
**Review:** The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

**Problem:** The time-independent temperature, \( T \), of a bar of length \( L \) with insulated horizontal sides and vertical extremes kept at fixed temperatures \( T_0, T_L \), is the solution of the BVP:

\[
T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,
\]

**Remark:** The heat transfer occurs only along the \( x \)-axis.
Solving the Heat Equation (Sect. 10.5).

- **The Heat Equation.**
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

**The Heat Equation.**

Remarks:
- The unknown of the problem is \( u(t, x) \), the temperature of the bar at the time \( t \) and position \( x \).
- The temperature does not depend on \( y \) or \( z \).
- The one-dimensional Heat Equation is:

\[
\frac{\partial}{\partial t} u(t, x) = k \frac{\partial^2}{\partial x^2} u(t, x),
\]

where \( k > 0 \) is the heat conductivity, units: \([k] = \frac{(\text{distance})^2}{\text{(time)}}\).
- The Heat Equation is a Partial Differential Equation, PDE.
Solving the Heat Equation (Sect. 10.5).

- The Heat Equation.
- **The Initial-Boundary Value Problem.**
- The separation of variables method.
- An example of separation of variables.

The Initial-Boundary Value Problem.

**Definition**

The **IBVP for the one-dimensional Heat Equation** is the following:

Given a constant \( k > 0 \) and a function \( f : [0, L] \to \mathbb{R} \) with \( f(0) = f(L) = 0 \), find \( u : [0, \infty) \times [0, L] \to \mathbb{R} \) solution of

\[
\partial_t u(t, x) = k \partial_x^2 u(t, x),
\]

I.C.: \( u(0, x) = f(x) \),

B.C.: \( u(t, 0) = 0, \quad u(t, L) = 0 \).
The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:
One looks for solutions $u$ given by an infinite series of simpler functions, $u_n$, that is,

$$u(t, x) = \sum_{n=1}^{\infty} c_n u_n(t, x),$$

where $u_n$ is simpler than $u$ is the sense,

$$u_n(t, x) = v_n(t) w_n(x).$$

Here $c_n$ are constants, $n = 1, 2, \cdots$. 
The separation of variables method.

Step 2:
Introduce the series expansion for $u$ into the Heat Equation,

$$\partial_t u - k \partial_x^2 u = 0 \Rightarrow \sum_{n=1}^{\infty} c_n [\partial_t u_n - k \partial_x^2 u_n] = 0.$$

A sufficient condition for the equation above is: To find $u_n$, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k \partial_x^2 u_n = 0.$$

Step 3:
Find $u_n(t, x) = v_n(t) \, w_n(x)$ solution of the IBVP

$$\partial_t u_n - k \partial_x^2 u_n = 0.$$

I.C.: $u_n(0, x) = w_n(x),$

B.C.: $u_n(t, 0) = 0, \quad u_n(t, L) = 0.$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for $u_n$ into: (a) IVP for $v_n$; (b) BVP for $w_n$. 

Notice:

$$\partial_t u_n(t, x) = \partial_t [v_n(t) \, w_n(x)] = w_n(x) \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t, x) = \partial_x^2 [v_n(t) \, w_n(x)] = v_n(t) \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$ is given by

$$w_n(x) \frac{dv_n}{dt}(t) = k \, v_n(t) \frac{d^2 w_n}{dx^2}(x)$$

$$\frac{1}{k \, v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x).$$

Depends only on $t$ = Depends only on $x$. 
The separation of variables method.

Recall: \[ \frac{1}{k \, v_n(t)} \frac{dv_n(t)}{dt} = \frac{1}{w_n(x)} \frac{d^2w_n(x)}{dx^2}. \]

Depends only on \( t \) = Depends only on \( x \).

- The Heat Equation has the following property: The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants \( \lambda_m \) holds
  \[ \frac{1}{k \, v_n(t)} \frac{dv_n(t)}{dt} = -\lambda_n, \quad \frac{1}{w_n(x)} \frac{d^2w_n(x)}{dx^2} = -\lambda_n. \]
- We have transformed the original PDE into infinitely many ODEs parametrized by \( n \), positive integer.

The separation of variables method.

**Summary Step 4:** The original IBVP for the Heat Equation, PDE, is transformed into:

(a) The IVP for \( v_n \),

\[ \frac{1}{k \, v_n(t)} \frac{dv_n(t)}{dt} = -\lambda_n, \quad \text{I.C.: } v_n(0) = 1. \]

(b) The BVP for \( w_n \),

\[ \frac{1}{w_n(x)} \frac{d^2w_n(x)}{dx^2} = -\lambda_n, \quad \text{B.C.: } w_n(0) = 0, \quad w_n(L) = 0. \]

**Step 5:**

(a) Solve the IVP for \( v_n \).

(b) Solve the BVP for \( w_n \).
The separation of variables method.

Step 5(a): Solving the IVP for $v_n$.

\[ v'_n(t) + k\lambda_n v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1. \]

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

\[ e^{k\lambda_n t}v'_n(t) + k\lambda_n e^{k\lambda_n t} v_n(t) = 0 \implies \left[ e^{k\lambda_n t}v_n(t) \right]' = 0. \]

\[ e^{k\lambda_n t}v_n(t) = c_n \implies v_n(t) = c_n e^{-k\lambda_n t}. \]

\[ 1 = v_n(0) = c \implies v_n(t) = e^{-k\lambda_n t}. \]

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for $w_n$:

Find the eigenvalues $\lambda_n$ and the non-zero eigenfunctions $w_n$ solutions of the BVP

\[ w''_n(x) + \lambda_n w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0. \]

We know that this problem has solution only for $\lambda_n > 0$.

Denote: $\lambda_n = \mu^2_n$. Proposing $w_n(x) = e^{\mu_n x}$, we get that

\[ p(r_n) = r_n^2 + \mu^2_n = 0 \implies r_{n\pm} = \pm \mu_n i \]

The real-valued general solution is

\[ w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x). \]
The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

\[ 0 = w_n(0) = c_1 \Rightarrow w_n(x) = c_2 \sin(\mu_n x). \]

\[ 0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \Rightarrow \sin(\mu_n L) = 0. \]

\[ \mu_n L = n\pi \Rightarrow \mu_n = \frac{n\pi}{L} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2. \]

Choosing $c_2 = 1$, we get $w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

We conclude that: $u_n(t, x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \cdots$.

The separation of variables method.

Step 6: Recall: $u_n(t, x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$.

Compute the solution to the IBVP for the Heat Equation,

\[ u(t, x) = \sum_{n=1}^{\infty} c_n u_n(t, x). \]

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \]

By construction, this solution satisfies the boundary conditions,

\[ u(t, 0) = 0, \quad u(t, L) = 0. \]

Given a function $f$ with $f(0) = f(L) = 0$, the solution $u$ above satisfies the initial condition $f(x) = u(0, x)$ iff holds

\[ f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right). \]
The separation of variables method.

Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right). \]

This is a Sine Series for \( f \). The coefficients \( c_n \) are computed in the usual way. Recall the orthogonality relation

\[ \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases} \]

Multiply the equation for \( u \) by \( \sin\left(\frac{m\pi x}{L}\right) \) nd integrate,

\[ \sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) \, dx. \]

\[ c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right). \]

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x). \]

where

- \( v_n \): Solution of an IVP.
- \( w_n \): Solution of a BVP, an eigenvalue-eigenfunction problem.
- \( c_n \): Fourier Series coefficients.

Remark:

The separation of variables method does not work for every PDE.
Example of separation of variables.

**Example**

Find the solution to the IBVP

\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

**Solution:** Let \( u_n(t, x) = v_n(t) w_n(x) \). Then

\[ 4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \implies \frac{4v'_n(t)}{v_n(t)} = \frac{w''_n(x)}{w_n(x)} = -\lambda_n. \]

The equations for \( v_n \) and \( w_n \) are

\[ v'_n(t) + \frac{\lambda_n}{4} v_n(t) = 0, \quad w''_n(x) + \lambda_n w_n(x) = 0. \]

We solve for \( v_n \) with the initial condition \( v_n(0) = 1 \).

\[ e^{\frac{\lambda_n}{4} t} v'_n(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0 \implies \left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0. \]
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \[ [e^{\frac{\lambda_n}{4}} t v_n(t)]' = 0. \]
Therefore,
\[ v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}. \]

Next the BVP: \[ w_n''(x) + \lambda_n w_n(x) = 0, \quad w_n(0) = w_n(L) = 0. \]
Since \( \lambda_n > 0 \), introduce \( \lambda_n = \mu_n^2 \). The characteristic polynomial is
\[ p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i. \]
The general solution, \( w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x) \).
The boundary conditions imply
\[ 0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x). \]

An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \[ v_n(t) = e^{-\frac{\lambda_n}{4} t}, \quad \text{and} \quad w_n(x) = c_2 \sin(\mu_n x). \]

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \( \mu_n 2 = n\pi \), that is, \( \mu_n = \frac{n\pi}{2} \). Choosing \( c_2 = 1 \), we conclude,
\[ \lambda_m = \left( \frac{n\pi}{2} \right)^2, \quad w_n(x) = \sin\left( \frac{n\pi x}{2} \right). \]

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left( \frac{n\pi}{4} \right)^2 t} \sin\left( \frac{n\pi x}{2} \right). \]
An example of separation of variables.

Example
Find the solution to the IBVP \( 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)
\[
u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.\]

Solution: Recall: \( u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \)

The initial condition is \( 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \)

The orthogonality of the sine functions implies
\[
3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.
\]
If \( m \neq 1, \) then \( 0 = c_m \frac{2}{2}, \) that is, \( c_m = 0 \) for \( m \neq 1. \) Therefore,
\[
3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.
\]

An example of separation of variables.

Example
Find the solution to the IBVP \( 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)
\[
u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.\]

Solution: We conclude that
\[
u(t, x) = 3 e^{-\left(\frac{\pi}{4}\right)^2 t} \sin\left(\frac{\pi x}{2}\right).
\]