Green’s Theorem on a plane. (Sect. 16.4)

- Review of Green’s Theorem on a plane.
- Sketch of the proof of Green’s Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.
Review: Green’s Theorem on a plane

Theorem

Given a field \( \mathbf{F} = \langle F_x, F_y \rangle \) and a loop \( C \) enclosing a region \( R \subseteq \mathbb{R}^2 \) described by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \), with unit tangent vector \( \mathbf{u} \) and exterior normal vector \( \mathbf{n} \), then holds:

▸ The counterclockwise line integral \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds \) satisfies:

\[
\int_{t_0}^{t_1} [F_x(t)x'(t) + F_y(t)y'(t)] \, dt = \int\int_R (\partial_x F_y - \partial_y F_x) \, dx \, dy.
\]

▸ The counterclockwise line integral \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) satisfies:

\[
\int_{t_0}^{t_1} [F_x(t)y'(t) - F_y(t)x'(t)] \, dt = \int\int_R (\partial_x F_x + \partial_y F_y) \, dx \, dy.
\]
Review: Green’s Theorem on a plane

Circulation-tangential form:
\[ \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy. \]

Flux-normal form:
\[ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy. \]

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.
Green’s Theorem on a plane. (Sect. 16.4)

- Review of Green’s Theorem on a plane.
- **Sketch of the proof of Green’s Theorem.**
- Divergence and curl of a function on a plane.
- Area computed with a line integral.
Sketch of the proof of Green’s Theorem

We want to prove that for every differentiable vector field \( \mathbf{F} = \langle F_x, F_y \rangle \) the Green Theorem in tangential form holds,

\[
\int_C \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] dt = \int_R \left( \partial_x F_y - \partial_y F_x \right) dx dy.
\]
Sketch of the proof of Green’s Theorem

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] \, dt = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy.$$

We only consider a simple domain like the one in the pictures.
Sketch of the proof of Green’s Theorem

We want to prove that for every differentiable vector field \( \mathbf{F} = \langle F_x, F_y \rangle \) the Green Theorem in tangential form holds,

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\]

We only consider a simple domain like the one in the pictures.

Using the picture on the left we show that

\[
\int_C F_x(t) x'(t) dt = \iint_R (-\partial_y F_x) dx dy;
\]
Sketch of the proof of Green’s Theorem

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] dt = \iint_R \left( \partial_x F_y - \partial_y F_x \right) dx \, dy.$$

We only consider a simple domain like the one in the pictures.

Using the picture on the left we show that

$$\int_C F_x(t) x'(t) \, dt = \iint_R (- \partial_y F_x) \, dx \, dy;$$

and using the picture on the right we show that

$$\int_C F_y(t) y'(t) \, dt = \iint_R (\partial_x F_y) \, dx \, dy.$$
Sketch of the proof of Green’s Theorem

Show that for \( F_x(t) = F_x(x(t), y(t)) \) holds

\[
\int_C F_x(t) x'(t) \, dt = \iint_R (-\partial_y F_x) \, dx \, dy;
\]

where

- The path \( C \) can be described by the curves \( r_0 \) and \( r_1 \) given by
  
  \[
  r_0(t) = \langle t, g_0(t) \rangle, \quad t \in [x_0, x_1]
  \]
  
  \[
  r_1(t) = \langle x_1 + x_0 - t, g_1(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1]
  \]

- Therefore, \( r'_0(t) = \langle 1, g'_0(t) \rangle, \quad t \in [x_0, x_1] \)
  
  \[
  r'_1(t) = \langle -1, -g'_1(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1]
  \]

- Recall: \( F_x(t) = F_x(x(t), y(t)) \) on \( r_0 \), and \( F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \) on \( r_1 \).
Sketch of the proof of Green’s Theorem

Show that for \( F_x(t) = F_x(x(t), y(t)) \) holds

\[
\int_C F_x(t) x'(t) \, dt = \iint_R (- \partial_y F_x) \, dx \, dy;
\]

The path \( C \) can be described by the curves \( r_0 \) and \( r_1 \) given by

\[
\begin{align*}
    r_0(t) &= \langle t, g_0(t) \rangle, \\
    r_1(t) &= \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle
\end{align*}
\]

\( t \in [x_0, x_1] \)
Sketch of the proof of Green’s Theorem

Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t) x'(t) \, dt = \iint_R \left( -\frac{\partial y}{\partial x} F_x \right) \, dx \, dy;$$

The path $C$ can be described by the curves $r_0$ and $r_1$ given by

$$r_0(t) = \langle t, g_0(t) \rangle, \quad t \in [x_0, x_1]$$
$$r_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1].$$

Therefore,

$$r_0'(t) = \langle 1, g_0'(t) \rangle, \quad t \in [x_0, x_1]$$
$$r_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1].$$
Sketch of the proof of Green’s Theorem

Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$
\int_C F_x(t) x'(t) \, dt = \iint_R (-\partial_y F_x) \, dx \, dy;
$$

The path $C$ can be described by the curves $r_0$ and $r_1$ given by

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  r_0(t) &= \langle t, g_0(t) \rangle, & t \in [x_0, x_1] \\
  r_1(t) &= \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle, & t \in [x_0, x_1].
\end{align*}
$$

Therefore,

$$
\begin{align*}
  r'_0(t) &= \langle 1, g'_0(t) \rangle, & t \in [x_0, x_1] \\
  r'_1(t) &= \langle -1, -g'_1(x_1 + x_0 - t) \rangle, & t \in [x_0, x_1].
\end{align*}
$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $r_0$. 

Sketch of the proof of Green’s Theorem

Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t) x'(t) dt = \iint_R (-\partial_y F_x) \, dx \, dy;$$

The path $C$ can be described by the curves $r_0$ and $r_1$ given by

$$r_0(t) = \langle t, g_0(t) \rangle, \quad t \in [x_0, x_1]$$
$$r_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1].$$

Therefore,

$$r_0'(t) = \langle 1, g_0'(t) \rangle, \quad t \in [x_0, x_1]$$
$$r_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t) \rangle, \quad t \in [x_0, x_1].$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $r_0,$

and $F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))$ on $r_1.$
Sketch of the proof of Green’s Theorem

\[ \int_C F_x(t)x'(t) \, dt = \int_{x_0}^{x_1} F_x(t, g_0(t)) \, dt \]

\[ - \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \, dt \]
Sketch of the proof of Green’s Theorem

\[ \int_C F_x(t)x'(t) \, dt = \int_{x_0}^{x_1} F_x(t, g_0(t)) \, dt \]

\[ - \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \, dt \]

Substitution in the second term: \( \tau = x_1 + x_0 - t \), so \( d\tau = -dt \).

\[ - \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \, dt = \]

\[ - \int_{x_1}^{x_0} F_x(\tau, g_1(\tau)) (-d\tau) \]
Sketch of the proof of Green’s Theorem

\[ \int_C F_x(t)x'(t) \, dt = \int_{x_0}^{x_1} F_x(t, g_0(t)) \, dt \]

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Sketch of the proof of Green’s Theorem

\[ \int_C F_x(t)x'(t) \, dt = \int_{x_0}^{x_1} F_x(t, g_0(t)) \, dt \]

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Therefore, \( \int_C F_x(t)x'(t) \, dt = \int_{x_0}^{x_1} \left[ F_x(t, g_0(t)) - F_x(t, g_1(t)) \right] \, dt. \)
Sketch of the proof of Green’s Theorem

\[
\int_C F_x(t) x'(t) \, dt = \int_{x_0}^{x_1} F_x(t, g_0(t)) \, dt \\
- \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \, dt
\]

Substitution in the second term: \( \tau = x_1 + x_0 - t \), so \( d\tau = -dt \).

\[
- \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t)) \, dt = \\
- \int_{x_1}^{x_0} F_x(\tau, g_1(\tau)) (-d\tau) = - \int_{x_0}^{x_1} F_x(\tau, g_1(\tau)) \, d\tau.
\]

Therefore,

\[
\int_C F_x(t) x'(t) \, dt = \int_{x_0}^{x_1} \left[ F_x(t, g_0(t)) - F_x(t, g_1(t)) \right] \, dt.
\]

We obtain:

\[
\int_C F_x(t) x'(t) \, dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} [-\partial_y F_x(t, y)] \, dy \, dt.
\]
Sketch of the proof of Green’s Theorem

Recall: \( \int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \left[-\partial_y F_x(t, y)\right]\,dy\,dt. \)
Sketch of the proof of Green’s Theorem

Recall: \[ \int_C F_x(t)x'(t)\, dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \left[ -\partial_y F_x(t, y) \right] \, dy \, dt. \]

This result is precisely what we wanted to prove:

\[ \int_C F_x(t)x'(t)\, dt = \iint_R (\partial_y F_x) \, dy \, dx. \]
Sketch of the proof of Green’s Theorem

Recall: \[ \int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} [-\partial_y F_x(t, y)] \,dy \,dt. \]

This result is precisely what we wanted to prove:

\[ \int_C F_x(t)x'(t)\,dt = \iint_R (-\partial_y F_x) \,dy \,dx. \]

We just mention that the result

\[ \int_C F_y(t)y'(t)\,dt = \iint_R (\partial_x F_y) \,dx \,dy. \]

is proven in a similar way using the parametrization of the \( C \) given in the picture.
Green’s Theorem on a plane. (Sect. 16.4)

- Review of Green’s Theorem on a plane.
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- **Divergence and curl of a function on a plane.**
- Area computed with a line integral.
Divergence and curl of a function on a plane

Definition

The \textit{curl} of a vector field \( \mathbf{F} = \langle F_x, F_y \rangle \) in \( \mathbb{R}^2 \) is the scalar

\[
(curl \, \mathbf{F})_z = \partial_x F_y - \partial_y F_x.
\]

The \textit{divergence} of a vector field \( \mathbf{F} = \langle F_x, F_y \rangle \) in \( \mathbb{R}^2 \) is the scalar

\[
\text{div} \, \mathbf{F} = \partial_x F_x + \partial_y F_y.
\]

Remark:
Both forms of Green's Theorem can be written as:

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (curl \, \mathbf{F}) \, dz \, dx \, dy.
\]

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \text{div} \, \mathbf{F} \, dx \, dy.
\]
Divergence and curl of a function on a plane

**Definition**
The *curl* of a vector field \( \mathbf{F} = \langle F_x, F_y \rangle \) in \( \mathbb{R}^2 \) is the scalar

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\text{div} \, \mathbf{F} = \partial_x F_x + \partial_y F_y.
\]

**Remark:** Both forms of Green’s Theorem can be written as:

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\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \iint_R (\text{curl} \, \mathbf{F})_z \, dx \, dy.
\]

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \text{div} \, \mathbf{F} \, dA.
\]
Divergence and curl of a function on a plane

Remark: What type of information about $\mathbf{F}$ is given in $(\text{curl } \mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and $\mathbf{F} = \langle -y, x \rangle$. Then $(\text{curl } \mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2$.

If we place a small ball at $(0, 0)$, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

Remark: The curl of a field measures its rotation.
Divergence and curl of a function on a plane

**Remark:** What type of information about $\mathbf{F}$ is given in $(\text{curl } \mathbf{F})_z$?

**Example:** Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

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If we place a small ball at $(0, 0)$, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

If we place a small ball at everywhere in the plane, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

**Remark:** The curl of a field measures its rotation.
Divergence and curl of a function on a plane

**Remark:** What type of information about \( \mathbf{F} \) is given in \( (\text{curl} \ \mathbf{F})_z \)?

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If we place a small ball at \((0, 0)\), the ball will spin around the \( z \)-axis with speed proportional to \((\text{curl } \mathbf{F})_z\).
Divergence and curl of a function on a plane

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If we place a small ball at $(0, 0)$, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

If we place a small ball at everywhere in the plane, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$. 
Divergence and curl of a function on a plane

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If we place a small ball at $(0,0)$, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

If we place a small ball at everywhere in the plane, the ball will spin around the $z$-axis with speed proportional to $(\text{curl } \mathbf{F})_z$.

Remark: The curl of a field measures its rotation.
Divergence and curl of a function on a plane

**Remark:** What type of information about $\mathbf{F}$ is given in $\nabla \cdot \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and $\mathbf{F} = \langle x, y \rangle \Rightarrow \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y = 2$.

The field $\mathbf{F}$ represents the gas as it is heated with a heat source at $(0, 0)$. The heated gas expands radially outward from $(0, 0)$. The $\nabla \cdot \mathbf{F}$ measures that expansion.

**Remark:** The divergence of a field measures its expansion.

Remarks:
▶ Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $(\nabla \times \mathbf{F})_z = 0$.
▶ Notice that for $\mathbf{F} = \langle -y, x \rangle$ we have $\nabla \cdot \mathbf{F} = 0$. 


Divergence and curl of a function on a plane

**Remark:** What type of information about $\mathbf{F}$ is given in $\text{div} \, \mathbf{F}$?

**Example:** Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle$$
Divergence and curl of a function on a plane

**Remark:** What type of information about $F$ is given in $\text{div } F$?

**Example:** Suppose $F$ is the velocity field of a gas and

$$F = \langle x, y \rangle \Rightarrow \text{div } F = \partial_x F_x + \partial_y F_y = 2.$$
Divergence and curl of a function on a plane

Remark: What type of information about $\mathbf{F}$ is given in $\text{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \implies \text{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$
Divergence and curl of a function on a plane

**Remark:** What type of information about \( F \) is given in \( \text{div} \ F \)?

**Example:** Suppose \( F \) is the velocity field of a gas and

\[
F = \langle x, y \rangle \quad \Rightarrow \quad \text{div} \ F = \partial_x F_x + \partial_y F_y = 2.
\]

The field \( F \) represents the gas as is heated with a heat source at \((0, 0)\). The heated gas expands in all directions, radially out from \((0, 0)\). The \( \text{div} \ F \) measures that expansion.
Divergence and curl of a function on a plane

Remark: What type of information about $\mathbf{F}$ is given in $\text{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \text{div} \mathbf{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y = 2.$$ 

The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out from $(0,0)$. The $\text{div} \mathbf{F}$ measures that expansion.

Remark: The divergence of a field measures its expansion.
Divergence and curl of a function on a plane

**Remark:** What type of information about $F$ is given in $\text{div} \ F$?

**Example:** Suppose $F$ is the velocity field of a gas and

$$F = \langle x, y \rangle \implies \text{div} \ F = \partial_x F_x + \partial_y F_y = 2.$$

The field $F$ represents the gas as is heated with a heat source at $(0, 0)$. The heated gas expands in all directions, radially out form $(0, 0)$. The $\text{div} \ F$ measures that expansion.

**Remark:** The **divergence** of a field measures its expansion.

**Remarks:**

- Notice that for $F = \langle x, y \rangle$ we have $(\text{curl} \ F)_z = 0$. 

...
Divergence and curl of a function on a plane

**Remark:** What type of information about $\mathbf{F}$ is given in $\text{div} \, \mathbf{F}$?

**Example:** Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \implies \text{div} \, \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$ 

The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out from $(0,0)$. The $\text{div} \, \mathbf{F}$ measures that expansion.

**Remark:** The divergence of a field measures its expansion.

**Remarks:**

- Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $(\text{curl} \, \mathbf{F})_z = 0$.
- Notice that for $\mathbf{F} = \langle -y, x \rangle$ we have $\text{div} \, \mathbf{F} = 0$. 
Green’s Theorem on a plane. (Sect. 16.4)

- Review of Green’s Theorem on a plane.
- Sketch of the proof of Green’s Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.
Area computed with a line integral

Remark: Any of the two versions of Green’s Theorem can be used to compute areas using a line integral.
Area computed with a line integral

Remark: Any of the two versions of Green’s Theorem can be used to compute areas using a line integral. For example:

\[
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Example

Use Green’s Theorem to find the area of the region enclosed by the ellipse \( \mathbf{r}(t) = \langle a \cos(t), b \sin(t) \rangle \), with \( t \in [0, 2\pi] \) and \( a, b \) positive.
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Solution: We use: \( A(R) = \oint_C x \, dy \).
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Since \( \int_0^{2\pi} \cos(2t) \, dt = 0 \), we obtain \( A(R) = \frac{ab}{2} 2\pi \).
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Example

Use Green’s Theorem to find the area of the region enclosed by the ellipse \( r(t) = \langle a \cos(t), b \sin(t) \rangle \), with \( t \in [0, 2\pi] \) and \( a, b \) positive.

Solution: We use: \( A(R) = \oint_C x \, dy \).

We need to compute \( r'(t) = \langle -a \sin(t), b \cos(t) \rangle \). Then,

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A(R) = \int_0^{2\pi} x(t) y'(t) \, dt = \int_0^{2\pi} a \cos(t) b \cos(t) \, dt.
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Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- The area of a surface in space.
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Review: Arc length and line integrals

- The integral of a function $f : [a, b] \rightarrow \mathbb{R}$ is
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  \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^n f(x_i^*) \Delta x.
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The flux of a function $\mathbf{F} : \{z = 0\} \cap \mathbb{R}^3 \to \{z = 0\} \cap \mathbb{R}^3$ along a loop $\mathbf{r} : [t_0, t_1] \to \{z = 0\} \cap \mathbb{R}^3$ is

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\iint_{R} f \, dA = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y.
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We will show how to compute:
- The area of a non-flat surface in space. (Today.)
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Two vectors tangent to the surface are $\frac{\partial}{\partial u} r(u, v) = \langle \frac{\partial}{\partial u} x(u, v), \frac{\partial}{\partial u} y(u, v), \frac{\partial}{\partial u} z(u, v) \rangle$, $\frac{\partial}{\partial v} r(u, v) = \langle \frac{\partial}{\partial v} x(u, v), \frac{\partial}{\partial v} y(u, v), \frac{\partial}{\partial v} z(u, v) \rangle$. 
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$$
Explicit, implicit, parametric equations of surfaces

Example

Find a parametric expression for the cone \( z = \sqrt{x^2 + y^2} \), and two tangent vectors.

Solution:

Use cylindrical coordinates:

\[
x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z.
\]

Parameters of the surface:

\[
u = r, \quad v = \theta.
\]

Then

\[
x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta), \quad z(r, \theta) = r.
\]

Using vector notation, a parametric equation of the cone is

\[
\mathbf{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r \rangle.
\]

Two tangent vectors to the cone are

\[
\frac{\partial}{\partial r} \mathbf{r} = \langle \cos(\theta), \sin(\theta), 1 \rangle,
\]

\[
\frac{\partial}{\partial \theta} \mathbf{r} = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle.
\]

\fi
Explicit, implicit, parametric equations of surfaces

Example
Find a parametric expression for the cone \( z = \sqrt{x^2 + y^2} \), and two tangent vectors.

Solution: Use cylindrical coordinates:
Example
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Solution: Use cylindrical coordinates: \( x = r \cos(\theta), \ y = r \sin(\theta), \ z = z \). Parameters of the surface: \( u = r, \ v = \theta \).
Explicit, implicit, parametric equations of surfaces

Example
Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

Solution: Use cylindrical coordinates: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $z = z$. Parameters of the surface: $u = r$, $v = \theta$. Then

$$x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta), \quad z(r, \theta) = r.$$
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the cone \( z = \sqrt{x^2 + y^2} \), and two tangent vectors.

Solution: Use cylindrical coordinates: \( x = r \cos(\theta), y = r \sin(\theta), z = z \). Parameters of the surface: \( u = r, v = \theta \). Then

\[
\begin{align*}
  x(r, \theta) &= r \cos(\theta), \\
  y(r, \theta) &= r \sin(\theta), \\
  z(r, \theta) &= r.
\end{align*}
\]

Using vector notation, a parametric equation of the cone is

\[
\mathbf{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r \rangle.
\]
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

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Two tangent vectors to the cone are $\partial_r \mathbf{r}$ and $\partial_\theta \mathbf{r}$,
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the cone \( z = \sqrt{x^2 + y^2} \), and two tangent vectors.

Solution: Use cylindrical coordinates: \( x = r \cos(\theta) \), \( y = r \sin(\theta) \), \( z = z \). Parameters of the surface: \( u = r \), \( v = \theta \). Then

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Explicit, implicit, parametric equations of surfaces

Example
Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere \( x^2 + y^2 + z^2 = R^2 \), and two tangent vectors.

Solution: Use spherical coordinates:
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere \( x^2 + y^2 + z^2 = R^2 \), and two tangent vectors.

Solution: Use spherical coordinates:
\[
x = \rho \cos(\theta) \sin(\phi), \quad y = \rho \sin(\theta) \sin(\phi), \quad z = \rho \cos(\phi).
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Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.

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$x = \rho \cos(\theta) \sin(\phi)$, $y = \rho \sin(\theta) \sin(\phi)$, $z = \rho \cos(\phi)$.
Parameters of the surface: $u = \theta$, $v = \phi$. 

$\begin{align*}
  x &= \rho \cos(\theta) \sin(\phi), \\
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  z &= \rho \cos(\phi).
\end{align*}$
Explicit, implicit, parametric equations of surfaces

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Parameters of the surface: \( u = \theta, \ v = \phi \).

\[
x = R \cos(\theta) \sin(\phi), \quad y = R \sin(\theta) \sin(\phi), \quad z = R \cos(\phi).
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Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere \( x^2 + y^2 + z^2 = R^2 \), and two tangent vectors.

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Explicit, implicit, parametric equations of surfaces

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$r(\theta, \phi) = R \langle \cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi) \rangle$.

Two tangent vectors to the paraboloid are $\partial_\theta r$ and $\partial_\phi r$. 
Explicit, implicit, parametric equations of surfaces

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Two tangent vectors to the paraboloid are \(\partial_\theta \mathbf{r}\) and \(\partial_\phi \mathbf{r}\),
\[ \partial_\theta \mathbf{r} = R \langle -\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0 \rangle, \]
Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere \( x^2 + y^2 + z^2 = R^2 \), and two tangent vectors.

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\]
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\]
Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- **The area of a surface in space.**
  - The surface is given in parametric form.
  - The surface is given in explicit form.
The area of a surface in parametric form

Theorem

Given a smooth surface \( S \) with parametric equation
\[
\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle
\]
for \( u \in [u_0, u_1] \) and \( v \in [v_0, v_1] \)
is given by
\[
A(S) = \int_{u_0}^{u_1} \int_{v_0}^{v_1} |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.
\]
The area of a surface in parametric form

**Theorem**

*Given a smooth surface* $S$ *with parametric equation* $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ *for* $u \in [u_0, u_1]$ *and* $v \in [v_0, v_1]$ *is given by*

$$A(S) = \int_{u_0}^{u_1} \int_{v_0}^{v_1} |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.$$

**Remark:**

The function $d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du$ represents the area of a small region on the surface. This is the generalization to surfaces of the arc-length formula for the length of a curve.
The area of a surface in parametric form

**Theorem**

*Given a smooth surface $S$ with parametric equation $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for $u \in [u_0, u_1]$ and $v \in [v_0, v_1]$ is given by*

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**Remark:** The function

$$d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.$$ 

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The area of a surface in parametric form

**Theorem**

*Given a smooth surface* $S$ *with parametric equation*

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**Remark:** The function

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represents the area of a small region on the surface.

This is the generalization to surfaces of the arc-length formula for the length of a curve.
The area of a surface in parametric form

Example
Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).
The area of a surface in parametric form

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Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).

Solution: Use cylindrical coordinates.
The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Use cylindrical coordinates. The surface in parametric form is

$$\mathbf{r}(r, \theta) = \langle r \cos(\theta), \ r \sin(\theta), \ r^2 \rangle.$$
The area of a surface in parametric form

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The tangent vectors to the surface \( \partial_r \mathbf{r}, \ \partial_\theta \mathbf{r} \) are
The area of a surface in parametric form

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\[
\partial_r r = \langle \cos(\theta), \sin(\theta), 2r \rangle,
\]
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Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

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$$\partial_r r = \langle \cos(\theta), \sin(\theta), \ 2r \rangle, \quad \partial_\theta r = \langle -r \sin(\theta), \ r \cos(\theta), \ 0 \rangle.$$
The area of a surface in parametric form

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\]

\[
\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos(\theta) & \sin(\theta) & 2r \\
-r \sin(\theta) & r \cos(\theta) & 0
\end{vmatrix}
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\]
\[
\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2 \cos(\theta), \ -2r^2 \sin(\theta), \ r \rangle.
\]
The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).

Solution: Recall: \( \partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle \).
The area of a surface in parametric form

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Solution: Recall: \( \partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle \).

\[
| \partial_r \mathbf{r} \times \partial_\theta \mathbf{r} | = \sqrt{4r^4 + r^2}
\]

This integral will be done later on by substitution.

The result is:

\[
A(S) = \frac{\pi}{6} \left[ \frac{17}{3} - 1 \right]
\]
The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 0 \) and \( z = 4 \).

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|\partial_r \mathbf{r} \times \partial_\theta \mathbf{r}| = \sqrt{4r^4 + r^2} = r\sqrt{1 + 4r^2}.
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A(S) = \int_0^{2\pi} \int_0^2 r \sqrt{1 + 4r^2} \, dr \, d\theta.
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Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- **The area of a surface in space.**
  - The surface is given in parametric form.
  - **The surface is given in explicit form.**
The area of a surface in space in explicit form

Theorem

Given a smooth function \( f : \mathbb{R}^3 \to \mathbb{R} \), the area of a level surface \( S = \{ f(x, y, z) = 0 \} \), over a closed, bounded region \( R \) in the plane \( \{ z = 0 \} \), is given by

\[
A(S) = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
\]

Remark: Eq. (7), page 949, in the textbook is more general than the equation above, since the region \( R \) can be located on any plane, not only the plane \( \{ z = 0 \} \) considered here. The vector \( p \) in the textbook is the vector normal to \( R \). In our case \( p = k \).
The area of a surface in space in explicit form

**Theorem**

Given a smooth function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, the area of a level surface $S = \{ f(x, y, z) = 0 \}$, over a closed, bounded region $R$ in the plane $\{ z = 0 \}$, is given by

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The vector $\mathbf{p}$ in the textbook is the vector normal to $R$. In our case $\mathbf{p} = \mathbf{k}$. 
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{z = 0\}$, is given by

$$ A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA. $$

Example

Find the area of $S = \{ z - 1 = 0 \}$ over $R$ in $\{z = 0\}$.

Solution:

This is simple: $f(x, y, z) = z - 1$, so $\nabla f = k$, hence $|\nabla f| |\nabla f \cdot k| = 1 \Rightarrow A(S) = \iint_{R} \, dA.

Remark:

The formula for $A(S)$ is reasonable: every flat horizontal surface $S$ over a flat horizontal region $R$ satisfies $A(S) = A(R)$. 

\[\text{\triangleright}\]
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

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The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat region $R$ in $\{z = 0\}$, is given by

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Example
Find the area of $S = \{z - 1 = 0\}$ over $R$ in $\{z = 0\}$.

Solution: This is simple: $f(x, y, z) = z - 1$, 

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat region $R$ in $\{z = 0\}$, is given by

$$A(S) = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$  

Example
Find the area of $S = \{z - 1 = 0\}$ over $R$ in $\{z = 0\}$.

Solution: This is simple: $f(x, y, z) = z - 1$, so $\nabla f = \mathbf{k}$,
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{f(x, y, z) = 0\} \) over a flat region \( R \) in \( \{z = 0\} \), is given by

\[
A(S) = \iiint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
\]

Example

Find the area of \( S = \{z - 1 = 0\} \) over \( R \) in \( \{z = 0\} \).

Solution: This is simple: \( f(x, y, z) = z - 1 \), so \( \nabla f = k \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot k|} = 1
\]
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int_\mathcal{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.
\]

Example
Find the area of \( S = \{ z = 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: This is simple: \( f(x, y, z) = z - 1 \), so \( \nabla f = \mathbf{k} \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = 1 \quad \Rightarrow \quad A(S) = \int_\mathcal{R} dA
\]
The area of a surface in space in explicit form

**Recall:** The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
\]

**Example**

Find the area of \( S = \{ z = 1 \} \) over \( R \) in \( \{ z = 0 \} \).

**Solution:** This is simple: \( f(x, y, z) = z - 1 \), so \( \nabla f = k \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot k|} = 1 \quad \Rightarrow \quad A(S) = \iint_R \, dx \, dy = A(R).
\]
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$ 

Example

Find the area of $S = \{ z - 1 = 0 \}$ over $R$ in $\{ z = 0 \}$.

Solution: This is simple: $f(x, y, z) = z - 1$, so $\nabla f = k$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot k|} = 1 \quad \Rightarrow \quad A(S) = \iint_{R} \, dx \, dy = A(R).$$

Remark: The formula for $A(S)$ is reasonable:
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int \int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
\]

Example

Find the area of \( S = \{ z - 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: This is simple: \( f(x, y, z) = z - 1 \), so \( \nabla f = k \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot k|} = 1 \quad \Rightarrow \quad A(S) = \int \int_{R} dx \, dy = A(R).
\]

Remark: The formula for \( A(S) \) is reasonable: Every flat horizontal surface \( S \) over a flat horizontal region \( R \) satisfies \( A(S) = A(R) \).
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int \int_R \frac{\left| \nabla f \right|}{| \nabla f \cdot k |} \, dA.
\]
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$ 

Example
Find the area of $S = \{ y + z - 1 = 0 \}$ over $R$ in $\{ z = 0 \}$. 

Remark: The formula for $A(S)$ is still reasonable: Every flat surface $S$ having an angle $\pi/4$ over a flat horizontal region $R$ satisfies $A(S) = \sqrt{2} A(R)$. 
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$ 

Example
Find the area of $S = \{ y + z - 1 = 0 \}$ over $R$ in $\{ z = 0 \}$.

Solution: The plane $S$ intersects the horizontal plane at a $\pi/4$ angle.
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \int\int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$  

Example

Find the area of $S = \{ y + z - 1 = 0 \}$ over $R$ in $\{ z = 0 \}$.

Solution: The plane $S$ intersects the horizontal plane at a $\pi/4$ angle. So, $f(x, y, z) = y + z - 1$, 

Remark: The formula for $A(S)$ is still reasonable: Every flat surface $S$ having an angle $\pi/4$ over a flat horizontal region $R$ satisfies $A(S) = \sqrt{2} A(R)$. 
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.
\]

Example
Find the area of \( S = \{ y + z - 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: The plane \( S \) intersects the horizontal plane at a \( \pi/4 \) angle. So, \( f(x, y, z) = y + z - 1 \), and \( \nabla f = \mathbf{j} + \mathbf{k} \),

\[
\nabla f = \mathbf{j} + \mathbf{k}.
\]
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \int \int_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.$$  

Example
Find the area of $S = \{ y + z - 1 = 0 \}$ over $R$ in $\{ z = 0 \}$.

Solution: The plane $S$ intersects the horizontal plane at a $\pi/4$ angle. So, $f(x, y, z) = y + z - 1$, and $\nabla f = j + k$, hence

$$\frac{|\nabla f|}{|\nabla f \cdot k|} = \sqrt{2}$$
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \iint_R \frac{\sqrt{\nabla f \cdot k}}{|\nabla f|} \, dA.
\]

Example
Find the area of \( S = \{ y + z - 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: The plane \( S \) intersects the horizontal plane at a \( \pi/4 \) angle. So, \( f(x, y, z) = y + z - 1 \), and \( \nabla f = j + k \), hence

\[
\frac{\sqrt{\nabla f}}{|\nabla f \cdot k|} = \sqrt{2} \Rightarrow A(S) = \iint_R \sqrt{2} \, dx \, dy
\]
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int \int_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA.
\]

Example
Find the area of \( S = \{ y + z − 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: The plane \( S \) intersects the horizontal plane at a \( \pi/4 \) angle. So, \( f(x, y, z) = y + z − 1 \), and \( \nabla f = \mathbf{j} + \mathbf{k} \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2} \Rightarrow A(S) = \int \int_R \sqrt{2} dx \, dy \Rightarrow A(S) = \sqrt{2} A(R).
\]

\( \triangle \)
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.
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Example
Find the area of \( S = \{ y + z - 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: The plane \( S \) intersects the horizontal plane at a \( \pi/4 \) angle. So, \( f(x, y, z) = y + z - 1 \), and \( \nabla f = \mathbf{j} + \mathbf{k} \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2} \Rightarrow A(S) = \iint_R \sqrt{2} \, dx \, dy \Rightarrow A(S) = \sqrt{2} \, A(R).
\]

Remark: The formula for \( A(S) \) is still reasonable:
The area of a surface in space in explicit form

Recall: The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat region \( R \) in \( \{ z = 0 \} \), is given by

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A(S) = \iiint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA.
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Find the area of \( S = \{ y + z - 1 = 0 \} \) over \( R \) in \( \{ z = 0 \} \).

Solution: The plane \( S \) intersects the horizontal plane at a \( \pi/4 \) angle. So, \( f(x, y, z) = y + z - 1 \), and \( \nabla f = \mathbf{j} + \mathbf{k} \), hence

\[
\frac{|\nabla f|}{|\nabla f \cdot k|} = \sqrt{2} \Rightarrow A(S) = \iiint_R \sqrt{2} \, dx \, dy \Rightarrow A(S) = \sqrt{2} \, A(R).
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Remark: The formula for \( A(S) \) is still reasonable: Every flat surface \( S \) having an angle \( \pi/4 \) over a flat horizontal region \( R \) satisfies \( A(S) = \sqrt{2} \, A(R) \).
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat horizontal region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \iint_{R} \left| \nabla f \right| \left| \nabla f \cdot k \right| dA.$$
The area of a surface in space in explicit form

**Recall:** The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat horizontal region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \int \int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} dA.$$  

**Remark:** The formula for $A(S)$ can be interpreted as follows:
The area of a surface in space in explicit form

**Recall:** The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat horizontal region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int\int_{R} \frac{|\nabla f|}{|\nabla f \cdot k|} \ dA.
\]

**Remark:** The formula for \( A(S) \) can be interpreted as follows:

The factor \( \frac{|\nabla f|}{|\nabla f \cdot k|} \) is the angle correction function.
The area of a surface in space in explicit form

**Recall:** The area of a level surface \( S = \{ f(x, y, z) = 0 \} \) over a flat horizontal region \( R \) in \( \{ z = 0 \} \), is given by

\[
A(S) = \int\int_R \frac{\vert \nabla f \vert}{\vert \nabla f \cdot k \vert} \, dA.
\]

**Remark:** The formula for \( A(S) \) can be interpreted as follows: The factor \( \frac{\vert \nabla f \vert}{\vert \nabla f \cdot k \vert} \) is the angle correction function needed to obtain the \( A(S) \) by correcting the \( A(R) \) by the relative inclination of \( S \) with respect to \( R \).
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat horizontal region $R$ in $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{\left| \nabla f \right|}{\left| \nabla f \cdot \mathbf{k} \right|} dA.$$

Remark: The formula for $A(S)$ can be interpreted as follows:
The factor $\frac{\left| \nabla f \right|}{\left| \nabla f \cdot \mathbf{k} \right|}$ is the angle correction function needed to obtain the $A(S)$ by correcting the $A(R)$.
The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{ f(x, y, z) = 0 \}$ over a flat horizontal region $R$ in $\{ z = 0 \}$, is given by

$$A(S) = \int \int_R \frac{\| \nabla f \|}{\| \nabla f \cdot k \|} dA.$$ 

Remark: The formula for $A(S)$ can be interpreted as follows: The factor $\frac{\| \nabla f \|}{\| \nabla f \cdot k \|}$ is the angle correction function needed to obtain the $A(S)$ by correcting the $A(R)$ by the relative inclination of $S$ with respect to $R$. 
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$. 

Recall:
$A(S) = \int \int_R |\nabla f| |\nabla f \cdot k| dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$. 
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution:
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution:

The surface is given by \( f = 0 \) with

\[
f(x, y, z) = x + 2y + 2z - 5.
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution:

The surface is given by $f = 0$ with

$$f(x, y, z) = x + 2y + 2z - 5.$$

The region $R$ is in the plane $z = 0$,

$$R = \left\{ (x, y, z) : z = 0, \ y \in [-1, 1], \ x \in [y^2, (2 - y^2)] \right\}.$$
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution:

The surface is given by $f = 0$ with

$$f(x, y, z) = x + 2y + 2z - 5.$$ 

The region $R$ is in the plane $z = 0$,

$$R = \left\{ (x, y, z) : z = 0, \ y \in [-1, 1] \ \text{and} \ x \in [y^2, (2 - y^2)] \right\}.$$

Recall: $A(S) = \iiint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA$. 
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution:

The surface is given by \( f = 0 \) with
\[
f(x, y, z) = x + 2y + 2z - 5.
\]
The region \( R \) is in the plane \( z = 0 \),
\[
R = \left\{ (x, y, z) : z = 0, \ y \in [-1, 1] \bigg| x \in [y^2, (2 - y^2)] \right\}.
\]
Recall: \( A(S) = \iint_R \frac{\left| \nabla f \right|}{\left| \nabla f \cdot \mathbf{k} \right|} dA \). Here \( \nabla f = \langle 1, 2, 2 \rangle \).
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \iint_{R} \frac{\left| \nabla f \right|}{\left| \nabla f \cdot k \right|} \, dA \). Here \( \nabla f = \langle 1, 2, 2 \rangle \).
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA \). Here \( \nabla f = \langle 1, 2, 2 \rangle \).

Therefore: \( |\nabla f| = \sqrt{1 + 4 + 4} = 3 \), and \( |\nabla f \cdot \mathbf{k}| = 2 \).
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA \). Here \( \nabla f = \langle 1, 2, 2 \rangle \).

Therefore: \( |\nabla f| = \sqrt{1 + 4 + 4} = 3 \), and \( |\nabla f \cdot k| = 2 \).

And the region \( R = \{(x, y) : y \in [-1, 1], \ x \in [y^2, (2 - y^2)]\} \).
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \iint_R \left| \nabla f \right| \left| \nabla f \cdot \mathbf{k} \right| dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$.

Therefore: $|\nabla f| = \sqrt{1 + 4 + 4} = 3$, and $|\nabla f \cdot \mathbf{k}| = 2$.
And the region $R = \{(x, y) : y \in [-1, 1], x \in [y^2, (2 - y^2)]\}$.
So we can write down the expression for $A(S)$ as follows,

$$A(S) = \iint_R \frac{3}{2} \, dx \, dy$$
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$.

Therefore: $|\nabla f| = \sqrt{1 + 4 + 4} = 3$, and $|\nabla f \cdot \mathbf{k}| = 2$.
And the region $R = \{(x, y) : y \in [-1, 1], x \in [y^2, (2 - y^2)]\}$.
So we can write down the expression for $A(S)$ as follows,

$$A(S) = \int\int_R \frac{3}{2} \, dx \, dy = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} \, dx \, dy.$$
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy \).
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \ dy$.

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \ dy
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy \).

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) dy
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy \).

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy
\]
\[
A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy$.

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy
\]

\[
A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy = 3 \left( y - \frac{y^3}{3} \right) \bigg|_{-1}^{1}
\]

$\Rightarrow A(S) = 4$. 
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy. \)

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy
\]

\[
A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy = 3 \left( y - \frac{y^3}{3} \right) \bigg|_{-1}^{1} = 3 \left( 1 - \frac{1}{3} + 1 - \frac{1}{3} \right)
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \( A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy \).

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy
\]

\[
A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy = 3 \left( y - \frac{y^3}{3} \right) \bigg|_{-1}^{1} = 3 \left( 1 - \frac{1}{3} + 1 - \frac{1}{3} \right)
\]

\[
A(S) = 3 \left( 2 - \frac{2}{3} \right)
\]
The area of a surface in space in explicit form

Example
Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy$.

\[
A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy
\]

\[
A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy = 3 \left( y - \frac{y^3}{3} \right) \bigg|_{-1}^{1} = 3 \left( 1 - \frac{1}{3} + 1 - \frac{1}{3} \right)
\]

\[
A(S) = 3 \left( 2 - \frac{2}{3} \right) = 3 \frac{4}{3}
\]
The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane \( x + 2y + 2z = 5 \) by the cylinder with walls \( x = y^2 \) and \( x = 2 - y^2 \).

Solution: \[ A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy. \]

\[ A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2) \, dy \]

\[ A(S) = 3 \int_{-1}^{1} (1 - y^2) \, dy = 3 \left( y - \frac{y^3}{3} \right) \bigg|_{-1}^{1} = 3 \left( 1 - \frac{1}{3} + 1 - \frac{1}{3} \right) \]

\[ A(S) = 3 \left( 2 - \frac{2}{3} \right) = 3 \cdot \frac{4}{3} \quad \Rightarrow \quad A(S) = 4. \]
The area of a surface in space in explicit form

Example

Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$. 

\[
\text{A}(S) = \iint_R \left| \nabla f \right| \left| \nabla f \cdot \mathbf{k} \right| \, dx \, dy,
\]
where

\[
\nabla f = \left\langle 2x, 2y, -1 \right\rangle,
\]

\[
\nabla f \cdot \mathbf{k} = -1,
\]

and

\[
\text{A}(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy.
\]

Since $R$ is a disk radius 2, it is convenient to use polar coordinates in $\mathbb{R}^2$.

We obtain

\[
\text{A}(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r \, dr \, d\theta.
\]
The area of a surface in space in explicit form

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Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: The surface is the level surface of the function $f(x, y, z) = x^2 + y^2 - z$. 
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**Solution:** The surface is the level surface of the function $f(x, y, z) = x^2 + y^2 - z$. The region $R$ is the disk $z = x^2 + y^2 \leq 4$. 

\[ \sqrt{1 + 4x^2 + 4y^2} \]
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A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy,
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Since \( R \) is a disk radius 2, it is convenient to use polar coordinates in \( \mathbb{R}^2 \). We obtain

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A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta.
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$$A(S) = 2\pi \int_0^2 \sqrt{1 + 4r^2} \, r \, dr,$$
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\[
A(S) = 2\pi \int_0^2 \sqrt{1 + 4r^2} \, r \, dr, \quad u = 1 + 4r^2, \quad du = 8r \, dr.
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A(S) = \frac{2\pi}{8} \int_1^{17} u^{1/2} \, du
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\]

\[
A(S) = \frac{2\pi}{8} \int_1^{17} u^{1/2} \, du = \frac{2\pi}{8} \frac{2}{3} \left( u^{3/2} \right|_1^{17} \right).
\]

We conclude:
\[
A(S) = \frac{\pi}{6} \left[ (17)^{3/2} - 1 \right].
\]

\( \triangleright \)
The area of a surface in space in explicit form

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We conclude: $A(S) = \frac{\pi}{6} [(17)^{3/2} - 1]$. ◇
The area of a surface in space in explicit form

**Remark:** The formula for the area of a surface in space can be generalized as follows.
The area of a surface in space in explicit form

Remark: The formula for the area of a surface in space can be generalized as follows.

Theorem

The area of a surface \( S \) given by \( f(x, y, z) = 0 \) over a closed and bounded plane region \( R \) in space is given by

\[
A(S) = \int\int_{R} \frac{|
abla f|}{|\nabla f \cdot p|} \, dA,
\]

where \( p \) is a unit vector normal to the region \( R \) and \( \nabla f \cdot p \neq 0 \).
The area of a surface in space in explicit form

Proof in a simple case: Assume that the surface is given in explicit form:

\[ S = \{(x, y, z) : z = g(x, y)\}, \]

On the one hand, a simple parametric form is to use \( u = x \), \( v = y \) and \( z(u, v) = g(u, v) \).

Hence

\[
\begin{align*}
\frac{\partial}{\partial x} r(x, y) &= \langle 1, 0, \frac{\partial}{\partial x} g \rangle \\
\frac{\partial}{\partial y} r(x, y) &= \langle 0, 1, \frac{\partial}{\partial y} g \rangle \\
\frac{\partial}{\partial x} r(x, y) \times \frac{\partial}{\partial y} r(x, y) &= \langle -\frac{\partial}{\partial x} g, -\frac{\partial}{\partial y} g, 1 \rangle
\end{align*}
\]

On the other hand, an implicit form for the surface is

\[ f(x, y, z) = g(x, y) - z \]

Therefore,

\[ \frac{\partial}{\partial x} f = \frac{\partial}{\partial x} g, \quad \frac{\partial}{\partial y} f = \frac{\partial}{\partial y} g, \quad \frac{\partial}{\partial z} f = -1. \]
The area of a surface in space in explicit form

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On the one hand, a simple parametric form is to use \( u = x, \ v = y \) and \( z(u, v) = g(u, v) \). Hence
\[
r(x, y) = \langle x, y, g(x, y) \rangle
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The area of a surface in space in explicit form

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\[ r(x, y) = \langle x, y, g(x, y) \rangle \quad \Rightarrow \quad \begin{cases} \partial_x r = \langle 1, 0, \partial_x g \rangle \\ \partial_y r = \langle 0, 1, \partial_y g \rangle, \end{cases} \]
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\partial_x \mathbf{r} = \langle 1, 0, \partial_x g \rangle \\
\partial_y \mathbf{r} = \langle 0, 1, \partial_y g \rangle,
\end{cases}
\]

\[
\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle
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f(x, y, z) = g(x, y) - z
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Therefore, \( \partial_x f = \partial_x g, \ \partial_y f = \partial_y g, \ \partial_z f = -1 \).
The area of a surface in space in explicit form

Proof in a simple case: Recall: $\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle$ and

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One can show (with chain rule)
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One can show (with chain rule) that \( \partial_x r \times \partial_y r \)
The area of a surface in space in explicit form

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One can show (with chain rule) that \( \partial_x \mathbf{r} \times \partial_y \mathbf{r} \) is given by

\[
\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \left\langle \frac{\partial_x f}{\partial z f}, \frac{\partial_x f}{\partial z f}, 1 \right\rangle
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$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \left\langle \frac{\partial_x f}{\partial z f}, \frac{\partial_x f}{\partial z f}, 1 \right\rangle \quad \Rightarrow \quad \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{1}{\partial_z f} \langle \partial_x f, \partial_y f, \partial_z f \rangle.$$
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$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \left\langle \frac{\partial_x f}{\partial_z f}, \frac{\partial_x f}{\partial_z f}, 1 \right\rangle \Rightarrow \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{1}{\partial_z f} \langle \partial_x f, \partial_y f, \partial_z f \rangle.$$  

That is, $\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{\nabla f}{\nabla f \cdot \mathbf{k}}.$
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\]

That is, \( \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{\nabla f}{\nabla f \cdot \mathbf{k}} \). We then obtain

\[
A(S) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} |\partial_x \mathbf{r} \times \partial_y \mathbf{r}| \, dy \, dx.
\]
The area of a surface in space in explicit form

Proof in a simple case: Recall: \( \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle \) and \( \partial_x f = \partial_x g, \quad \partial_y f = \partial_y g, \quad \partial_z f = -1. \)

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That is, \( \partial_x \mathbf{r} \times \partial_y \mathbf{r} = \frac{\nabla f}{\nabla f \cdot \mathbf{k}}. \) We then obtain

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A(S) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} |\partial_x \mathbf{r} \times \partial_y \mathbf{r}| \, dy \, dx = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.
\]
The area of a surface in space in explicit form

**Proof:** Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle $\Delta R$ in that partition.
The area of a surface in space in explicit form

**Proof:** Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle $\Delta R$ in that partition. We compute the area $\Delta P$.

![Diagram showing a partition in space with points at $(x_i, y_i, z_i)$ and $(x_i + \Delta x, y_i, \hat{z}_i)$, and vectors $u$, $v$, and $u \times v$.](image)

It is simple to see that $\Delta P = |u \times v|$, and $u = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle$, $v = \langle 0, \Delta y, (z_i - \hat{z}_i) \rangle$.

Therefore, $u \times v = \begin{vmatrix} i & j & k \\ \Delta x & 0 & (z_i - \hat{z}_i) \\ 0 & \Delta y & (z_i - \hat{z}_i) \end{vmatrix} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \hat{z}_i), \Delta x \Delta y \rangle$. 
The area of a surface in space in explicit form

**Proof:** Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle $\Delta R$ in that partition. We compute the area $\Delta P$.

It is simple to see that

$$\Delta P = |\mathbf{u} \times \mathbf{v}|,$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors defining the sides of the rectangle $\Delta R$. The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is given by the magnitude of their cross product.
The area of a surface in space in explicit form

Proof: Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle $\Delta R$ in that partition. We compute the area $\Delta P$.

It is simple to see that

$$\Delta P = |u \times v|,$$

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$$u = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle,$$
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Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle. \)
The area of a surface in space in explicit form

**Proof:** Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \bar{z}_i), \Delta x \Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \( (x_i, y_i, z_i) \) implies

\[
\begin{align*}
f(x, y, z) & \approx f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
\end{align*}
\]
The area of a surface in space in explicit form

Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \left\langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x\Delta y \right\rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies
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f(x, y, z) \approx f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
\]
Since \( f(x_i, y_i, z_i) = 0 \), \( f(x_i + \Delta x, y_i, \hat{z}_i) = 0 \), \( f(x_i, y_i + \Delta y, \bar{z}_i) = 0 \),
The area of a surface in space in explicit form

Proof: Recall: \( u \times v = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle \).

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f(x, y, z) \approx f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
\]

Since \( f(x_i, y_i, z_i) = 0 \), \( f(x_i + \Delta x, y_i, \hat{z}_i) = 0 \), \( f(x_i, y_i + \Delta y, \bar{z}_i) = 0 \),

\[
0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i)
\]
The area of a surface in space in explicit form

Proof: Recall: \( u \times v = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies

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f(x, y, z) \approx f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
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Since \( f(x_i, y_i, z_i) = 0 \), \( f(x_i + \Delta x, y_i, \hat{z}_i) = 0 \), \( f(x_i, y_i + \Delta y, \bar{z}_i) = 0 \),

\[
0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i) \quad \Rightarrow \quad (z_i - \hat{z}_i) = -\frac{(\partial_x f)_i}{(\partial_z f)_i} \Delta x,
\]
The area of a surface in space in explicit form

Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x\Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies

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f(x, y, z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).
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Since \( f(x_i, y_i, z_i) = 0, f(x_i + \Delta x, y_i, \hat{z}_i) = 0, f(x_i, y_i + \Delta y, \bar{z}_i) = 0, \)

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The area of a surface in space in explicit form

**Proof:** Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x\Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies
\[
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Since \( f(x_i, y_i, z_i) = 0, f(x_i + \Delta x, y_i, \hat{z}_i) = 0, f(x_i, y_i + \Delta y, \bar{z}_i) = 0, \)
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\]
The area of a surface in space in explicit form

Proof: Recall: $u \times v = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

The linearization of $f(x, y, z)$ at $(x_i, y_i, z_i)$ implies

$f(x, y, z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i)$. 

Since $f(x_i, y_i, z_i) = 0$, $f(x_i + \Delta x, y_i, \hat{z}_i) = 0$, $f(x_i, y_i + \Delta y, \bar{z}_i) = 0$,

$$0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i) \quad \Rightarrow \quad (z_i - \hat{z}_i) = -\frac{(\partial_x f)_i}{(\partial_z f)_i} \Delta x,$$

$$0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.$$ 

$$u \times v = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x \Delta y}{(\partial_z f)_i}.$$
The area of a surface in space in explicit form

**Proof:** Recall: \( u \times v = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x\Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies

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\]

\[
0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.
\]

\[
u \times v = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x\Delta y}{(\partial_z f)_i} \quad \Rightarrow \quad u \times v = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x\Delta y.
\]
The area of a surface in space in explicit form

Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \((x_i, y_i, z_i)\) implies
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\]
\[
0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.
\]
\[
\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x \Delta y}{(\partial_z f)_i} \Rightarrow \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x \Delta y.
\]
\[
\Delta P = \frac{|(\nabla f)_i|}{|(\nabla f \cdot \mathbf{k})_i|} \Delta x \Delta y.
\]
The area of a surface in space in explicit form

Proof: Recall: \( \mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x\Delta y \rangle \).

The linearization of \( f(x, y, z) \) at \( (x_i, y_i, z_i) \) implies

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Since \( f(x_i, y_i, z_i) = 0, f(x_i + \Delta x, y_i, \hat{z}_i) = 0, f(x_i, y_i + \Delta y, \bar{z}_i) = 0, \)

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\]

\[
0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.
\]

\[
\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \Delta x\Delta y \quad \Rightarrow \quad \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x\Delta y.
\]

\[
\Delta P = \frac{|(\nabla f)_i|}{|(\nabla f \cdot \mathbf{k})_i|} \Delta x\Delta y \quad \Rightarrow \quad A(S) = \iint_{R} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA. \quad \square
\]