Variable coefficients second order linear ODE (Sect. 3.2).

**Summary:** The study the main properties of solutions to second order, linear, variable coefficients, ODE.

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel’s theorem on the Wronskian.

**Review: Second order linear ODE.**

**Definition**
Given functions $a_1, a_0, b : \mathbb{R} \rightarrow \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$ y'' + a_1(t) y' + a_0(t) y = b(t) \quad (1) $$

is called a *second order linear* differential equation with *variable coefficients*.

**Theorem**
If the functions $y_1$ and $y_2$ are solutions to the homogeneous linear equation

$$ y'' + a_1(t) y' + a_0(t) y = 0, $$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants $c_1, c_2 \in \mathbb{R}$. 
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- **Existence and uniqueness of solutions.**
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Existence and uniqueness of solutions.

**Theorem (Variable coefficients)**

*If the functions \( a, b : (t_1, t_2) \rightarrow \mathbb{R} \) are continuous, the constants \( t_0 \in (t_1, t_2) \) and \( y_0, y_1 \in \mathbb{R} \), then there exists a unique solution \( y : (t_1, t_2) \rightarrow \mathbb{R} \) to the initial value problem*

\[
y'' + a_1(t) y' + a_0(t) y = b(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.
\]

**Remarks:**

- Unlike the first order linear ODE where we have an explicit expression for the solution, there is *no explicit expression* for the solution of second order linear ODE.
- **Two integrations** must be done to find solutions to second order linear. Therefore, initial value problems with *two initial conditions* can have a unique solution.
Existence and uniqueness of solutions.

Example
Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem
$$(t - 1)y'' - 3ty' + 4y = t(t - 1), \quad y(-2) = 2, \quad y'(-2) = 1.$$ 

Solution: We first write the equation above in the form given in the Theorem above,
$$y'' - \frac{3t}{t - 1} y' + \frac{4}{t - 1} y = t.$$ 

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are $I_1 = (-\infty, 1)$ and $I_2 = (1, +\infty)$. Since the initial condition belongs to $I_1$, the solution domain is $I_1 = (-\infty, 1)$.

Existence and uniqueness of solutions.

Remark: The rest of the class is dedicated to show:

If functions $y_1$, $y_2$ are not proportional to each other and they are solutions of the equation
$$y'' + a_1(t) y' + a_0(t) y = 0,$$
then any other solution to this equation is given by
$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$
for appropriate constants $c_1$, $c_2$.

Remark: Before we prove this statement we need few definitions:
- Proportional functions (linearly dependent).
- Wronskian.
- State a more precise and general result.
Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- **Linearly dependent and independent functions.**
- The Wronskian of two functions.
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**Linearly dependent and independent functions.**

**Definition**
Two continuous functions \( y_1, y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R} \) are called **linearly dependent, (ld)**, on the interval \( (t_1, t_2) \) iff there exists a constant \( c \) such that for all \( t \in I \) holds

\[
y_1(t) = c \, y_2(t).
\]

The two functions are called **linearly independent, (li)**, on the interval \( (t_1, t_2) \) iff they are not linearly dependent.

**Remarks:**
- \( y_1, y_2 : (t_1, t_2) \to \mathbb{R} \) are ld ⇔ there exist constants \( c_1, c_2 \), not both zero, such that \( c_1 \, y_1(t) + c_2 \, y_2(t) = 0 \) for all \( t \in (t_1, t_2) \).
- \( y_1, y_2 : (t_1, t_2) \to \mathbb{R} \) are li ⇔ the only constants \( c_1, c_2 \), solutions of \( c_1 \, y_1(t) + c_2 \, y_2(t) = 0 \) for all \( t \in (t_1, t_2) \) are \( c_1 = c_2 = 0 \).
- These definitions are not given in the textbook.
Example

(a) Show that \( y_1(t) = \sin(t) \), \( y_2(t) = 2 \sin(t) \) are ld.

(b) Show that \( y_1(t) = \sin(t) \), \( y_2(t) = t \sin(t) \) are li.

Solution:

Case (a): Trivial. \( y_2 = 2y_1 \).

Case (b): Find constants \( c_1 \), \( c_2 \) such that for all \( t \in \mathbb{R} \) holds

\[
c_1 \sin(t) + c_2 t \sin(t) = 0 \iff (c_1 + c_2 t) \sin(t) = 0.
\]

Evaluating at \( t = \pi/2 \) and \( t = 3\pi/2 \) we obtain

\[
c_1 + \frac{\pi}{2} c_2 = 0, \quad c_1 + \frac{3\pi}{2} c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.
\]

We conclude: The functions \( y_1 \) and \( y_2 \) are li.  ◀
The Wronskian of two functions.

Remark: The Wronskian is a function that determines whether two functions are ld or li.

Definition
The Wronskian of functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ is the function

$$W_{y_1y_2}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Remark:
- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1y_2}(t) = \det(A(t))$.
- An alternative notation is: $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

The Wronskian of two functions.

Example
Find the Wronskian of the functions:
(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)
(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

Solution:
Case (a): $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)2\cos(t) - \cos(t)2\sin(t) \Rightarrow W_{y_1y_2}(t) = 0.$$ 

Case (b): $W_{y_1y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)[\sin(t) + t\cos(t)] - \cos(t)t\sin(t).$$

We obtain $W_{y_1y_2}(t) = \sin^2(t).$
The Wronskian of two functions.

**Remark:** The Wronskian determines whether two functions are linearly dependent or independent.

**Theorem (Wronskian and linearly dependence)**

The continuously differentiable functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are linearly dependent iff $W_{y_1y_2}(t) = 0$ for all $t \in (t_1, t_2)$.

**Remark:** Importance of the Wronskian:

- Sometimes it is not simple to decide whether two functions are proportional to each other.
- The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel’s Theorem later on.)

The Wronskian of two functions.

**Example**

Show whether the following two functions form a l.d. or l.i. set:

$$y_1(t) = \cos(2t) - 2 \cos^2(t), \quad y_2(t) = \cos(2t) + 2 \sin^2(t).$$

**Solution:** Compute their Wronskian:

$$W_{y_1y_2}(t) = y_1 y_2' - y_1' y_2.$$

$$W_{y_1y_2}(t) = \begin{bmatrix} \cos(2t) - 2 \cos^2(t) \\ -2 \sin(2t) + 4 \sin(t) \cos(t) \end{bmatrix} \begin{bmatrix} -2 \sin(2t) + 4 \sin(t) \cos(t) \\ \cos(2t) + 2 \sin^2(t) \end{bmatrix}.$$

$$\sin(2t) = 2 \sin(t) \cos(t) \Rightarrow [-2 \sin(2t) + 4 \sin(t) \cos(t)] [\cos(2t) + 2 \sin^2(t)] = 0.$$

We conclude $W_{y_1y_2}(t) = 0$, so the functions $y_1$ and $y_2$ are li. <\>
The Wronskian of two functions.

Theorem (Variable coefficients)
Let \( y_1 \) and \( y_2 \) be continuously differentiable solutions of

\[
y'' + a_1(t)y' + a_0(t)y = 0, \quad (2)
\]

where \( a_1, a_2 \) are continuous functions. Then, the following statement holds: Every solution \( y \) of Eq. (2) can be decomposed as

\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

for appropriate constants \( c_1, c_2 \) iff functions \( y_1 \) and \( y_2 \) are linearly independent, that is, iff \( W_{y_1y_2} \neq 0 \).

Proof: See the textbook and the lecture notes.

Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- **General and fundamental solutions.**
- Abel’s theorem on the Wronskian.
Remark: The Theorem above justifies the following definitions.

**Definition**
Two solutions \( y_1, y_2 \) of the homogeneous equation

\[
y'' + a_1(t)y' + a_0(t)y = 0,
\]

are called **fundamental solutions** iff the functions \( y_1, y_2 \) are linearly independent, that is, iff \( W_{y_1y_2} \neq 0 \).

**Definition**
Given any two fundamental solutions \( y_1, y_2 \), and arbitrary constants \( c_1, c_2 \), the function

\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

is called the **fundamental solution** of Eq. (2).

**Example**
Show that \( y_1 = \sqrt{t} \) and \( y_2 = 1/t \) are fundamental solutions of

\[
2t^2 \, y'' + 3t \, y' - y = 0.
\]

**Solution:** First show that \( y_1 \) is a solution:

\[
y_1 = t^{1/2}, \quad y_1' = \frac{1}{2} t^{-1/2}, \quad y_1 = -\frac{1}{4} t^{-3/2},
\]

\[
2t^2 \left( -\frac{1}{4} t^{-3/2} \right) + 3t \left( \frac{1}{2} t^{-1/2} \right) - t^{1/2} = -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0.
\]

Now show that \( y_2 \) is a solution:

\[
y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2 = 2t^{-3},
\]

\[
2t^2 \left( 2t^{-3} \right) + 3t \left( -t^{-2} \right) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.
\]
General and fundamental solutions.

Example
Show that \( y_1 = \sqrt{t} \) and \( y_2 = \frac{1}{t} \) are fundamental solutions of

\[
2t^2 y'' + 3t y' - y = 0.
\]

Solution: We show that \( y_1, y_2 \) are linearly independent.

\[
W_{y_1y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix}.
\]

\[
W_{y_1y_2}(t) = -t^{1/2}t^2 - \frac{1}{2}t^{-1/2}t^{-1} = -t^{-3/2} - \frac{1}{2}t^{-3/2}
\]

\[
W_{y_1y_2}(t) = -\frac{3}{3}t^{-3/2} \Rightarrow y_1, y_2 \text{ li.}
\]

Variable coefficients second order linear ODE (Sect. 3.2).

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- **Abel’s theorem on the Wronskian.**
Abel’s theorem on the Wronskian.

**Theorem (Abel)**

If \( a_1, a_0 : (t_1, t_2) \to \mathbb{R} \) are continuous functions and \( y_1, y_2 \) are continuously differentiable solutions of the equation

\[
y'' + a_1(t) y' + a_0(t) y = 0,
\]

then the Wronskian \( W_{y_1y_2} \) is a solution of the equation

\[
W'_{y_1y_2}(t) + a_1(t) W_{y_1y_2}(t) = 0.
\]

Therefore, for any \( t_0 \in (t_1, t_2) \), the Wronskian \( W_{y_1y_2} \) is given by

\[
W_{y_1y_2}(t) = W_{y_1y_2}(t_0) e^{A(t)} \quad A(t) = \int_{t_0}^{t} a_1(s) \, ds.
\]

**Remarks:** One can know the Wronskian of two solutions without having the explicit expression of these solutions.

Abel’s theorem on the Wronskian.

**Example**

Find the Wronskian of two solutions of the equation

\[
t^2 y'' - t(t + 2) y' + (t + 2) y = 0, \quad t > 0.
\]

**Solution:** Write the equation as in Abel’s Theorem,

\[
y'' - \left( \frac{2}{t} + 1 \right) y' + \left( \frac{2}{t^2} + \frac{1}{t} \right) y = 0.
\]

Abel’s Theorem says that the Wronskian satisfies the equation

\[
W'_{y_1y_2}(t) - \left( \frac{2}{t} + 1 \right) W_{y_1y_2}(t) = 0.
\]

This is a first order, linear equation for \( W_{y_1y_2} \). The integrating factor method implies

\[
A(t) = -\int_{t_0}^{t} \left( \frac{2}{s} + 1 \right) \, ds = -2 \ln \left( \frac{t}{t_0} \right) - (t - t_0).
\]
Abel’s theorem on the Wronskian.

Example
Find the Wronskian of two solutions of the equation

\[ t^2 y'' - t(t + 2) y' + (t + 2) y = 0, \quad t > 0. \]

Solution: \( A(t) = -2 \ln\left(\frac{t}{t_0}\right) - (t - t_0) = \ln\left(\frac{t_0}{t^2}\right) - (t - t_0). \)

The integrating factor is \( \mu = \frac{t_0^2}{t^2} e^{-(t-t_0)}. \) Therefore,

\[ \left[ \mu(t) W_{y_1y_2}(t) \right]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1y_2}(t) - \mu(t_0) W_{y_1y_2}(t_0) = 0 \]

so, the solution is \( W_{y_1y_2}(t) = W_{y_1y_2}(t_0) \frac{t_0}{t^2} e^{(t-t_0)}. \)

Denoting \( c = \left( W_{y_1y_2}(t_0)/t_0^2 \right) e^{-t_0}, \) then \( W_{y_1y_2}(t) = c t^2 e^t. \) \( \square \)