Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number $\lambda$ and a non-zero function $y$ solutions to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix $A$, find $\lambda$ and a non-zero $n$-vector $v$ solutions of

$$Av - \lambda v = 0.$$ 

Differences:
- $A$ $\rightarrow$ $\begin{cases} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{cases}$
- $v$ $\rightarrow$ \{a function $y$\}. 
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

Remarks: We will show that:

1. If $\lambda \leq 0$, then the BVP has no solution.
2. If $\lambda > 0$, then there exist infinitely many eigenvalues $\lambda_n$ and eigenfunctions $y_n$, with $n$ any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

3. Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0, \ y'(L) = 0$; or for $y'(0) = 0, \ y'(L) = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$ 

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0.$$ 

Since $y = 0$, there are NO non-zero solutions for $\lambda = 0.$
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu.$$ 

The general solution is

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$ 

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

$$0 = y(L) = c_1 e^{\mu L} + c_2 e^{-\mu L}.$$ 

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}.$$ 

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix $Z$ is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$. 

Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$. 

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Recall: $y(x) = c_1 e^{\mu x} + c_2 e^{\mu x}$ and

$$c_1 + c_2 = 0, \quad c_1 e^{\mu L} + c_2 e^{-\mu L} = 0.$$ 

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}.$$ 

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix $Z$ is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$. 

Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$. 

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \implies r_\pm = \pm \mu i.$$  

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$  

The boundary condition are

$$0 = y(0) = c_1, \quad y(x) = c_2 \sin(\mu x).$$
$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \implies \sin(\mu L) = 0.$$  

Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \implies \mu_n L = n\pi \implies \mu_n = \frac{n\pi}{L}.$$  

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \quad \triangleq$$
On the Separation of Variables Method (Sect. 10.5).

- Review: The eigenvalue-eigenfunction problem.
- Example: Solving a Heat Equation.
- Not every PDE can be solved with SVM.
- Example: Solving a Wave Equation.

Review: The separation of variables method.

**Summary:** IBVP for the Heat Equation.

**Propose:**

\[ u(t, x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x). \]

where

- \( v_n \): Solution of an IVP.
- \( w_n \): Solution of a BVP, an eigenvalue-eigenfunction problem.
- \( c_n \): Fourier Series coefficients.

**Remark:**
The separation of variables method does not work for every PDE.
Example: Solving a Heat Equation.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Let \( u_n(t, x) = v_n(t) w_n(x) \). Then
\[ 4w_n(x) \frac{d v_n}{d t}(t) = v_n(t) \frac{d^2 w_n}{d x^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n. \]

The equations for \( v_n \) and \( w_n \) are
\[ v_n'(t) + \frac{\lambda_n}{4} v_n(t) = 0, \quad w_n''(x) + \lambda_n w_n(x) = 0. \]

We solve for \( v_n \) with the initial condition \( v_n(0) = 1 \).
\[ e^{\frac{\lambda_n}{4} t} v_n'(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0 \quad \Rightarrow \quad \left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0. \]
Example: Solving a Heat Equation.

Find the solution to the IBVP \(4\partial_t u = \partial^2_x u, \quad t > 0, \quad x \in [0, 2],\)
\[ u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \[ [e^{\frac{\lambda_n}{4} t} v_n(t)]' = 0. \] Therefore,
\[ v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}. \]

Next the BVP: \(w_n''(x) + \lambda_n w_n(x) = 0, \) with \(w_n(0) = w_n(2) = 0.\)
Since \(\lambda_n > 0,\) introduce \(\lambda_n = \mu_n^2.\) The characteristic polynomial is
\[ p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i. \]

The general solution, \(w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).\)

The boundary conditions imply
\[ 0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x). \]

Example: Solving a Heat Equation.

Find the solution to the IBVP \(4\partial_t u = \partial^2_x u, \quad t > 0, \quad x \in [0, 2],\)
\[ u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \(v_n(t) = e^{-\frac{\lambda_n}{4} t} \) and \(w_n(x) = c_2 \sin(\mu_n x).\)

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \(\mu_n 2 = n\pi,\) that is, \(\mu_n = \frac{n\pi}{2}.\) Choosing \(c_2 = 1,\) we conclude,
\[ \lambda_m \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad v_n(t) = e^{-\left(\frac{n\pi}{4}\right)^2 t}. \]

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]
Example: Solving a Heat Equation.

Example
Find the solution to the IBVP $4\partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2],
\quad u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$

Solution: Recall
\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is
\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies
\[ \int_0^2 3 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 c_n \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,
\[ 3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3. \]

Example: Solving a Heat Equation.

Example
Find the solution to the IBVP $4\partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2],
\quad u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$

Solution: We conclude that
\[ u(t, x) = 3 e^{-\left(\frac{\pi}{4}\right)^2 t} \sin\left(\frac{\pi x}{2}\right). \]
On the Separation of Variables Method (Sect. 10.5).

- Review: The eigenvalue-eigenfunction problem.
- Review: The Separation of Variables Method (SVM).
- Example: Solving a Heat Equation.
- **Not every PDE can be solved with SVM.**
- Example: Solving a Wave Equation.

---

**Not every PDE can be solved with SVM.**

**Example**

Determine whether the Separation of Variables Method can be used to solve
\[
\left(t + \frac{x}{c}\right) \partial_t^2 u(t, x) + k^2 \partial_x^2 u(t, x) = 0.
\]

**Solution:** If \( u(t, x) = v(t) w(x) \), then
\[
\partial_t^2 u(t, x) = w(x) \partial_t^2 v(t), \quad \partial_x^2 u(t, x) = v(t) \partial_x^2 w(x).
\]

Therefore, \( \left(t + \frac{x}{c}\right) w(x) \partial_t^2 v(t) = -k^2 v(t) \partial_x^2 w(x) \),

\[
\left(t + \frac{x}{c}\right) \frac{\partial_t^2 v(t)}{v(t)} = -k^2 \frac{\partial_x^2 w(x)}{w(x)}.
\]

**Function of \( t \) and \( x \) = Function only of \( x \)**

We conclude: **The SVM can not be used in this equation.**
Example: Solving a Wave Equation.

Example
Find the solution to the IBVP $\partial_t^2 u = c^2 \partial_x^2 u, \ t > 0, \ x \in [0, 3\pi],$
$u(0, x) = \sin(x), \ \partial_t u(0, x) = 0, \ u(t, 0) = 0, \ u(t, 3\pi) = 0.$

Remarks:
- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:
  (1) Initial position of the string.
  (2) Initial velocity of the string.
- There are two boundary conditions:
  (1) The string is fixed at both the end points.
Example: Solving a Wave Equation.

Example
Find the solution to the IBVP \( \partial_t^2 u = c^2 \partial_x^2 u, \ t > 0, \ x \in [0, 3\pi], \)
\[ u(0, x) = \sin(x), \ \partial_t u(0, x) = 0, \ u(t, 0) = 0, \ u(t, 3\pi) = 0. \]

Solution: Let \( u_n(t, x) = v_n(t) \ w_n(x). \) Then
\[ w_n(x) \frac{d^2 v_n(t)}{dt^2}(t) = c^2 v_n(t) \frac{d^2 w_n(x)}{dx^2}(x) \Rightarrow \frac{v_n''(t)}{c^2 v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n. \]
The equations for \( v_n \) and \( w_n \) are
\[ v_n''(t) + \lambda_n c^2 v_n(t) = 0, \quad w_n''(x) + \lambda_n w_n(x) = 0. \]

We first find the solution \( w_n \) to the BVP:
\[ w_n''(x) + \lambda_n w_n(x) = 0, \quad w_n(0) = 0, \quad w_n(3\pi) = 0. \]

Example: Solving a Wave Equation.

Example
Find the solution to the IBVP \( \partial_t^2 u = c^2 \partial_x^2 u, \ t > 0, \ x \in [0, 3\pi], \)
\[ u(0, x) = \sin(x), \ \partial_t u(0, x) = 0, \ u(t, 0) = 0, \ u(t, 3\pi) = 0. \]

Solution: Recall: \( w_n''(x) + \lambda_n w_n(x) = 0, \ w_n(0) = w_n(3\pi) = 0. \)
Since \( \lambda_n > 0, \) introduce \( \lambda_n = \mu_n^2. \) The characteristic polynomial is
\[ p(r) = r^2 + \mu_n^2 = 0 \ \Rightarrow \ r_{n\pm} = \pm \mu_n i. \]
The general solution, \( w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x). \)
The boundary conditions imply
\[ 0 = w_n(0) = c_1, \ \Rightarrow \ w_n(x) = c_2 \sin(\mu_n x). \]
\[ 0 = w_n(3\pi) = c_2 \sin(3\pi \mu_n), \ c_2 \neq 0, \ \Rightarrow \ \sin(3\pi \mu_n) = 0. \]
Example: Solving a Wave Equation.

Example
Find the solution to the IBVP \( \partial_t^2 u = c^2 \partial_x^2 u, \ t > 0, \ x \in [0, 3\pi], \)
\[ u(0, x) = \sin(x), \quad \partial_t u(0, x) = 0, \quad u(t, 0) = 0, \quad u(t, 3\pi) = 0. \]

Solution: Recall: \( w_n(x) = c_2 \sin(\mu_n x) \) and \( \sin(3\pi \mu_n) = 0. \)
Then, \( 3\pi \mu_n = n\pi, \) that is, \( \mu_n = \frac{n}{3}. \) Choosing \( c_2 = 1, \) we conclude,
\[ \lambda_m = \left( \frac{n}{3} \right)^2, \quad w_n(x) = \sin\left( \frac{nx}{3} \right). \]

We now find the general solution of \( v_n''(t) + c^2 \mu_n^2 v_n(t) = 0. \)
Then the solution to the Wave Equation will be
\[ u(t, x) = \sum_{n=1}^{\infty} v_n(t) \sin\left( \frac{nx}{3} \right). \]
Example: Solving a Wave Equation.

Example
Find the solution to the IBVP $\partial_t^2 u = c^2 \partial_x^2 u$, $t > 0$, $x \in [0, 3\pi]$,
$u(0, x) = \sin(x)$, $\partial_t u(0, x) = 0$, $u(t, 0) = 0$, $u(t, 3\pi) = 0$.

Solution: $u(t, x) = \sum_{n=1}^{\infty} \left[ c_n \cos\left(\frac{cnt}{3}\right) + d_n \sin\left(\frac{cnt}{3}\right) \right] \sin\left(\frac{nx}{3}\right)$.

First initial condition: $\sin(x) = u(0, x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx}{3}\right)$.

Multiply by $\sin(mx/3)$ on both sides above and integrate
$$\int_0^{3\pi} \sin(x) \sin\left(\frac{mx}{3}\right) dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\left(\frac{nx}{3}\right) \sin\left(\frac{mx}{3}\right) dx.$$ 

The orthogonality of the sine functions implies: If $m \neq 3$, then $0 = c_m \frac{3\pi}{2}$, that is, $c_m = 0$ for $m \neq 3$. Hence, $\sin(x) = c_3 \sin\left(\frac{3x}{3}\right)$, and then $c_3 = 1$.

Example: Solving a Wave Equation.

Example
Find the solution to the IBVP $\partial_t^2 u = c^2 \partial_x^2 u$, $t > 0$, $x \in [0, 3\pi]$,
$u(0, x) = \sin(x)$, $\partial_t u(0, x) = 0$, $u(t, 0) = 0$, $u(t, 3\pi) = 0$.

Solution: After the first initial condition we get
$u(t, x) = \cos(ct) \sin(x) + \sum_{n=1}^{\infty} d_n \sin\left(\frac{cnt}{3}\right) \sin\left(\frac{nx}{3}\right)$.

Second initial condition: $0 = \partial_t u(0, x) = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \sin\left(\frac{nx}{3}\right)$.

Multiply by $\sin\left(\frac{mx}{3}\right)$ on both sides above, then for $m = 1, 2, \cdots$,
$$0 = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \int_0^{3\pi} \sin\left(\frac{nx}{3}\right) \sin\left(\frac{mx}{3}\right) dx \Rightarrow d_m = 0.$$ 

We conclude: $u(t, x) = \cos(ct) \sin(x)$. \triangleq