The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]

- Solutions to the Euler equation near \(x_0\).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

Overview: Equations with singular points.

Recall: The point \(x_0 \in \mathbb{R}\) is a singular point of the equation

\[P(x) y'' + Q(x) y' + R(x) y = 0\]

iff holds \(P(x_0) = 0\).

Remarks:

- We are interested in finding solutions to the equation above arbitrary close to a singular point \(x_0\).
- The order of the differential equation changes in a neighborhood of a singular point.
- In the limit \(x \to x_0\) the following could happen:
  - (a) The two linearly independent solutions remain bounded.
  - (b) Only one solution remains bounded.
  - (c) None solution remains bounded.
Overview: Equations with singular points.

Remarks:

- If the singular point of a differential equation is not so singular, in a sense to be made precise later on, then it is known how to find solutions to such equation.
- Singular points where the singular behavior of the solution is somehow mild, in a sense to be made precise later, will be called regular-singular points.
- The main example of an equation with a regular-singular point is the Euler differential equation.

The Euler equation (Sect. 5.4).

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The Euler equation

Definition
Given real constants $p_0$, $q_0$, the \textit{Euler differential equation} for the unknown $y$ with singular point at $x_0 \in \mathbb{R}$ is given by

$$(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$$ 

Remarks:
- The Euler equation has variable coefficients.
- Functions $y(x) = e^{rx}$ are \textit{not} solutions of the Euler equation.
- The point $x_0 \in \mathbb{R}$ is a singular point of the equation.
- The particular case $x_0 = 0$ is given by

$$x^2 y'' + p_0 x y' + q_0 y = 0.$$ 

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:

$$(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.$$ 

- \textbf{Solutions to the Euler equation near $x_0$.}
- The roots of the indicial polynomial.
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Solutions to the Euler equation near $x_0$.

**Summary of the main idea:**

- The main idea to find solution to the constant coefficients equation $y'' + a_1 y' + a_0 y = 0$ was to look for functions of the form $y(x) = e^{rx}$. The exponential cancels out from the equation and we obtain an equation only for $r$ without $x$,

$$
(r^2 + a_1 r + a_0)e^{rx} = 0 \iff (r^2 + a_1 r + a_0) = 0. \tag{1}
$$

- In the case of the Euler equation $x^2 y'' + p_0 x y' + q_0 y = 0$ the exponential functions $e^{rx}$ do not have the property given in Eq. (1), since

$$(x^2 r^2 + p_0 x r + q_0) e^{rx} = 0 \iff x^2 r^2 + p_0 x r + q_0 = 0,$$

but the later equation still involves the variable $x$.

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Solutions to the Euler equation near $x_0$.

**Summary of the main idea:** Look for solutions like $y(x) = x^r$. These function have the following property:

$$
y'(x) = r x^{r-1} \Rightarrow x y'(x) = r x^r;
$$

$$
y''(x) = r(r-1) x^{r-2} \Rightarrow x^2 y''(x) = r(r-1) x^r.
$$

Introduce $y = x^r$ into Euler’s equation $x^2 y'' + p_0 x y' + q_0 y = 0$, for $x \neq 0$ we obtain

$$
[r(r-1) + p_0 r + q_0] x^r = 0 \iff r(r-1) + p_0 r + q_0 = 0.
$$

The last equation involves only $r$, not $x$.

This equation is called the **indicial equation**, and is also called the **Euler characteristic equation**.
Solutions to the Euler equation near $x_0$.

Theorem (Euler equation)

Given $p_0$, $q_0$, $x_0 \in \mathbb{R}$, consider the Euler equation

$$\begin{align*}
(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y &= 0. \tag{2}
\end{align*}$$

Let $r_+$, $r_-$ be solutions of $r(r - 1) + p_0 r + q_0 = 0$.

(a) If $r_+ \neq r_-$, then a real-valued general solution of Eq. (2) is

$$y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}, \quad x \neq x_0, \quad c_0, \ c_1 \in \mathbb{R}.$$ 

(b) If $r_+ = r_-$, then a real-valued general solution of Eq. (2) is

$$y(x) = c_0 + c_1 \ln(|x - x_0|) |x - x_0|^{r_+}, \quad x \neq x_0, \quad c_0, \ c_1 \in \mathbb{R}.$$ 

Given $x_0 \neq x_1$, $y_0$, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$\begin{align*}
(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y &= 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.
\end{align*}$$

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:

$$\begin{align*}
(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y &= 0.
\end{align*}$$

- Solutions to the Euler equation near $x_0$.
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Different real roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' + 4x y' + 2y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ xy'(x) = rx^r, \quad x^2 y''(x) = r(r-1)x^r. \]

Introduce \( y(x) = x^r \) into Euler equation,
\[ [r(r-1) + 4r + 2] x^r = 0 \iff r(r-1) + 4r + 2 = 0. \]

The solutions of \( r^2 + 3r + 2 = 0 \) are given by
\[ r_{\pm} = \frac{1}{2} [-3 \pm \sqrt{9-8}] \implies r_+ = -1 \quad r_- = -2. \]

The general solution is \( y(x) = c_1 |x|^{-1} + c_2 |x|^{-2}. \)
Repeated roots.

**Example**

Find the general solution of \( x^2 y'' - 3x y' + 4y = 0. \)

**Solution:** We look for solutions of the form \( y(x) = x^r, \)

\[
x y'(x) = rx^r, \quad x^2 y''(x) = r(r - 1)x^r.
\]

Introduce \( y(x) = x^r \) into Euler equation,

\[
[r(r - 1) - 3r + 4] x^r = 0 \quad \Leftrightarrow \quad r(r - 1) - 3r + 4 = 0.
\]

The solutions of \( r^2 - 4r + 4 = 0 \) are given by

\[
r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \quad \Rightarrow \quad r_+ = r_- = 2.
\]

Two linearly independent solutions are

\[
y_1(x) = x^2, \quad y_2 = x^2 \ln(|x|).
\]

The general solution is \( y(x) = c_1 x^2 + c_2 x^2 \ln(|x|). \)

\[
\triangleq
\]

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \( x_0. \)
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  - Different real roots.
  - Repeated roots.
  - **Different complex roots.**
Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13 y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \), \( x y'(x) = r x^r \), \( x^2 y''(x) = r(r-1) x^r \).

Introduce \( y(x) = x^r \) into Euler equation
\[
[r(r-1) - 3r + 13] x^r = 0 \quad \iff \quad r(r-1) - 3r + 13 = 0.
\]
The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are
\[
\{ r_+ = 2 + 3i \quad r_- = 2 - 3i. \}
\]
The general solution is \( y(x) = c_1 |x|^{(2+3i)} + c_2 |x|^{(2-3i)}. \)

\[ \triangle \]

Different complex roots.

Theorem
If \( p_0, q_0 \in \mathbb{R} \) satisfy that \( [(p_0 - 1)^2 - 4q_0] < 0 \), then the indicial polynomial \( p(r) = r(r-1) + p_0 r + q_0 \) of the Euler equation
\[ x^2 y'' + p_0 x y' + q_0 y = 0 \] (3)
has complex roots \( r_+ = \alpha + i\beta \) and \( r_- = \alpha - i\beta \), where
\[
\alpha = -\frac{(p_0 - 1)}{2}, \quad \beta = \frac{1}{2} \sqrt{4q_0 - (p_0 - 1)^2}.
\]
Furthermore, a fundamental set of solution to Eq. (3) is
\[
\tilde{y}_1(x) = |x|^{(\alpha+i\beta)}, \quad \tilde{y}_2(x) = |x|^{(\alpha-i\beta)},
\]
while another fundamental set of solutions to Eq. (3) is
\[
y_1(x) = |x|^\alpha \cos(\beta \ln |x|), \quad y_2(x) = |x|^\alpha \sin(\beta \ln |x|).\]
Different complex roots.

**Proof:** Given \( \tilde{y}_1 = |x|^{(\alpha+i\beta)} \) and \( \tilde{y}_2 = |x|^{(\alpha-i\beta)} \), introduce

\[
y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_1 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).
\]

Use another Euler equation to rewrite \( \tilde{y}_1 \) and \( \tilde{y}_2 \),

\[
\tilde{y}_1 = |x|^{(\alpha+i\beta)} = |x|^\alpha x^{i\beta} = |x|^\alpha e^{i\beta \ln(|x|)} = |x|^\alpha e^{i\beta \ln(|x|)}.
\]

\[
\tilde{y}_1 = |x|^\alpha \left[ \cos(\beta \ln |x|) + 1 \sin(\beta \ln |x|) \right],
\]

\[
\tilde{y}_2 = |x|^\alpha \left[ \cos(\beta \ln |x|) - 1 \sin(\beta \ln |x|) \right].
\]

We conclude that

\[
y_1(x) = |x|^\alpha \cos(\beta \ln |x|), \quad y_2(x) = |x|^\alpha \sin(\beta \ln |x|).
\]

\[\square\]

Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

\[x^2 y'' - 3x y' + 13 y = 0.\]

**Solution:** The indicial equation is \( r(r - 1) - 3r + 13 = 0 \).

The solutions of the indicial equations are

\[r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i.\]

A complex-valued general solution is

\[y(x) = \tilde{c}_1 |x|^{(2+3i)} + \tilde{c}_2 |x|^{(2-3i)} \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.\]

A real-valued general solution is

\[y(x) = c_1 |x|^2 \cos(3 \ln |x|) + c_2 |x|^2 \sin(3 \ln |x|), \quad c_1, c_2 \in \mathbb{R}. \]

\[\triangle\]