Review 2 for Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks, webwork.
- Exam covers:
  - Linear equations (2.1).
  - Separable equations (2.2).
  - Homogeneous equations (2.2).
  - Modeling (2.3).
  - Non-linear equations (2.4).
  - Bernoulli equation (2.4).
  - Exact equations (2.6).
  - Exact equations with integrating factors (2.6).
Example

Find the integrating factor that converts the equation below into an exact equation, where

\[
\left( x^3 e^y + \frac{x}{y} \right)y' + (2x^2 e^y + 1) = 0.
\]
Example
Find the integrating factor that converts the equation below into an exact equation, where
\[(x^3e^y + \frac{x}{y}) y' + (2x^2e^y + 1) = 0.\]

Solution: We first verify if the equation is not exact.
\[N = \left(x^3e^y + \frac{x}{y}\right)\]
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\[
N = \left(x^3 e^y + \frac{x}{y}\right) \Rightarrow \partial_x N = 3x^2 e^y + \frac{1}{y}.
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So the equation is not exact. We now compute

\[
\frac{\partial_y M - \partial_x N}{N}
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So the equation is not exact. We now compute
\[
\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)}
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\[
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**Solution:** We first verify if the equation is not exact.

\[
N = \left( x^3 e^y + \frac{x}{y} \right) \implies \partial_x N = 3x^2 e^y + \frac{1}{y}.
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\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left( 3x^2 e^y + \frac{1}{y} \right)}{x^3 e^y + \frac{x}{y}} = \frac{-x^2 e^y - \frac{1}{y}}{x \left( x^2 e^y + \frac{1}{y} \right)}.
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Solution: Recall: \( \frac{\partial_y M - \partial_x N}{N} = -\frac{1}{x} \). Therefore,

\[
\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x}
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\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \implies \ln(\mu) = -\ln(x) = \ln\left(\frac{1}{x}\right) \implies \mu(x) = \frac{1}{x}.
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So the equation \( \left( x^2e^y + \frac{1}{y} \right) y' + \left( 2xe^y + \frac{1}{x} \right) = 0 \) is exact.
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\tilde{N} = \left(x^2 e^y + \frac{1}{y}\right) \quad \Rightarrow \quad \partial_x \tilde{N} = 2xe^y,
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Example

Find every solution $y$ of the equation

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Example
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\left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2x e^y + \frac{1}{x} \right) = 0.
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Solution: The equation is exact.
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Find every solution $y$ of the equation
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Solution: The equation is exact. We need to find the potential function $\psi$. 
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$$\partial_y \psi = N, \quad \partial_x \psi = M.$$
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From the first equation we get:

$$\partial_y \psi = x^2 e^y + \frac{1}{y}.$$
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$$2xe^y + g'(x) = \partial_x \psi$$
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\left( x^2e^y + \frac{1}{y} \right) y' + \left( 2x e^y + \frac{1}{x} \right) = 0.
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Solution: Recall: \( g'(x) = \frac{1}{x} \).
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Find every solution $y$ of the equation

$$\left(x^2 e^y + \frac{1}{y}\right)y' + \left(2x e^y + \frac{1}{x}\right) = 0.$$

Solution: Recall: $g'(x) = \frac{1}{x}$. Therefore $g(x) = \ln(x)$. 
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Find every solution $y$ of the equation

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\left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2xe^y + \frac{1}{x} \right) = 0.
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Solution: Recall: $g'(x) = \frac{1}{x}$. Therefore $g(x) = \ln(x)$.

The potential function is $\psi = x^2 e^y + \ln(y) + \ln(x)$. 

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.
Example

Find every solution $y$ of the equation

$\left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2x e^y + \frac{1}{x} \right) = 0$.

Solution: Recall: $g'(x) = \frac{1}{x}$. Therefore $g(x) = \ln(x)$.

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The solution $y$ satisfies $x^2 e^{y(x)} + \ln(y(x)) + \ln(x) = c$. $\triangle$
Example

Find every solution $y$ of the equation

$$(x^2e^y + \frac{1}{y})y' + (2xe^y + \frac{1}{x}) = 0.$$ 

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The solution $y$ satisfies $x^2e^{y(x)} + \ln(y(x)) + \ln(x) = c$. 

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.

$$2xe^y + x^2e^y y' + \frac{1}{y} y' + \frac{1}{x} = 0.$$
Example
Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]
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Solution: The equation is: Not linear.
It is a Bernoulli equation: \( y' - 4xy = 4xy^n \), with \( n = 1/2 \).
Example
Find every solution of the initial value problem

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Solution: The equation is: Not linear.
It is a Bernoulli equation: \( y' - 4xy = 4xy^n \), with \( n = 1/2 \).
It is separable: \( \frac{y'}{y + \sqrt{y}} = 4x \).
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It is a Bernoulli equation: \( y' - 4x y = 4x y^n \), with \( n = 1/2 \).
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The equation is not homogeneous.
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The equation is not homogeneous. It is not exact.
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The equation is not homogeneous. It is not exact.

Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method.
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$$y' = 4x(y + \sqrt{y}), \quad y(0) = 4.$$ 

Solution: The equation is: Not linear.
It is a Bernoulli equation: $$y' - 4x y = 4x y^n,$$ with $$n = 1/2.$$ It is separable: $$\frac{y'}{y + \sqrt{y}} = 4x.$$ The equation is not homogeneous. It is not exact.

Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method. Indeed

$$\int \frac{y'}{y + \sqrt{y}} \, dt = \int 4x \, dx + c$$
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\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

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It is a Bernoulli equation: \( y' - 4x\,y = 4x\,y^n \), with \( n = 1/2 \).

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\int \frac{y'}{y + \sqrt{y}} \, dt = \int 4x \, dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c.
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The equation is not homogeneous. It is not exact.

Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method. Indeed

\[ \int \frac{y'}{y + \sqrt{y}} \, dt = \int 4x \, dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c. \]

The integral on the left-hand side requires an integration table.
Example

Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: We find solutions using the Bernoulli method.
Review 2 for Exam 1.

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Change the unknowns: \( v = 1/y^{n-1} \), with \( n = 1/2 \).
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Review 2 for Exam 1.

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\[ v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2} \frac{y'}{y^{1/2}}. \]
Example

Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: We find solutions using the Bernoulli method.

\[ y' - 4x \cdot y = 4x \cdot y^{1/2} \quad \Rightarrow \quad \frac{y'}{y^{1/2}} - 4x \cdot y^{1/2} = 4x. \]

Change the unknowns: \( v = 1/y^{n-1} \), with \( n = 1/2 \). That is,

\[ v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2} \frac{y'}{y^{1/2}}. \]

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The coefficient function is \( a(x) = -2x \),
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Find every solution of the initial value problem

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The coefficient function is \( a(x) = -2x \), so \( A(x) = -x^2 \),
Review 2 for Exam 1.

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Find every solution of the initial value problem

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Change the unknowns: \(v = 1/y^{n-1}\), with \(n = 1/2\). That is,

$$v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2} \frac{y'}{y^{1/2}}.$$

$$2v' - 4xv = 4x \quad \Rightarrow \quad v' - 2xv = 2x.$$  

The coefficient function is \(a(x) = -2x\), so \(A(x) = -x^2\), and the integrating factor is \(\mu(x) = e^{-x^2}\).
Example
Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: Recall: \( v' - 2xv = 2x \) and \( \mu(x) = e^{-x^2} \).
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\[
e^{-x^2}v = \int 2xe^{-x^2} \, dx + c
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We conclude that \( v = c e^{x^2} - 1. \)
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We conclude that \( v = c e^{x^2} - 1 \). The initial condition for \( y \) implies the initial condition for \( v \),

\[ v(0) = 2 = c - 1 \quad \Rightarrow \quad c = 3. \]
Example

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\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

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\[ 2 = v(0) = c - 1 \quad \Rightarrow \quad c = 3 \]
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We conclude that \( v = c e^{x^2} - 1 \). The initial condition for \( y \) implies the initial condition for \( v \), that is, \( v(x) = \sqrt{y(x)} \) implies \( v(0) = 2 \).

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2 = v(0) = c - 1 \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad v(x) = 3e^{x^2} - 1.
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We finally find \( y = v^2 \),
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We finally find \( y = v^2 \), that is, \( y(x) = (3e^{x^2} - 1)^2 \). \( \triangle \)
Example
Find the domain of the function $y$ solution of the IVP

\[ \frac{dy}{dt} = -\frac{2t}{y}, \quad y(1) = 2. \]
Example
Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$

Solution: We first need to find the solution $y$. 

\[\int y y' dt = \int -\frac{2t}{y} dt + c \]

\[y^2 = -t^2 + c \]

\[y(1) = 2 \]

\[c = 3 \]

\[y(t) = \sqrt{2(3 - t^2)} \]

The domain of the solution $y$ is $D = (-\sqrt{3}, \sqrt{3})$. The points $\pm \sqrt{3}$ do not belong to the domain of $y$, since $y'$ and the differential equation are not defined there.
Example
Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$ 

Solution: We first need to find the solution $y$.
The equation is separable.
Review 2 for Exam 1.

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$$y \, y' = -2t$$
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y y' = -2t \quad \Rightarrow \quad \int y y' \, dt = \int -2t \, dt + c
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$$y \, y' = -2t \quad \Rightarrow \quad \int y \, y' \, dt = \int -2t \, dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c$$
Example

Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$ 

Solution: We first need to find the solution $y$. The equation is separable.

$$y \ y' = -2t \quad \Rightarrow \quad \int y \ y' \ dt = \int -2t \ dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c$$

$$\frac{4}{2} = \frac{y^2(1)}{2} = -1 + c$$
Review 2 for Exam 1.

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Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(1) = 2.$$ 

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The equation is **separable**.

$$yy' = -2t \quad \Rightarrow \quad \int yy' \, dt = \int -2t \, dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c$$

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Find the domain of the function \( y \) solution of the IVP
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y' = \frac{-2t}{y}, \quad y(1) = 2.
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y y' = -2t \quad \Rightarrow \quad \int y y' \, dt = \int -2t \, dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c
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The domain of the solution \( y \) is \( D = (-\sqrt{3}, \sqrt{3}) \).
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Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(t_0) = y_0.$$
Example
Find the domain of the function \( y \) solution of the IVP

\[
y' = -\frac{2t}{y}, \quad y(t_0) = y_0.
\]

Solution: The solution \( y \) is given as above, \( \frac{y^2}{2} = -t^2 + c \).
Example

Find the domain of the function $y$ solution of the IVP

\[ y' = -\frac{2t}{y}, \quad y(t_0) = y_0. \]

Solution: The solution $y$ is given as above, $\frac{y^2}{2} = -t^2 + c$. The initial condition implies

\[ \frac{y_0^2}{2} = \frac{y^2(t_0)}{2} \]
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\[ \frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \quad \Rightarrow \quad c = \frac{y_0^2}{2} + t_0^2 \]
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$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \quad \Rightarrow \quad c = \frac{y_0^2}{2} + t_0^2 \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$
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Find the domain of the function $y$ solution of the IVP

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$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \Rightarrow c = \frac{y_0^2}{2} + t_0^2 \Rightarrow \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$ 

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2)} + y_0^2.$
Example
Find the domain of the function \( y \) solution of the IVP

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\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \quad \Rightarrow \quad c = \frac{y_0^2}{2} + t_0^2 \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.
\]

The solution to the IVP is \( y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2} \).

The domain of the solution depends on the initial condition \( t_0, y_0 \):

\[
D = \left(-\sqrt{t_0^2 + y_0^2}, +\sqrt{t_0^2 + y_0^2}\right).
\]
Example
Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(t_0) = y_0.$$  

Solution: The solution $y$ is given as above, $\frac{y^2}{2} = -t^2 + c$.  
The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \quad \Rightarrow \quad c = \frac{y_0^2}{2} + t_0^2 \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}. $$

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}$.  

The domain of the solution depends on the initial condition $t_0, y_0$:

$$D = \left( -\sqrt{t_0^2 + \frac{y_0^2}{2}}, +\sqrt{t_0^2 + \frac{y_0^2}{2}} \right).$$
Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$. 
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Solution: The equation is not linear,
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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable.
Review 2 for Exam 1.

Example
Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by $(1/x)$.)

\[
\frac{2x + 3y}{3x + 4y} = \frac{\frac{2}{x} + \frac{3y}{x}}{\frac{3x}{x} + \frac{4y}{x}}.
\]

Is it exact? 

\[
(3x + 4y) y' + (2x + 3y) = 0 \implies \frac{\partial}{\partial x} N = \frac{\partial}{\partial y} M.
\]

So the equation is exact.

We choose here the exact equation method. 

(Finding the potential function is sometimes simpler that solving homogeneous Eqs.)

We need to find the potential function $\psi$:

\[
\frac{\partial}{\partial y} \psi = N \implies \psi = 3xy + 2y^2 + g(x).
\]

\[
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\]

We conclude:

\[
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\]

\[
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\]
Example

Find every solution $y$ to the equation $y' = \frac{-2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by $(1/x)$.)

Is it exact? $(3x + 4y) y' + (2x + 3y) = 0$ implies $\partial_x N = 3 = \partial_y M$.

So the equation is exact.
Review 2 for Exam 1.

Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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We conclude: \( \psi(x, y) = 3xy + 2y^2 + x^2 \).
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We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$, and $\psi(x, y(x)) = c$. \(\triangleleft\)
Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$. 
Example
Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation.
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\[
y' = -\left(\frac{2x + 3y}{3x + 4y}\right) \left(\frac{1}{x}\right) \left(\frac{1}{x}\right)
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Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x + 3y)}{(3x + 4y)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = -\frac{2 + 3\left(\frac{y}{x}\right)}{3 + 4\left(\frac{y}{x}\right)}.$$
Review 2 for Exam 1.

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The change \( v = y/x \) implies \( y = xv \) and \( y' = v + xv' \).
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The change $v = y/x$ implies $y = xv$ and $y' = v + xv'$. Hence

$$v + xv' = \frac{2 + 3v}{3 + 4v}.$$
Review 2 for Exam 1.

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The change \( v = y/x \) implies \( y = xv \) and \( y' = v + x \cdot v' \). Hence

\[
v + x \cdot v' = \frac{2 + 3v}{3 + 4v} \quad \Rightarrow \quad x \cdot v' = \frac{2 + 3v}{3 + 4v} - v
\]
Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

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We conclude that $v$ satisfies $\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}$. 

Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: Recall: \[
\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}.
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Review 2 for Exam 1.

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Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: Recall: $\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}$.

This equation is complicated to integrate.

$$\int \frac{3 v'}{2 - 4v^2} \, dx + \int \frac{4v v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c$$
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The usual substitution \( u = v(x) \) implies \( du = v' \, dx \),
Review 2 for Exam 1.

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\[
\int \frac{3 \, du}{2 - 4u^2} + \int \frac{4u \, du}{2 - 4u^2} = \ln(x) + c.
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The first integral on the left-hand side requires integration tables.
Review 2 for Exam 1.

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The first integral on the left-hand side requires integration tables. This is why the exact method is simpler to use in this case. \( \blacktriangle \)
Second order linear homogeneous ODE (Sect. 3.3).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Definition
Any two solutions $y_1, y_2$ of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called *fundamental solutions* iff the functions $y_1, y_2$ are linearly independent, that is, iff $W_{y_1y_2} \neq 0$.

Remark: Fundamental solutions are not unique.

Definition Given any two fundamental solutions $y_1, y_2$, and arbitrary constants $c_1, c_2$, the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is called the *general solution* of the differential equation above.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Theorem (Constant coefficients)

Given real constants \( a_1, a_0 \), consider the homogeneous, linear differential equation on the unknown \( y : \mathbb{R} \to \mathbb{R} \) given by

\[
y'' + a_1 y' + a_0 y = 0. \tag{1}
\]

Let \( r_+, r_- \) be the roots of the characteristic polynomial \( p(r) = r^2 + a_1 r + a_0 \), and let \( c_0, c_1 \) be arbitrary constants. Then, any solution of Eq. (1) belongs to only one of the following cases:

(a) If \( r_+ \neq r_- \), the general solution is \( y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t} \).

(b) If \( r_+ = r_- \in \mathbb{R} \), the general solution is \( y(t) = (c_1 + c_2 t) e^{r_+ t} \).

Furthermore, given real constants \( t_0, y_1 \) and \( y_2 \), there is a unique solution to the initial value problem given by Eq. (1) and the initial conditions

\[
y(t_0) = y_1, \quad y'(t_0) = y_2.
\]
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

**Example**

Find the general solution of the equation \( y'' - y' - 6y = 0 \).

Solution:

Since solutions have the form \( e^{rt} \), we need to find the roots of the characteristic polynomial \( p(r) = r^2 - r - 6 \), that is,

\[
 r \pm = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2}.
\]

\( r^+ \) and \( r^- \) are real-valued.

A fundamental solution set is formed by \( y_1(t) = e^{3t} \), \( y_2(t) = e^{-2t} \).

The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is, \( y(t) = c_1 e^{3t} + c_2 e^{-2t} \), \( c_1, c_2 \in \mathbb{R} \).

\( \triangleright \)

Remark: Since \( c_1, c_2 \in \mathbb{R} \), then \( y \) is real-valued.
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Example
Find the general solution of the equation \( y'' - y' - 6y = 0 \).

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So, \( r_{\pm} \) are real-valued.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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So, $r_{\pm}$ are real-valued. A fundamental solution set is formed by

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    r_{\pm} = \frac{1}{2} (1 \pm \sqrt{1 + 24}) = \frac{1}{2} (1 \pm 5) \Rightarrow r_+ = 3, \quad r_- = -2.
\]

So, \( r_{\pm} \) are real-valued. A fundamental solution set is formed by

\[
    y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}.
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The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions,
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

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Find the general solution of the equation \( y'' - y' - 6y = 0 \).

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Remark: Since \( c_1, c_2 \in \mathbb{R} \), then \( y \) is real-valued.
Second order linear homogeneous ODE.

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.
Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants $a_1, a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1r + a_0$ of the equation

$$y'' + a_1 y' + a_0 y = 0$$

(2)

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}.$$ 

Furthermore, a fundamental set of solutions to Eq. (2) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

while another fundamental set of solutions to Eq. (2) is

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$
Two main sets of fundamental solutions.

Example
Find the general solution of the equation $y'' - 2y' + 6y = 0$. 

Solution:
We first find the roots of the characteristic polynomial,
$r^2 - 2r + 6 = 0 
\Rightarrow r = \frac{1 \pm i\sqrt{5}}{2}$.

A fundamental solution set is
$\tilde{y}_1(t) = e^{(1+i \sqrt{5})t}$,
$\tilde{y}_2(t) = e^{(1-i \sqrt{5})t}$.

These are complex-valued functions.

The general solution is
$y(t) = \tilde{c}_1 e^{(1+i \sqrt{5})t} + \tilde{c}_2 e^{(1-i \sqrt{5})t}$,
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- One way to find the real-valued general solution is to find real-valued fundamental solutions.
Second order linear homogeneous ODE.

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
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A real-valued fundamental and general solutions.

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The Theorem above says that a real-valued fundamental set is
\[ y_1(t) = e^t \cos(\sqrt{5} \ t), \quad y_2(t) = e^t \sin(\sqrt{5} \ t). \]
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Hence, the complex-valued general solution can also be written as

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We just restricted the coefficients \( c_1, c_2 \) to be real-valued. \( \triangleq \)
A real-valued fundamental and general solutions.

Example
Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$. 
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Show that \( y_1(t) = e^t \cos(\sqrt{5} \, t) \) and \( y_2(t) = e^t \sin(\sqrt{5} \, t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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Now, recalling \( e^{(1\pm i\sqrt{5})t} = e^t e^{\pm i\sqrt{5}t} \)
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Show that $y_1(t) = e^t \cos(\sqrt{5} \, t)$ and $y_2(t) = e^t \sin(\sqrt{5} \, t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

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The Euler formula and its complex-conjugate formula

$$e^{i\sqrt{5}t} = \left[ \cos(\sqrt{5} t) + i \sin(\sqrt{5} t) \right],$$
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Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

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A real-valued fundamental and general solutions.

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Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

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Summary:
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- Therefore, $y_1, y_2$ form a fundamental set.
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Example

Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

Solution: \( y_1(t) = e^t \cos(\sqrt{5} t), \ y_2(t) = e^t \sin(\sqrt{5} t) \).

Summary:

- These functions are solutions of the differential equation.
- They are not proportional to each other, Hence li.
- Therefore, \( y_1, y_2 \) form a fundamental set.
- The general solution of the equation is

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y(t) = [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)] \ e^t.
\]
A real-valued fundamental and general solutions.

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- $y$ is complex-valued for $c_1, c_2 \in \mathbb{C}$. 
A real-valued fundamental and general solutions.

Remark:

- The proof of the Theorem follow exactly the same ideas given in the example above.

\[ y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t). \]
A real-valued fundamental and general solutions.

Remark:

- The proof of the Theorem follow exactly the same ideas given in the example above.
- One has to replace the roots of the characteristic polynomial
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  1 + i\sqrt{5} \quad \rightarrow \quad \alpha + i\beta, \quad 1 - i\sqrt{5} \quad \rightarrow \quad \alpha - i\beta.
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- The proof of the Theorem follow exactly the same ideas given in the example above.
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Example
Find real-valued fundamental solutions to the equation
\[ y'' + 2y' + 6y = 0. \]
A real-valued fundamental and general solutions.

Example
Find real-valued fundamental solutions to the equation

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Solution:
The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \)
A real-valued fundamental and general solutions.

Example
Find real-valued fundamental solutions to the equation

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The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \) are

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Differential equations like the one in this example describe physical processes related to damped oscillations. For example pendulums with friction.
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of \( y'' + 5y = 0 \).

\[
\text{Solution: The characteristic polynomial is } p(r) = r^2 + 5.
\]

Its roots are \( r = \pm \sqrt{5}i \). This is the case \( \alpha = 0 \), and \( \beta = \sqrt{5} \).

Real-valued fundamental solutions are \( y_1(t) = \cos(\sqrt{5}t) \), \( y_2(t) = \sin(\sqrt{5}t) \).

The real-valued general solution is \( y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) \), with \( c_1, c_2 \in \mathbb{R} \).

\[\text{ Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.}\]
A real-valued fundamental and general solutions.

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A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of $y'' + 5y = 0$.

Solution: The characteristic polynomial is $p(r) = r^2 + 5$.
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Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.
Second order linear homogeneous ODE.

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.
Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

\[ \frac{d}{dt} I(t) + R I(t) + \frac{1}{C} \int_0^t I(s) \, ds = 0. \]

Derivate both sides above:
\[ \frac{d^2}{dt^2} I(t) + \frac{2R}{L} I'(t) + \frac{1}{LC} I(t) = 0. \]

Divide by $L$:
\[ I''(t) + 2 \left( \frac{R}{L} \right)^2 I'(t) + \frac{1}{LC} I(t) = 0. \]

Introduce $\alpha = \frac{R}{L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then
\[ I''(t) + 2 \alpha I'(t) + \omega^2 I(t) = 0. \]
Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

The electric current flowing in such circuit satisfies:

$$LI'(t) + RI(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.$$
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Derivate both sides above: $L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0$. 
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$I(t)$: electric current.
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Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \).
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Case (a) $R = 0$. 

Remark: When the circuit has no resistance, the current oscillates without dissipation.
Application: The RLC circuit.

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Case (a) $R = 0$. This implies $\alpha = 0$, so $r_{\pm} = \pm i\omega$. 

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Application: The RLC circuit.

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Case (b) \( R < \sqrt{4L/C} \).
Example

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R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC}
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\[
l_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t),
\]

\[
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\]

The resistance \( R \) damps the current oscillations.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$ 

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$l_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad l_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$
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Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos\left(\sqrt{\omega^2 - \alpha^2} \ t\right), \quad I_2(t) = e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2} \ t\right).$$

R C L

I (t) : electric current.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: Recall: \( r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \).

Case (b) \( R < \sqrt{4L/C} \). This implies
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Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \Leftrightarrow \frac{R^2}{4L^2} < \frac{1}{LC} \Leftrightarrow \alpha^2 < \omega^2.$$ 

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad I_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$

The resistance $R$ damps the current oscillations.