Power series solutions near regular points (Sect. 5.2).

- We study: \( P(x) y'' + Q(x) y' + R(x) y = 0 \).
- Review of power series.
- Regular point equations.
- Solutions using power series.
- Examples of the power series method.

Review of power series.

**Definition**

The **power series** of a function \( y : \mathbb{R} \to \mathbb{R} \) centered at \( x_0 \in \mathbb{R} \) is

\[
y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.
\]

**Example**

- \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \). Here \( x_0 = 0 \) and \( |x| < 1 \).
- \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \). Here \( x_0 = 0 \) and \( x \in \mathbb{R} \).
- The Taylor series of \( y : \mathbb{R} \to \mathbb{R} \) centered at \( x_0 \in \mathbb{R} \) is

\[
y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n = y(x_0) + y'(x_0)(x - x_0) + \cdots.
\]
Review of power series.

Example
Find the Taylor series of \( y(x) = \sin(x) \) centered at \( x_0 = 0 \).

Solution: \( y(x) = \sin(x) \), \( y(0) = 0 \). \( y'(x) = \cos(x) \), \( y'(0) = 1 \).

\( y''(x) = -\sin(x) \), \( y''(0) = 0 \). \( y'''(x) = -\cos(x) \), \( y'''(0) = -1 \).

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{(2n+1)} \]

Remark: The Taylor series of \( y(x) = \cos(x) \) centered at \( x_0 = 0 \) is

\[ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{(2n)} \].

Review of power series.

Remark: The power series of a function may not be defined on the whole domain of the function.

Example
The function \( y(x) = \frac{1}{1-x} \) is defined for \( x \in \mathbb{R} - \{1\} \).

The power series

\[ y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \]

converges only for \( |x| < 1 \).
Review of power series.

Definition
The power series \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely iff the series \( \sum_{n=0}^{\infty} |a_n| |x - x_0|^n \) converges.

Example
The series \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges, but it does not converge absolutely, since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

Review of power series.

Definition
The radius of convergence of a power series

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

is the number \( \rho \geq 0 \) that satisfies both

(a) the series converges absolutely for \( |x - x_0| < \rho \);
(b) the series diverges for \( |x - x_0| > \rho \).

\[
\begin{array}{ccc}
\text{diverges} & \text{converges absolutely} & \text{diverges} \\
\hline
\hline
x_0 & \hline
\hline
\rho
\end{array}
\]
Review of power series.

Example

(1) \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) has radius of convergence \( \rho = 1 \).

(2) \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) has radius of convergence \( \rho = \infty \).

(3) \( \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \) has radius \( \rho = \infty \).

(4) \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \) has radius of convergence \( \rho = \infty \).

Review of power series.

Theorem (Ratio test)

Given the power series \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \), introduce the number \( L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \). Then, the following statements hold:

(1) The power series converges in the domain \( |x - x_0| L < 1 \).

(2) The power series diverges in the domain \( |x - x_0| L > 1 \).

(3) The power series may or may not converge at \( |x - x_0| L = 1 \).

Therefore, if \( L \neq 0 \), then \( \rho = \frac{1}{L} \) is the series radius of convergence; if \( L = 0 \), then the radius of convergence is \( \rho = \infty \).
Review of power series.

Remarks: On summation indices:

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \]

\[ y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{m=-3}^{\infty} a_{m+3} (x - x_0)^{m+3}. \]

\[ y'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + \cdots \]

\[ y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{m=0}^{\infty} (m + 1) a_{m+1} (x - x_0)^m \]

where \( m = n - 1 \), that is, \( n = m + 1 \).

Power series solutions near regular points (Sect. 5.2).

- We study: \( P(x) y'' + Q(x) y' + R(x) y = 0 \).
- Review of power series.
- **Regular point equations.**
- Solutions using power series.
- Examples of the power series method.
Regular point equations.

Problem: We look for solutions $y$ of the variable coefficients equation

$$P(x) y'' + Q(x) y' + R(x) y = 0.$$ 

around $x_0 \in \mathbb{R}$ where $P(x_0) \neq 0$ using a power series representation of the solution centered at $x_0$, that is,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$ 

Definition

Given continuous functions $P, Q, R : (x_1, x_2) \to \mathbb{R}$, a point $x_0 \in (x_1, x_2)$ is called a regular point of the equation

$$P(x) y'' + Q(x) y' + R(x) y = 0.$$ 

iff $P(x_0) \neq 0$. The point $x_0$ is called a singular point iff $P(x_0) = 0$.

Remark: The equation order does not change near regular points.

Power series solutions near regular points (Sect. 5.2).

- We study: $P(x) y'' + Q(x) y' + R(x) y = 0$.
- Review of power series.
- Regular point equations.
- Solutions using power series.
- Examples of the power series method.
Solutions using power series.

Summary for regular points:

(1) Propose a power series representation of the solution centered at $x_0$, given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n; \quad (1)$$

(2) Introduce Eq. (1) into the differential equation

$$P(x) y'' + Q(x) y' + R(x) y = 0.$$

(3) Find a recurrence relation among the coefficients $a_n$;

(4) Solve the recurrence relation in terms of free coefficients;

(5) If possible, add up the resulting power series for the solution $y$.

Power series solutions near regular points (Sect. 5.2).

- We study: $P(x) y'' + Q(x) y' + R(x) y = 0$.
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- **Examples of the power series method.**
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$  

Solution: Recall: The solution is $y(x) = a_0 e^{-c x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} = \sum_{n=1}^{\infty} n a_n x^{(n-1)}.$$  

Change the summation index: $m = n - 1$, so $n = m + 1$.

$$y'(x) = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n.$$  

Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c y = 0, \quad c \in \mathbb{R}.$$  

Solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and $y'(x) = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$.

Introduce $y$ and $y'$ into the differential equation,

$$\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n + \sum_{n=0}^{\infty} c a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n + 1) a_{n+1} + c a_n] x^n = 0.$$  

The recurrence relation is $(n + 1) a_{n+1} + c a_n = 0$ for all $n \geq 0$.  

Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c\, y = 0, \quad c \in \mathbb{R}.\]

Solution: Recurrence relation: $(n + 1)a_{n+1} + c\, a_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1} a_n$. That is,

- $n = 0$, $a_1 = -c\, a_0 \Rightarrow a_1 = -c\, a_0$,
- $n = 1$, $2a_2 = -c\, a_1 \Rightarrow a_2 = \frac{c^2}{2!}\, a_0$,
- $n = 2$, $3a_3 = -c\, a_2 \Rightarrow a_3 = -\frac{c^3}{3!}\, a_0$,
- $n = 3$, $4a_4 = -c\, a_3 \Rightarrow a_4 = \frac{c^4}{4!}\, a_0$.

If we recall the power series $e^{ax} = \sum_{n=0}^\infty \frac{(ax)^n}{n!}$, then, we conclude that the solution is $y(x) = a_0\, e^{-c\, x}$.
Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
$$y'' + y = 0.$$ 

Solution: Recall: The characteristic polynomial is $r^2 + 1 = 0$, hence the general solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

We re-obtain this solution using the power series method:

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m,$$

where $m = n - 1$, so $n = m + 1$;

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m.$$

where $m = n - 2$, so $n = m + 2$. 

Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
$$y'' + y = 0.$$ 

Solution: Introduce $y$ and $y''$ into the differential equation,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{(n+2)} + a_n] x^n = 0.$$ 

The recurrence relation is $(n + 2)(n + 1) a_{(n+2)} + a_n = 0, \quad n \geq 0.$

Equivalently: $(n + 2)(n + 1) a_{(n+2)} = -a_n,$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: $(n + 2)(n + 1) a_{(n+2)} = -a_n, \ n \geq 0.$

For $n$ even: $n = 0, \ (2)(1)a_2 = -a_0 \ \Rightarrow \ a_2 = \frac{-1}{2!} a_0,$

$n = 2, \ (4)(3)a_4 = -a_2 \ \Rightarrow \ a_4 = \frac{1}{4!} a_0,$

$n = 4, \ (6)(5)a_6 = -a_4 \ \Rightarrow \ a_6 = \frac{1}{6!} a_0.$

We obtain: $a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \ for \ k \geq 0.$

Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: $a_{2k} = \frac{(-1)^k}{(2k)!} a_0$ and $(n + 2)(n + 1) a_{(n+2)} = -a_n.$

For $n$ odd: $n = 1, \ (3)(2)a_3 = -a_1 \ \Rightarrow \ a_3 = \frac{-1}{3!} a_1,$

$n = 3, \ (5)(4)a_5 = -a_3 \ \Rightarrow \ a_5 = \frac{1}{5!} a_1,$

$n = 5, \ (7)(6)a_7 = -a_5 \ \Rightarrow \ a_7 = \frac{-1}{7!} a_1.$

We obtain $a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1, \ for \ k \geq 0.$
Examples of the power series method.

Example
Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$ 

Solution: Recall: $a_{2k} = \frac{(-1)^k}{(2k)!} a_0$ and $a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1$.

Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}.$$ 

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

Solution: We propose: $y = \sum_{n=0}^{\infty} a_n (x - 2)^n$.

It is convenient to rewrite the function $xy$ as follows,

$$xy = \sum_{n=0}^{\infty} a_n x(x - 2)^n = \sum_{n=0}^{\infty} a_n [(x - 2) + 2] (x - 2)^n,$$

$$xy = \sum_{n=0}^{\infty} a_n (x - 2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x - 2)^n.$$ 

We relabel the first sum: $\sum_{n=0}^{\infty} a_n (x - 2)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x - 2)^n$. 

$\triangle$
Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

\[ y'' - x y = 0. \]

Solution: We relabel the \( y'' \),

\[ y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^{n}. \]

Introduce \( y'' \) and \( xy \) in the differential equation

\[ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{n-1}(x-2)^n = 0 \]

\[ (2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2a_n - a_{n-1}](x-2)^n = 0. \]

The recurrence relation for the coefficients \( a_n \) is:

\[ a_2 - a_0 = 0, \quad (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1. \]

Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation

\[ y'' - x y = 0. \]

Solution: The recurrence relation is:

\[ a_2 - a_0 = 0, \quad (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1. \]

We solve this recurrence relation for the first four coefficients,

\[ n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0, \]

\[ n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3}, \]

\[ n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}. \]

\[ y \approx a_0 + a_1(x-2) + a_0(x-2)^2 + \left( \frac{a_0}{6} + \frac{a_1}{3} \right)(x-2)^3 + \left( \frac{a_0}{6} + \frac{a_1}{12} \right)(x-2)^4. \]
Examples of the power series method.

Example
Find the first three terms of the power series expansion around the point \( x_0 = 2 \) of each fundamental solution to the differential equation
\[
y'' - x y = 0.
\]
Solution: The first terms in the power series expression for \( y \) are
\[
y \simeq a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left( \frac{a_0}{6} + \frac{a_1}{3} \right)(x - 2)^3 + \left( \frac{a_0}{6} + \frac{a_1}{12} \right)(x - 2)^4.
\]
\[
y = a_0 \left[ 1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3 + \frac{1}{6}(x - 2)^4 + \cdots \right] + a_1 \left[ (x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4 + \cdots \right]
\]
So the first three terms on each fundamental solution are given by
\[
y_1 \simeq 1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3, \quad y_2 \simeq (x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4.
\]