Boundary Value Problems (Sect. 10.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Two-point Boundary Value Problem.

Definition

A **two-point BVP** is the following: Given functions $p$, $q$, $g$, and constants $x_1 < x_2$, $y_1, y_2$, $b_1, b_2$, $\tilde{b}_1, \tilde{b}_2$,

find a function $y$ solution of the differential equation

$$y'' + p(x) y' + q(x) y = g(x),$$

together with the extra, **boundary conditions**,

$$b_1 y(x_1) + b_2 y'(x_1) = y_1,$$
$$\tilde{b}_1 y(x_2) + \tilde{b}_2 y'(x_2) = y_2.$$

Remarks:

- Both $y$ and $y'$ might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,

$$y'' + a_1 y' + a_0 y = g(x).$$
Two-point Boundary Value Problem.

Example
Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find $y$ solution of
\[ y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2. \]

(2) Find $y$ solution of
\[ y'' + a_1 y' + a_0 y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2. \]

(3) Find $y$ solution of
\[ y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2. \]

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Example from physics.

**Problem:** The equilibrium (time independent) temperature of a bar of length $L$ with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures $T_0$, $T_L$ is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

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Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation
\[ y'' + a_1 y' + a_0 y = g(t), \]

together with the initial conditions
\[ y(t_0) = y_1, \quad y'(t_0) = y_2. \]

Remark: In physics:
- $y(t)$: Position at time $t$.
- Initial conditions: Position and velocity at the initial time $t_0$.

Comparison: IVP vs BVP.

Review: BVP:
Find the function values $y(x)$ solutions of the differential equation
\[ y'' + a_1 y' + a_0 y = g(x), \]

together with the initial conditions
\[ y(x_1) = y_1, \quad y(x_2) = y_2. \]

Remark: In physics:
- $y(x)$: A physical quantity (temperature) at a position $x$.
- Boundary conditions: Conditions at the boundary of the object under study, where $x_1 \neq x_2$. 
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Existence, uniqueness of solutions to BVP.

**Review:** The initial value problem.

**Theorem (IVP)**

Consider the homogeneous initial value problem:

\[ \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \]

and let \( r_\pm \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.

**Summary:** The IVP above always has a unique solution, no matter what \( y_0 \) and \( y_1 \) we choose.
Existence, uniqueness of solutions to BVP.

Theorem (BVP)
Consider the homogeneous boundary value problem:
\[ y'' + a_1 y' + a_0 y = 0, \quad y(0) = y_0, \quad y(L) = y_1, \]
and let \( r_\pm \) be the roots of the characteristic polynomial
\[ p(r) = r^2 + a_1 r + a_0. \]

(A) If \( r_+ \neq r_- \), real, then for every choice of \( L \neq 0 \) and \( y_0, y_1 \), there exists a unique solution \( y \) to the BVP above.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \beta \neq 0 \), and \( \alpha, \beta \in \mathbb{R} \), then the solutions to the BVP above belong to one of these possibilities:
(1) There exists a unique solution.
(2) There exists no solution.
(3) There exist infinitely many solutions.

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case \( r_+ \neq r_- \). The general solution is
\[ y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}. \]
The initial conditions determine \( c_1 \) and \( c_2 \) as follows:
\[ y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0} \]
\[ y_1 = y'(t_0) = c_1 r_- e^{r_- t_0} + c_2 r_+ e^{r_+ t_0} \]
Using matrix notation,
\[
\begin{bmatrix}
  e^{r_- t_0} & e^{r_+ t_0} \\
  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= \begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}.
\]
The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where
\[
Z = \begin{bmatrix}
  e^{r_- t_0} & e^{r_+ t_0} \\
  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix} \Rightarrow Z \begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

**Proof of IVP:**
Recall: 
\[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \neq 0 \iff r_+ \neq r_- . \]

Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so
\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} . \]

We conclude that for every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the IVP above has a unique solution. \( \square \)

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Existence, uniqueness of solutions to BVP.

**Proof of BVP:** The general solution is
\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R} . \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:
\[ y_0 = y(0) = c_1 + c_2 . \]
\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} . \]

Using matrix notation,
\[ \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} . \]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where
\[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} . \]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows \[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i \beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then \[ \det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,

1. If \( \beta L \neq n\pi \), then BVP has a unique solution.
2. If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions. \( \square \)

Existence, uniqueness of solutions to BVP.

Example

Find \( y \) solution of the BVP \[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1. \]

Solution: The characteristic polynomial is \[ p(r) = r^2 + 1 \Rightarrow r_\pm = \pm i. \]

The general solution is \[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]

The boundary conditions are \( 1 = y(0) = c_1 \), \( -1 = y(\pi) = -c_1 \) \Rightarrow \( c_1 = 1 \), \( c_2 \) free.

We conclude: \( y(x) = \cos(x) + c_2 \sin(x) \), with \( c_2 \in \mathbb{R} \).

The BVP has infinitely many solutions. \( \triangleright \)
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$  

Solution: The characteristic polynomial is
$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are
$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1.$$  

The BVP has no solution.

Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is
$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are
$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$  

We conclude: $y(x) = \cos(x) + \sin(x)$.  

The BVP has a unique solution.
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Particular case of BVP: Eigenvalue-eigenfunction problem.

**Problem:**
Find a number $\lambda$ and a non-zero function $y$ solutions to the boundary value problem
\[ y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0. \]

**Remark:** This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix $A$, find $\lambda$ and a non-zero $n$-vector $v$ solutions of
\[ Av - \lambda v = 0. \]

**Differences:**
- $A \rightarrow \begin{cases} 
\text{computing a second derivative and} \\
\text{applying the boundary conditions.} 
\end{cases}$
- $v \rightarrow \{\text{a function } y\}$. 
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP
\[ y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0. \]

Remarks: We will show that:
(1) If $\lambda \leq 0$, then the BVP has no solution.
(2) If $\lambda > 0$, then there exist infinitely many eigenvalues $\lambda_n$ and eigenfunctions $y_n$, with $n$ any positive integer, given by
\[ \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad y_n(x) = \sin \left( \frac{n\pi x}{L} \right), \]
(3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0, \ y'(L) = 0$; or for $y'(0) = 0, \ y'(L) = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP
\[ y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0. \]

Solution: Case $\lambda = 0$. The equation is
\[ y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x. \]

The boundary conditions imply
\[ 0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0. \]

Since $y = 0$, there are NO non-zero solutions for $\lambda = 0.$
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every \( \lambda \in \mathbb{R} \) and non-zero functions \( y \) solutions of the BVP

\[
y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.
\]

Solution: Case \( \lambda < 0 \). Introduce the notation \( \lambda = -\mu^2 \). The characteristic equation is

\[
p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu.
\]

The general solution is

\[
y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.
\]

The boundary condition are

\[
0 = y(0) = c_1 + c_2,
\]

\[
0 = y(L) = c_1 e^{\mu L} + c_2 e^{-\mu L}.
\]

Solution: Recall: \( y(x) = c_1 e^{\mu x} + c_2 e^{\mu x} \) and

\[
c_1 + c_2 = 0, \quad c_1 e^{\mu L} + c_2 e^{-\mu L} = 0.
\]

We need to solve the linear system

\[
\begin{bmatrix}
1 & 1 \\
e^{\mu L} & e^{-\mu L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix} \quad \Leftrightarrow \quad Z \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad Z = \begin{bmatrix}
1 & 1 \\
e^{\mu L} & e^{-\mu L}
\end{bmatrix}
\]

Since \( \text{det}(Z) = e^{-\mu L} - e^{\mu L} \neq 0 \) for \( L \neq 0 \), matrix \( Z \) is invertible, so the linear system above has a unique solution \( c_1 = 0 \) and \( c_2 = 0 \).

Since \( y = 0 \), there are NO non-zero solutions for \( \lambda < 0 \).
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$  

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$  

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$  

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0.$$  

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$  

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Recall: $c_1 = 0, \quad c_2 \neq 0, \quad$ and $\sin(\mu L) = 0$.  

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$  

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \quad \triangle$$