Exact equations (Sect. 2.6).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

Exact differential equations.

**Definition**
Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions $M, N : R \to \mathbb{R}$, denoted as $(t, u) \mapsto M(t, u)$ and $(t, u) \mapsto N(t, u)$, the differential equation in the unknown function $y : (t_1, t_2) \to \mathbb{R}$ given by

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is called **exact** iff for every point $(t, u) \in R$ holds

$$\partial_t N(t, u) = \partial_u M(t, u)$$

**Recall:** we use the notation: $\partial_t N = \frac{\partial N}{\partial t}$, and $\partial_u M = \frac{\partial M}{\partial u}$. 
Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ 2ty(t) \, y'(t) + 2t + y^2(t) = 0. \]

Solution: We first identify the functions \( N \) and \( M \),

\[
2ty(t) \, y'(t) + 2t + y^2(t) = 0 \implies \begin{cases} N(t, u) = 2tu, \\ M(t, u) = 2t + u^2. \end{cases}
\]

The equation is exact iff \( \partial_t N = \partial_u M \). Since

\[
N(t, u) = 2tu \implies \partial_t N(t, u) = 2u,
\]

\[
M(t, u) = 2t + u^2 \implies \partial_u M(t, u) = 2u.
\]

We conclude: \( \partial_t N(t, u) = \partial_u M(t, u) \).

Remark: The ODE above is not separable and non-linear.

Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ \sin(t)y'(y) + t^2e^{y(t)}y'(t) - y'(t) = -y(t) \cos(t) - 2te^{y(t)}. \]

Solution: We first identify the functions \( N \) and \( M \), if we write

\[
[\sin(t) + t^2e^{y(t)} - 1] \, y'(t) + [y(t) \cos(t) + 2te^{y(t)}] = 0,
\]

we can see that

\[
N(t, u) = \sin(t) + t^2e^u - 1 \implies \partial_t N(t, u) = \cos(t) + 2te^u,
\]

\[
M(t, u) = u \cos(t) + 2te^u \implies \partial_u M(t, u) = \cos(t) + 2te^u.
\]

The equation is exact, since \( \partial_t N(t, u) = \partial_u M(t, u) \).
Exact differential equations.

Example
Show whether the linear differential equation below is exact,

\[ y'(t) = -a(t) y(t) + b(t), \quad a(t) \neq 0. \]

Solution: We first find the functions \( N \) and \( M \),

\[
y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad \begin{cases} 
N(t, u) = 1, \\
M(t, u) = a(t)u - b(t).
\end{cases}
\]

The differential equation is not exact, since

\[
N(t, u) = 1 \implies \partial_t N(t, u) = 0,
\]

\[
M(t, u) = a(t)u - b(t) \implies \partial_u M(t, u) = a(t).
\]

This implies that \( \partial_t N(t, u) \neq \partial_u M(t, u) \).

Exact equations (Sect. 2.6).

- Exact differential equations.
- **The Poincaré Lemma.**
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The Poincaré Lemma.

**Remark:** The coefficients $N$ and $M$ of an exact equations are the derivatives of a potential function $\psi$.

**Lemma (Poincaré)**

*Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subseteq \mathbb{R}^2$, the continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy the equation*

$$\partial_t N(t, u) = \partial_u M(t, u)$$

*iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, such that for all $(t, u) \in R$ holds*

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$

**Proof:** ($\Leftarrow$) Simple: \[
\begin{align*}
\partial_t N &= \partial_t \partial_u \psi, \\
\partial_u M &= \partial_u \partial_t \psi,
\end{align*}\]

$$\Rightarrow \partial_t N = \partial_u M.$$

($\Rightarrow$) Difficult: Poincaré, 1880.

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The Poincaré Lemma.

**Example**

Show that the function $\psi(t, u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t) y'(t) + 2t + y^2(t) = 0.$$

**Solution:** We already saw that the differential equation above is exact, since the functions $M$ and $N$,

$$\begin{align*}
N(t, u) &= 2tu, \\
M(t, u) &= 2t + u^2
\end{align*}$$

$$\Rightarrow \partial_t N = 2u = \partial_u M.$$

The potential function is $\psi(t, u) = t^2 + tu^2$, since

$$\partial_t \psi = 2t + u^2 = M, \quad \partial_u \psi = 2tu = N.$$

**Remark:** The Poincaré Lemma only states necessary and sufficient conditions on $N$ and $M$ for the existence of $\psi$. 

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Exact equations (Sect. 2.6).

- Exact differential equations.
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Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0 \quad (1)$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (1).

Proof: $0 = N(t, y) y' + M(t, y) = \partial_y \psi(t, y) \frac{dy}{dt} + \partial_t \psi(t, y)$. 

$$0 = \frac{d}{dt} \psi(t, y(t)) \iff \psi(t, y(t)) = c. \qed$$
Implicit solutions and the potential function.

Example
Find all solutions $y$ to the equation

$$[\sin(t) + t^2e^y - 1] y'(t) + y(t) \cos(t) + 2te^y = 0.$$ 

Solution: Recall: The equation is exact,

$$N(t, u) = \sin(t) + t^2e^u - 1 \quad \Rightarrow \quad \partial_t N(t, u) = \cos(t) + 2te^u,$$

$$M(t, u) = u \cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t, u) = \cos(t) + 2te^u,$$

hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists $\psi$,

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$

These are actually equations for $\psi$. From the first one,

$$\psi(t, u) = \int [\sin(t) + t^2e^u - 1] \, du + g(t).$$
Exact equations (Sect. 2.6).

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**Remark:**
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

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**Generalization: The integrating factor method.**

**Theorem (Integrating factor)**

Let $M, N : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, with $N \neq 0$. If the equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, and if the function

$$\frac{1}{N(t, u)} [\partial_u M(t, u) - \partial_t N(t, u)]$$

does not depend on the variable $u$, then the equation

$$\mu(t) [N(t, y(t)) y'(t) + M(t, y(t))] = 0$$

is exact, where $\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t, u)} [\partial_u M(t, u) - \partial_t N(t, u)]$. 
Example
Find all solutions \( y \) to the differential equation
\[
[t^2 + t y(t)] \ y'(t) + [3 t y(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]

hence \( \partial_t N \neq \partial_u M \). We now verify whether the extra condition in Theorem above holds:
\[
\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{(t^2 + tu)} [(3t + 2u) - (2t + u)]
\]
\[
\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t(t + u)} (t + u) = \frac{1}{t}.
\]

We find a function \( \mu \) solution of
\[
\frac{\mu'}{\mu} = \frac{[\partial_u M - \partial_t N]}{N},
\]
that is
\[
\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu(t)) = \ln(t) \quad \Rightarrow \quad \mu(t) = t.
\]

Therefore, the equation below is exact:
\[
[t^3 + t^2 y(t)] \ y'(t) + [3 t^2 y(t) + t y^2(t)] = 0.
\]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[ [t^2 + t y(t)] y'(t) + [3 t y(t) + y^2(t)] = 0. \]
Solution: $[t^3 + t^2 y(t)] y'(t) + [3 t^2 y(t) + t y^2(t)] = 0.$

This equation is exact:
\[
\tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,
\]
\[
\tilde{M}(t, u) = 3t^2 u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,
\]
that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists $\psi$ such that
\[
\partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u).
\]
From the first equation above we obtain
\[
\partial_u \psi = t^3 + t^2 u \quad \Rightarrow \quad \psi(t, u) = \int (t^3 + t^2 u) \, du + g(t).
\]

Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[ [t^2 + t y(t)] y'(t) + [3 t y(t) + y^2(t)] = 0. \]
Solution: $\psi(t, u) = \int (t^3 + t^2 u) \, du + g(t)$.

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,
\[
\partial_t \psi(t, u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 u + tu^2,
\]
So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a potential function is $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2$.

And every solution $y$ satisfies $t^3 y(t) + \frac{1}{2} t^2 [y(t)]^2 = c$. \(\triangledown\)