Exam 1.

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Review 2 for Exam 1.

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Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x+3y)}{(3x+4y)} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

Review 2 for Exam 1.

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$$y' = -\frac{(2x+3y)}{(3x+4y)}\,\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = -\frac{2+3\left(\frac{y}{x}\right)}{3+4\left(\frac{y}{x}\right)}.$$

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We conclude that v satisfies $\dfrac{3+4v}{2-4v^2}\, v' = \dfrac{1}{x}$.

Review 2 for Exam 1.

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This equation is complicated to integrate.

$$\int \frac{3\, v'}{2-4v^2}\, dx + \int \frac{4v\, v'}{2-4v^2}\, dx = \int \frac{1}{x}\, dx + c$$

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This is why the exact method is simpler to use in this case. $\triangleleft$

Second order linear homogeneous ODE (Sect. 3.3).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Review: On solutions of $y'' + a_1 \, y' + a_0 \, y = 0$.

Definition

Any two solutions y_1, y_2 of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called *fundamental solutions* iff the functions y_1, y_2 are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

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Remark: Fundamental solutions are not unique.

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Definition

Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the function

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

is called the *general solution* of the differential equation above.

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1\, y' + a_0\, y = 0. \tag{1}$$

Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, any solution of Eq. (1) belongs to only one of the following cases:

(a) *If $r_+ \neq r_-$, the general solution is $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$.*

(b) *If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_1 + c_2 t) e^{r_+ t}$.*

Furthermore, given real constants t_0, y_1 and y_2, there is a unique solution to the initial value problem given by Eq. (1) and the initial conditions

$$y(t_0) = y_1, \qquad y'(t_0) = y_2.$$

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example

Find the general solution of the equation $y'' - y' - 6y = 0$.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example

Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution: Since solutions have the form e^{rt}, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$,

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

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Find the general solution of the equation $y'' - y' - 6y = 0$.

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$$r_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 + 24}\right)$$

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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So, $r_\pm$ are real-valued. A fundamental solution set is formed by

$$y_1(t) = e^{3t}, \qquad y_2(t) = e^{-2t}.$$

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The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions,

Review: On solutions of $y'' + a_1 \, y' + a_0 \, y = 0$.

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The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is,

$$y(t) = c_1 \, e^{3t} + c_2 \, e^{-2t}, \qquad c_1, \, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is,

$$y(t) = c_1\, e^{3t} + c_2\, e^{-2t}, \qquad c_1,\, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Remark: Since c_1, $c_2 \in \mathbb{R}$, then y is real-valued.

Second order linear homogeneous ODE.

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0.$
- Characteristic polynomial with complex roots.
  - **Two main sets of fundamental solutions.**
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants $a_1, a_0 \in \mathbb{R}$ *satisfy that* $a_1^2 - 4a_0 < 0$*, then the characteristic polynomial* $p(r) = r^2 + a_1 r + a_0$ *of the equation*

$$y'' + a_1\, y' + a_0\, y = 0 \tag{2}$$

has complex roots $r_+ = \alpha + i\beta$ *and* $r_- = \alpha - i\beta$*, where*

$$\alpha = -\frac{a_1}{2}, \qquad \beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}.$$

Furthermore, a fundamental set of solutions to Eq. (2) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

while another fundamental set of solutions to Eq. (2) is

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

Two main sets of fundamental solutions.

Example

Find the general solution of the equation $y'' - 2y' + 6y = 0$.

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Find the general solution of the equation $y'' - 2y' + 6y = 0$.

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Two main sets of fundamental solutions.

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Find the general solution of the equation $y'' - 2y' + 6y = 0$.

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- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.

Second order linear homogeneous ODE.

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - **A real-valued fundamental and general solutions.**
- Application: The RLC circuit.

A real-valued fundamental and general solutions.

Example

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The Theorem above says that a real-valued fundamental set is

$$y_1(t) = e^t \cos(\sqrt{5}\,t), \qquad y_2(t) = e^t \sin(\sqrt{5}\,t).$$

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We just restricted the coefficients c_1, c_2 to be real-valued. $\triangleleft$

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$$e^{i\sqrt{5}t} = \big[\cos(\sqrt{5}\,t) + i\sin(\sqrt{5}\,t)\big],$$

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Remark:

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$$1 + i\sqrt{5} \quad \rightarrow \quad \alpha + i\beta, \qquad 1 - i\sqrt{5} \quad \rightarrow \quad \alpha - i\beta.$$

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Solution:

The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

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These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

A real-valued fundamental and general solutions.

Example

Find real-valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution:

The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

$$r_{\pm} = \frac{1}{2}\big[-2 \pm \sqrt{4 - 24}\big] = \frac{1}{2}\big[-2 \pm \sqrt{-20}\big] \;\Rightarrow\; r_{\pm} = -1 \pm i\sqrt{5}.$$

These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

Real-valued fundamental solutions are

$$y_1(t) = e^{-t}\,\cos(\sqrt{5}\,t), \qquad y_2(t) = e^{-t}\,\sin(\sqrt{5}\,t). \qquad \lhd$$

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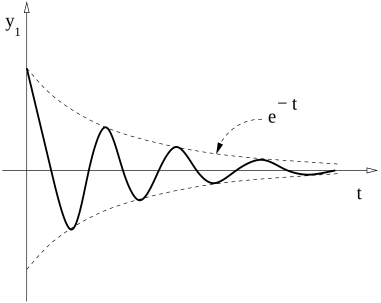

Differential equations like the one in this example describe physical processes related to damped oscillations. For example pendulums with friction.

A real-valued fundamental and general solutions.

Example

Find the real-valued general solution of $y'' + 5\,y = 0$.

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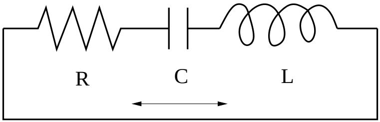
Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.

Second order linear homogeneous ODE.

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- **Application: The RLC circuit.**

Application: The RLC circuit.

Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



I (t) : electric current.

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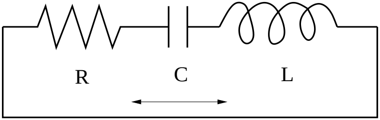

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The electric current flowing in such circuit satisfies:

$$L\, I'(t) + R\, I(t) + \frac{1}{C} \int_{t_0}^{t} I(s)\, ds = 0.$$

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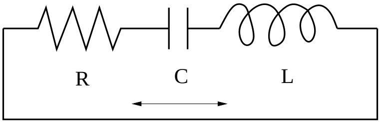

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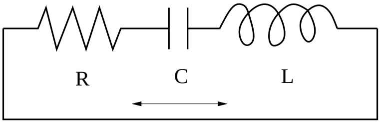

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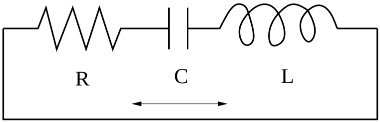

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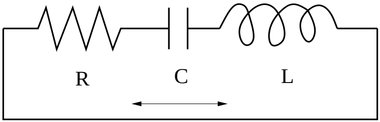

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Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Remark: When the circuit has no resistance, the current oscillates without dissipation.

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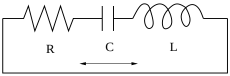

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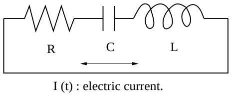

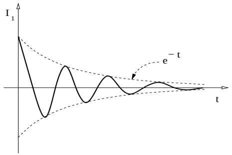

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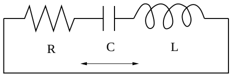

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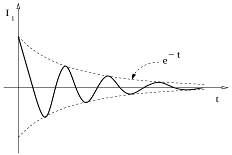

The resistance R damps the current oscillations.