Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

Review: Green's Theorem on a plane

Theorem

Given a field $\mathbf{F} = \langle F_x, F_y \rangle$ *and a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *described by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$*, with unit tangent vector* $\mathbf{u}$ *and exterior normal vector* $\mathbf{n}$*, then holds:*

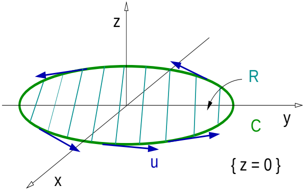
- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *satisfies:*

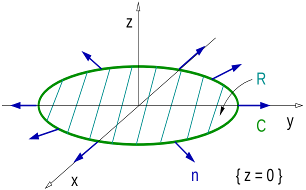
$$\int_{t_0}^{t_1} \left[F_x(t)\, x'(t) + F_y(t)\, y'(t) \right] dt = \iint_R \left(\partial_x F_y - \partial_y F_x \right) dx\, dy.$$

- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *satisfies:*

$$\int_{t_0}^{t_1} \left[F_x(t)\, y'(t) - F_y(t)\, x'(t) \right] dt = \iint_R \left(\partial_x F_x + \partial_y F_y \right) dx\, dy.$$

Review: Green's Theorem on a plane





Circulation-tangential form:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \left(\partial_x F_y - \partial_y F_x\right) dx\, dy.$$

Flux-normal form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\partial_x F_x + \partial_y F_y\right) dx\, dy.$$

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

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- **Sketch of the proof of Green's Theorem.**
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Sketch of the proof of Green's Theorem

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \big[F_x(t)\,x'(t) + F_y(t)\,y'(t)\big]\,dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$

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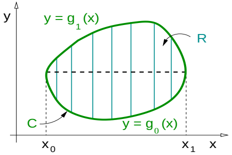
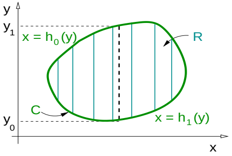
We only consider a simple domain like the one in the pictures.





Sketch of the proof of Green's Theorem

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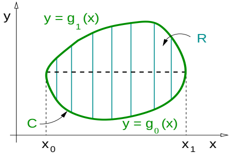

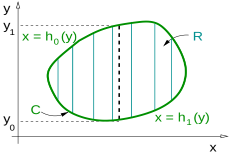

Using the picture on the left we show that

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R \left(-\partial_y F_x \right) dx\, dy;$$

Sketch of the proof of Green's Theorem

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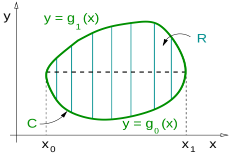
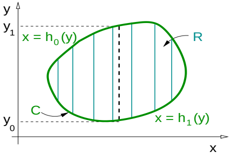




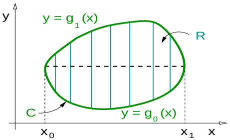
Using the picture on the left we show that

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R \left(-\partial_y F_x \right) dx\, dy;$$

and using the picture on the right we show that

$$\int_C F_y(t)\, y'(t)\, dt = \iint_R \left(\partial_x F_y \right) dx\, dy.$$

Sketch of the proof of Green's Theorem



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

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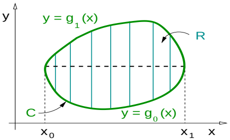
Sketch of the proof of Green's Theorem



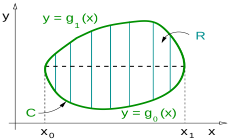
Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t) \rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle \qquad t \in [x_0, x_1].$$

Sketch of the proof of Green's Theorem



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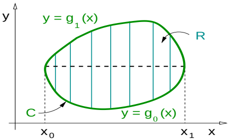
Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t) \rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t) \rangle \qquad t \in [x_0, x_1].$$

Sketch of the proof of Green's Theorem



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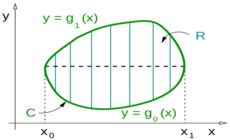
Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,

Sketch of the proof of Green's Theorem



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Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,
and $F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))$ on $\mathbf{r}_1$.

Sketch of the proof of Green's Theorem

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Sketch of the proof of Green's Theorem

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

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Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau)$$

Sketch of the proof of Green's Theorem

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Therefore, $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \left[F_x(t, g_0(t)) - F_x(t, g_1(t))\right] dt.$

Sketch of the proof of Green's Theorem

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

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Therefore, $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \big[F_x(t, g_0(t)) - F_x(t, g_1(t))\big]\,dt.$

We obtain: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t, y)\big]\,dy\,dt.$

Sketch of the proof of Green's Theorem

Recall: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \left[-\partial_y F_x(t,y)\right] dy\,dt.$

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This result is precisely what we wanted to prove:

$$\int_C F_x(t)x'(t)\,dt = \iint_R \left(-\partial_y F_x\right) dy\,dx.$$

Sketch of the proof of Green's Theorem

Recall: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \left[-\partial_y F_x(t,y)\right] dy\,dt.$

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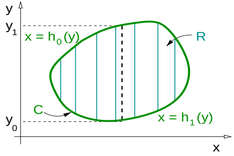
We just mention that the result

$$\int_C F_y(t)\,y'(t)\,dt = \iint_R \left(\partial_x F_y\right) dx\,dy.$$

is proven in a similar way using the parametrization of the C given in the picture.



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Divergence and curl of a function on a plane

Definition

The *curl* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\left(\operatorname{curl}\mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x.$$

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\operatorname{div}\mathbf{F} = \partial_x F_x + \partial_y F_y.$$

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Remark: Both forms of Green's Theorem can be written as:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \left(\operatorname{curl} \mathbf{F}\right)_z dx\, dy.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dx\, dy.$$

Divergence and curl of a function on a plane

Remark: What type of information about **F** is given in $(\operatorname{curl} \mathbf{F})_z$?

Divergence and curl of a function on a plane

Remark: What type of information about $\mathbf{F}$ is given in $(\mathrm{curl}\,\mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle$$

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$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\operatorname{curl} \mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$

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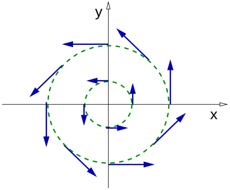

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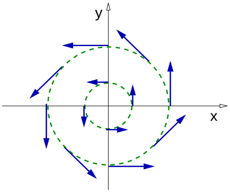

If we place a small ball at $(0, 0)$, the ball will spin around the z-axis with speed proportional to $(\mathrm{curl}\,\mathbf{F})_z$.

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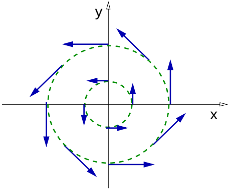

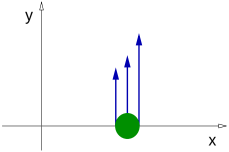

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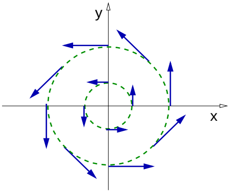

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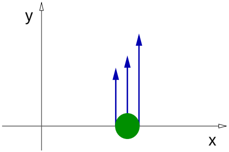

If we place a small ball at everywhere in the plane, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl}\mathbf{F})_z$.

Divergence and curl of a function on a plane

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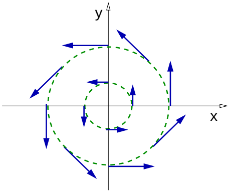

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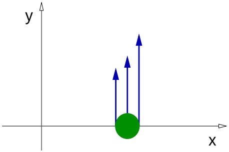

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Remark: The curl of a field measures its rotation.

Divergence and curl of a function on a plane

Remark: What type of information about **F** is given in $\operatorname{div} \mathbf{F}$?

Divergence and curl of a function on a plane

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Example: Suppose **F** is the velocity field of a gas and

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Divergence and curl of a function on a plane

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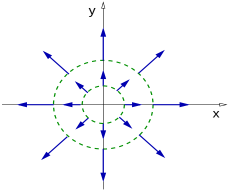

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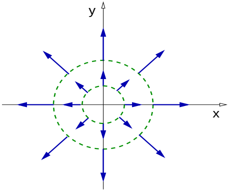

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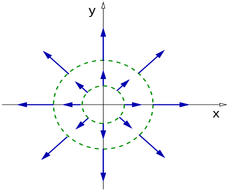

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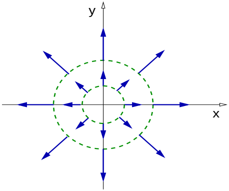

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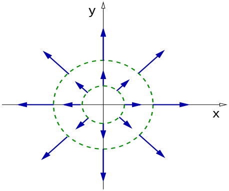

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Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- **Area computed with a line integral.**

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Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral.

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◁

Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- The area of a surface in space.
  - The surface is given in parametric form.
  - The surface is given in explicit form.

Review: Arc length and line integrals

- The integral of a function $f : [a, b] \to \mathbb{R}$ is
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Surface area and surface integrals. (Sect. 16.5)

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- **Review: Double integral of a scalar function.**
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Review: Double integral of a scalar function

- The double integral of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on a region $R \subset \mathbb{R}^2$, which is the volume under the graph of f and above the $z = 0$ plane, and is given by

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- Explicit form, $z = f(x, y)$;
- Implicit form, $F(x, y, z) = 0$;

Explicit, implicit, parametric equations of surfaces

Review: Curves on $\mathbb{R}^2$ can be defined in:

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Explicit, implicit, parametric equations of surfaces

Example

Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

Explicit, implicit, parametric equations of surfaces

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Solution: Use cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$. Parameters of the surface: $u = r$, $v = \theta$.

Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the cone $z = \sqrt{x^2 + y^2}$, and two tangent vectors.

Solution: Use cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$. Parameters of the surface: $u = r$, $v = \theta$. Then

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta), \quad z(r,\theta) = r.$$

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Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.

Explicit, implicit, parametric equations of surfaces

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Find a parametric expression for the sphere $x^2 + y^2 + z^2 = R^2$, and two tangent vectors.

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$x = \rho \cos(\theta) \sin(\phi)$, $y = \rho \sin(\theta) \sin(\phi)$, $z = \rho \cos(\phi)$.

Explicit, implicit, parametric equations of surfaces

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Explicit, implicit, parametric equations of surfaces

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$$\mathbf{r}(\theta, \phi) = R \langle \cos(\theta) \sin(\phi), \; \sin(\theta) \sin(\phi), \; \cos(\phi) \rangle.$$

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Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- **The area of a surface in space.**
  - **The surface is given in parametric form.**
  - The surface is given in explicit form.

The area of a surface in parametric form

Theorem

Given a smooth surface S with parametric equation $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for $u \in [u_0, u_1]$ and $v \in [v_0, v_1]$ is given by

$$A(S) = \int_{u_0}^{u_1} \int_{v_0}^{v_1} |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.$$

The area of a surface in parametric form

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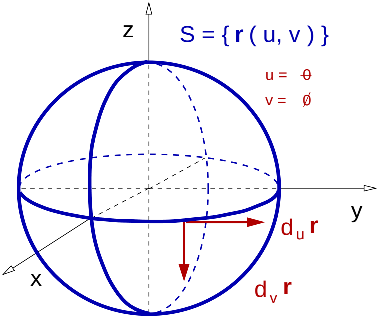

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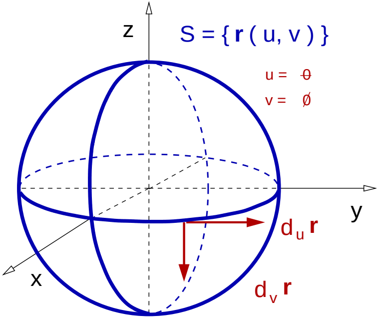

Remark: The function

$$d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.$$

represents the area of a small region on the surface.

The area of a surface in parametric form

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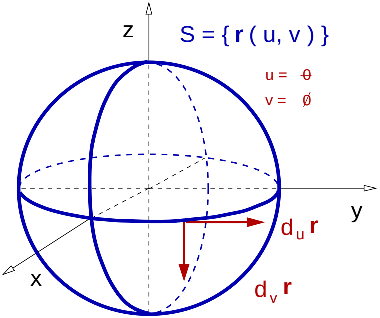

Remark: The function

$$d\sigma = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, dv \, du.$$

represents the area of a small region on the surface.

This is the generalization to surfaces of the arc-length formula for the length of a curve.

The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

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Solution: Use cylindrical coordinates.

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Solution: Use cylindrical coordinates. The surface in parametric form is

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$$\partial_r \mathbf{r} \times \partial_\theta \mathbf{r}$$

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$$\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix}$$

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$$\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r\rangle.$$

The area of a surface in parametric form

Example

Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Recall: $\partial_r \mathbf{r} \times \partial_\theta \mathbf{r} = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle$.

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Find an expression for the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

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$$|\partial_r \mathbf{r} \times \partial_\theta \mathbf{r}| = \sqrt{4r^4 + r^2}$$

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$$|\partial_r \mathbf{r} \times \partial_\theta \mathbf{r}| = \sqrt{4r^4 + r^2} = r\sqrt{1 + 4r^2}.$$

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$$A(S) = \int_0^{2\pi} \int_0^2 r\, \sqrt{1 + 4r^2}\, dr\, d\theta.$$

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$$A(S) = \int_0^{2\pi} \int_0^2 r\,\sqrt{1 + 4r^2}\,dr\,d\theta.$$

This integral will be done later on by substitution. The result is:

$$A(S) = \frac{\pi}{6}\left[(17)^{3/2} - 1\right].$$

◁

Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- Explicit, implicit, parametric equations of surfaces.
- **The area of a surface in space.**
  - The surface is given in parametric form.
  - **The surface is given in explicit form.**

The area of a surface in space in explicit form

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, *the area of a level surface* $S = \{f(x, y, z) = 0\}$, *over a closed, bounded region* R *in the plane* $\{z = 0\}$, *is given by*

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA.$$

The area of a surface in space in explicit form

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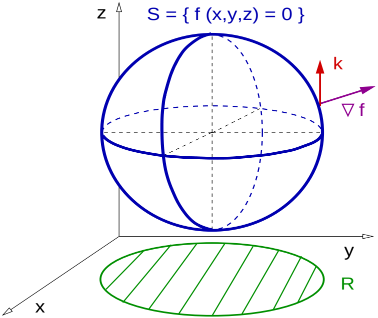

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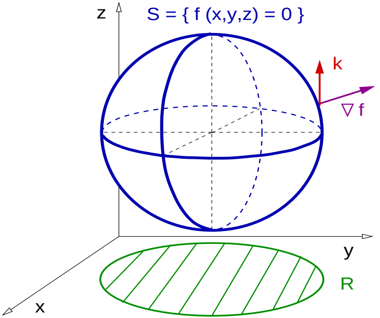

Remark: Eq. (7), page 949, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The area of a surface in space in explicit form

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Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, *the area of a level surface* $S = \{f(x, y, z) = 0\}$, *over a closed, bounded region* R *in the plane* $\{z = 0\}$, *is given by*

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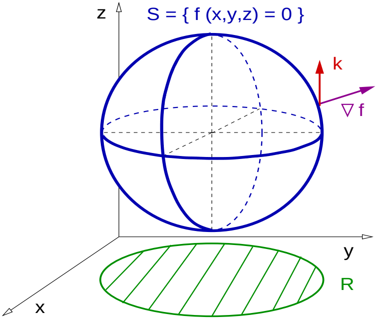

Remark: Eq. (7), page 949, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

The area of a surface in space in explicit form

Recall: The area of a level surface $S = \{f(x, y, z) = 0\}$ over a flat region R in $\{z = 0\}$, is given by

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Example

Find the area of $S = \{z - 1 = 0\}$ over R in $\{z = 0\}$.

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Remark: The formula for $A(S)$ is reasonable:

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Remark: The formula for $A(S)$ is reasonable: Every flat horizontal surface S over a flat horizontal region R satisfies $A(S) = A(R)$.

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Solution: The plane S intersects the horizontal plane at a $\pi/4$ angle.

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◁

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Remark: The formula for $A(S)$ is still reasonable:

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Remark: The formula for $A(S)$ is still reasonable: Every flat surface S having an angle $\pi/4$ over a flat horizontal region R satisfies $A(S) = \sqrt{2}\, A(R)$.

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The factor $\dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}$ is the angle correction function

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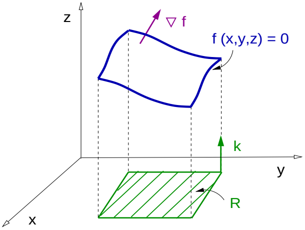
Remark: The formula for $A(S)$ can be interpreted as follows:
The factor $\dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}$ is the angle correction function needed to obtain the $A(S)$ by correcting the $A(R)$ by the relative inclination of S with respect to R.



The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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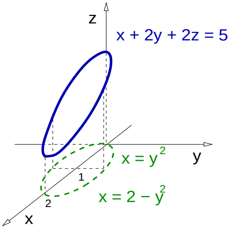

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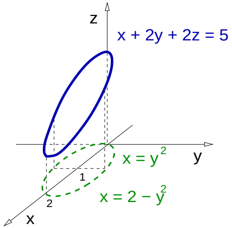

The surface is given by $f = 0$ with

$$f(x, y, z) = x + 2y + 2z - 5.$$

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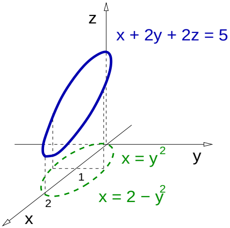

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The region R is in the plane $z = 0$,

$$R = \left\{ \begin{array}{c} (x, y, z) \;:\; z = 0,\; y \in [-1, 1] \\ x \in [y^2, (2 - y^2)] \end{array} \right\}.$$

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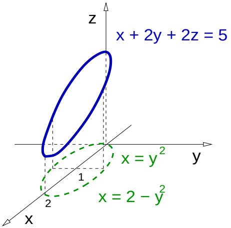

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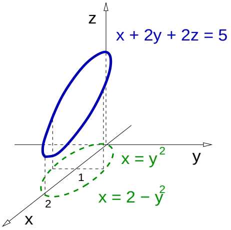

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Recall: $A(S) = \iint_R \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$.

The area of a surface in space in explicit form

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Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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Solution: $A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA$. Here $\nabla f = \langle 1, 2, 2 \rangle$.

Therefore: $|\nabla f| = \sqrt{1 + 4 + 4} = 3$, and $|\nabla f \cdot \mathbf{k}| = 2$.

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So we can write down the expression for $A(S)$ as follows,

$$A(S) = \iint_R \frac{3}{2}\, dx\, dy$$

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Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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So we can write down the expression for $A(S)$ as follows,

$$A(S) = \iint_R \frac{3}{2} \, dx \, dy = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx \, dy.$$

The area of a surface in space in explicit form

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Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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$$A(S) = \frac{3}{2} \int_{-1}^{1} \left(2 - y^2 - y^2\right) dy$$

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$$A(S) = 3 \int_{-1}^{1} \left(1 - y^2\right) dy$$

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$$A(S) = 3\int_{-1}^{1}\bigl(1 - y^2\bigr)\,dy = 3\Bigl(y - \frac{y^3}{3}\Bigr)\Big|_{-1}^{1}$$

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$$A(S) = 3 \int_{-1}^{1} (1 - y^2)\, dy = 3\Big(y - \frac{y^3}{3}\Big)\Big|_{-1}^{1} = 3\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big)$$

$$A(S) = 3\Big(2 - \frac{2}{3}\Big)$$

The area of a surface in space in explicit form

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} \, dx \, dy$.

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Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

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Since R is a disk radius 2, it is convenient to use polar coordinates in $\mathbb{R}^2$. We obtain

$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

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Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

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$$A(S) = \frac{2\pi}{8} \int_1^{17} u^{1/2}\, du$$

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We conclude: $A(S) = \frac{\pi}{6} \big[(17)^{3/2} - 1 \big]$. $\triangleleft$

The area of a surface in space in explicit form

Remark: The formula for the area of a surface in space can be generalized as follows.

The area of a surface in space in explicit form

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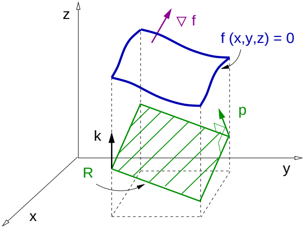
Theorem

The area of a surface S given by $f(x, y, z) = 0$ over a closed and bounded plane region R in space is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA,$$



where $\mathbf{p}$ *is a unit vector normal to the region* R *and* $\nabla f \cdot \mathbf{p} \neq 0$.

The area of a surface in space in explicit form

Proof in a simple case: Assume that the surface us given in explicit form:
$$S = \{(x, y, z) \ : \ z = g(x, y)\},$$

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On the one hand, a simple parametric form is to use $u = x$, $v = y$ and $z(u, v) = g(u, v)$. Hence

$$\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$$

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$$\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle \quad \Rightarrow \quad \begin{cases} \partial_x \mathbf{r} = \langle 1, 0, \partial_x g \rangle \\ \partial_y \mathbf{r} = \langle 0, 1, \partial_y g \rangle, \end{cases}$$

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$$\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1\rangle$$

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Therefore, $\quad \partial_x f = \partial_x g, \quad \partial_y f = \partial_y g, \quad \partial_z f = -1.$

The area of a surface in space in explicit form

Proof in a simple case: Recall: $\partial_x \mathbf{r} \times \partial_y \mathbf{r} = \langle -\partial_x g, -\partial_y g, 1 \rangle$ and

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$$A(S) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} |\partial_x \mathbf{r} \times \partial_y \mathbf{r}| \, dy \, dx$$

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□

The area of a surface in space in explicit form

Proof: Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle ΔR in that partition.

The area of a surface in space in explicit form

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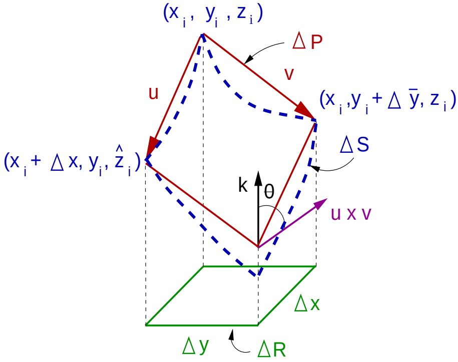

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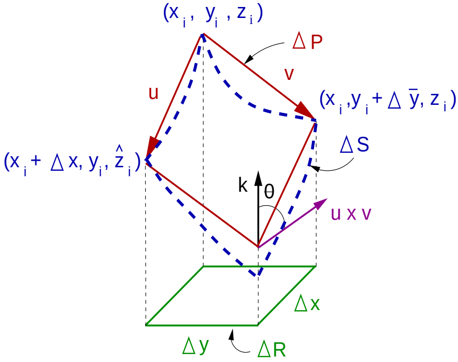

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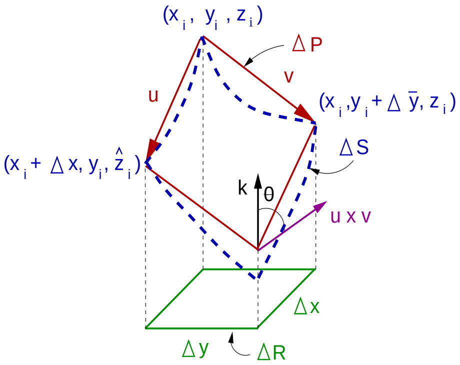

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and

$$\mathbf{u} = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle,$$
$$\mathbf{v} = \langle 0, \Delta y, (z_i - \bar{z}_i) \rangle.$$

Therefore,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & (z_i - \hat{z}_i) \\ 0 & \Delta y & (z_i - \bar{z}_i) \end{vmatrix} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \bar{z}_i), \Delta x \Delta y \rangle.$$

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Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

The linearization of $f(x,y,z)$ at (x_i, y_i, z_i) implies

$$f(x,y,z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).$$

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Since $f(x_i, y_i, z_i) = 0$, $f(x_i + \Delta x, y_i, \hat{z}_i) = 0$, $f(x_i, y_i + \Delta y, \bar{z}_i) = 0$,

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The area of a surface in space in explicit form

Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

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$$\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x \Delta y}{(\partial_z f)_i} \Rightarrow \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x \Delta y.$$

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Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

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$$\Delta P = \frac{|(\nabla f)_i|}{|(\nabla f \cdot \mathbf{k})_i|} \Delta x \Delta y$$

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