Modeling with first order equations (Sect. 2.3).

- The mathematical modeling of natural processes.
- Main example: Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.

The mathematical modeling of natural processes.

Remarks:
- Physics describes natural processes with mathematical constructions, called physical theories.
- More often than not these physical theories contain differential equations.
- Natural processes are described through solutions of differential equations.
- Usually a physical theory, constructed to describe all known natural processes, predicts yet unknown natural processes.
- If the prediction is verified by an experiment or observation, one says that we have unveiled a secret from nature.
Salt in a water tank.

Problem: Study the mass conservation law.

Particular situation: Salt concentration in water.

Main ideas of the test:
- Assuming the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- We study the predictions of this mathematical description.
- If the description agrees with the observation of the natural process, then we conclude that the conservation of mass law holds for salt in water.

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The experimental device.

Definitions:

- \( r_i(t) \), \( r_o(t) \): Rates in and out of water entering and leaving the tank at the time \( t \).
- \( q_i(t) \), \( q_o(t) \): Salt concentration of the water entering and leaving the tank at the time \( t \).
- \( V(t) \): Water volume in the tank at the time \( t \).
- \( Q(t) \): Salt mass in the tank at the time \( t \).

Units:

\[
[r_i(t)] = [r_o(t)] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i(t)] = [q_o(t)] = \frac{\text{Mass}}{\text{Volume}}.
\]

\[
[V(t)] = \text{Volume}, \quad [Q(t)] = \text{Mass}.
\]
The main equations.

Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

\[ \frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \tag{1} \]

\[ \frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \tag{2} \]

\[ q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \tag{3} \]

\[ r_i, \ r_o : \ \text{Constants.} \tag{4} \]
The main equations.

Remarks:
\[
\frac{dV}{dt} = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o],
\]
\[
\frac{dQ}{dt} = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o],
\]
\[
[r_i q_i - r_o q_o] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.
\]

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Analysis of the mathematical model.

Eqs. (4) and (1) imply

\[ V(t) = (r_i - r_o) t + V_0, \]

(5)

where \( V(0) = V_0 \) is the initial volume of water in the tank.

Eqs. (3) and (2) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \]

(6)

Eqs. (5) and (6) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

(7)

Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \)

Notation: \( a(t) = \frac{r_o}{(r_i - r_o) t + V_0}, \) and \( b(t) = r_i q_i(t). \)

The main equation of the description is given by

\[ Q'(t) = -a(t) Q(t) + b(t). \]

Linear ODE for \( Q. \) Solution: Integrating factor method.

\[ Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) \, ds \right] \]

with \( Q(0) = Q_0, \) where \( \mu(t) = e^{A(t)} \) and \( A(t) = \int_0^t a(s) \, ds. \)
The mathematical modeling of natural processes.

Main example: Salt in a water tank.

The experimental device.

The main equations.

Analysis of the mathematical model.

Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = -a(t)Q(t) + b(t) \).
In this case:
\[
a(t) = \frac{r_o}{(r_i - r_o)t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,
\]
\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = rq_i = b_0.
\]
We need to solve the IVP:
\[
Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) = -a_0 Q + b_0 \), \( Q(0) = Q_0 \).

Integrating factor method:
\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 \, ds \right].
\]
\[
\int_0^t \mu(s) b_0 \, ds = \frac{b_0}{a_0} \left( e^{a_0 t} - 1 \right) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} \left( e^{a_0 t} - 1 \right) \right].
\]

So: \( Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0} \). But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0 \).

We conclude: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \).

Particular cases:
\[
\begin{align*}
\( Q_0 \) \frac{V_0}{V_0} > q_i; \\
\( Q_0 \) \frac{V_0}{V_0} = q_i, \text{ so } Q(t) = Q_0; \\
\( Q_0 \) \frac{V_0}{V_0} < q_i.
\end{align*}
\]

Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \).

Particular cases:
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\end{align*}
\]
Predictions for particular situations.

**Example**
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

**Solution:** This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,

\[
\frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \Rightarrow e^{-rt_1/V_0} = \frac{1}{100}.
\]

Predictions for particular situations.

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find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

**Solution:** Recall: \( e^{-rt_1/V_0} = \frac{1}{100} \). Then,

\[
-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100) \Rightarrow \frac{r}{V_0} t_1 = \ln(100).
\]

We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \).

In this case: \( t_1 = 100 \ln(100) \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) are constants. If \( r = 5 \times 10^6 \) gal/year, \( q_i(t) = 2 + \sin(2t) \) grams/gal, \( V_0 = 10^6 \) gal, \( Q_0 = 0 \), find \( Q(t) \).

Solution: Recall: \( Q'(t) = -a(t)\,Q(t) + b(t) \). In this case:

\[
 a(t) = \frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,
\]

\[
 b(t) = r_i\,q_i(t) \quad \Rightarrow \quad b(t) = r[2 + \sin(2t)].
\]

We need to solve the IVP: \( Q'(t) = -a_0\,Q(t) + b(t), \; Q(0) = 0. \)

\[
 Q(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)\,b(s)\,ds, \quad \mu(t) = e^{a_0 t},
\]

We conclude: \( Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] \, ds. \)