Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- The flux across a plane curve.
Vector fields on a plane and in space

Definition
A vector field on a plane or in space is a vector-valued function \( \mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), respectively.
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Examples from physics:
- Electric and magnetic fields.
- The gravitational field of the Earth.
- The velocity field in a fluid or gas.
- The variation of temperature in a room. (Gradient field.)
- Magnetic field of a small magnet.
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The gradient field of a scalar-valued function

Remark:

- Given a scalar-valued function \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R} \), with \( n = 2, 3 \), its gradient vector, \( \nabla f = \langle \partial_x f, \partial_y f \rangle \) or \( \nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle \), respectively, is a vector field in a plane or in space.

Example
Find and sketch a graph of the gradient field of the function \( f(x, y) = x^2 + y^2 \).

Solution:
We know the graph of \( f \) is a paraboloid. The gradient field is \( \nabla f = \langle 2x, 2y \rangle \).
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The line integral of a vector field along a curve

Definition
The *line integral* of a vector-valued function \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), along the curve associated with the function \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds
\]

where \( \hat{\mathbf{r}}(s) \) is the arc length parametrization of the function \( \mathbf{r} \), and \( s(t_0) = s_0, s(t_1) = s_1 \) are the arc lengths at the points \( t_0, t_1 \).
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Example

Remark: It is common the notation $\hat{\mathbf{r}}' = \mathbf{T}$, where $\mathbf{T}$ is the tangent vector.
The line integral of a vector field along a curve

Definition
The line integral of a vector-valued function $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3$ is given by

$$\int_C F \cdot d\mathbf{r} = \int_{s_0}^{s_1} F(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0, t_1$.

Example

Remark: It is common the notation

$$\hat{\mathbf{r}}' = \mathbf{T},$$

since $\mathbf{T}$ is tangent to the curve and unit, since $s$ is the curve arc-length parameter.
Line integrals in space

**Theorem (General parametrization formula)**

The line integral of a continuous function $\mathbf{F} : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ along a differentiable curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3$ can be written as

$$
\int_{s_0}^{s_1} \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s) \, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
$$

where $\mathbf{r}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0$, $t_1$. 

**Proof:**

Recall the curve arc-length function $s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau$. Then

$$
ds = |\mathbf{r}'(t)| \, dt.
$$

Also, $\mathbf{r}(s(s(t))) = \mathbf{r}(t)$.

And finally

$$
\mathbf{r}'(s) \, ds = \frac{d\mathbf{r}}{dt} \, dt = |\mathbf{r}'(t)| \, dt.
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This substitution provides the equation in the Theorem.
Line integrals in space

Theorem (General parametrization formula)

The line integral of a continuous function \( \mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3 \) along a differentiable curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) can be written as

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where \( \hat{\mathbf{r}}(s) \) is the arc length parametrization of the function \( \mathbf{r} \), and \( s(t_0) = s_0, \ s(t_1) = s_1 \) are the arc lengths at the points \( t_0, t_1 \).

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The line integral of a continuous function $\mathbf{F} : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ along a differentiable curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3$ can be written as

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where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0$, $t_1$.

**Proof:** Recall the curve arc-length function $s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau$. Then $ds = |\mathbf{r}'(t)| \, dt$. 
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Proof: Recall the curve arc-length function \( s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau \).

Then \( ds = |\mathbf{r}'(t)| \, dt \). Also, \( \hat{\mathbf{r}}(s(t)) = \mathbf{r}(t) \).
The line integral of a continuous function $F: D \subset \mathbb{R}^3 \to \mathbb{R}^3$ along a differentiable curve $r: [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ can be written as

$$\int_{s_0}^{s_1} F(\hat{r}(s)) \cdot \hat{r}'(s) \, ds = \int_{t_0}^{t_1} F(r(t)) \cdot r'(t) \, dt,$$

where $\hat{r}(s)$ is the arc length parametrization of the function $r$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0$, $t_1$.

**Proof:** Recall the curve arc-length function $s(t) = \int_{t_0}^{t} |r'(\tau)| \, d\tau$. Then $ds = |r'(t)| \, dt$. Also, $\hat{r}(s(t)) = r(t)$. And finally

$$\hat{r}'(s)$$
The line integral of a continuous function \( F : D \subset \mathbb{R}^3 \to \mathbb{R}^3 \) along a differentiable curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) can be written as
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Then \( ds = |\mathbf{r}'(t)| \, dt \). Also, \( \hat{\mathbf{r}}(s(t)) = \mathbf{r}(t) \). And finally
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\hat{\mathbf{r}}'(s) = \frac{d\hat{\mathbf{r}}}{ds}(s)
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Line integrals in space

Theorem (General parametrization formula)
The line integral of a continuous function \( \mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3 \) along a differentiable curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) can be written as

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Line integrals in space

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$$\frac{d\hat{s}}{ds}(s) = \frac{d\mathbf{r}}{dt}(t) \frac{dt}{ds} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$
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Work done by a force on a particle

Definition
If the vector valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the work done by the force on the particle.
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Example
A mass \( m \) projectile near the Earth surface.
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- The movement takes place on a plane, and \( \mathbf{F} = \langle 0, -mg \rangle \).
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Example
A mass $m$ projectile near the Earth surface.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leq 0$ in the first half of the trajectory, and $W \geq 0$ on the second half.
Work done by a force on a particle

Example
Find the work done by the force \( \mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle \) on a particle moving along the curve with \( \mathbf{r}(t) = \langle t, t^2, t^4 \rangle \), \( t \in [0, 1] \).
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Solution:
First: Evaluate \( \mathbf{F} \) along \( \mathbf{r} \).
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Solution:
First: Evaluate \( \mathbf{F} \) along \( \mathbf{r} \). This is: \( \mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle \).
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Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

Solution:
First: Evaluate $\mathbf{F}$ along $\mathbf{r}$. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle$.
Second: Compute $\mathbf{r}'(t)$.
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Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

Solution:
First: Evaluate $\mathbf{F}$ along $\mathbf{r}$. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle$.
Second: Compute $\mathbf{r}'(t)$. This is: $\mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle$. 

So, $W = \int_0^1 \left[ (3t^2 - 3t) + (6t^5) + (4t^3) \right] dt = (t^3 - 3t^2 + t^6 + t^4) \bigg|_0^1 = 1 - 3 + 1 + 1 = 3$. We conclude: The work done is $W = 3$. 

$\blacksquare$
Work done by a force on a particle

Example
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Solution:
First: Evaluate \( \mathbf{F} \) along \( \mathbf{r} \). This is: \( \mathbf{F}(t) = \langle 3t^2 - 3t, 3t^4, 1 \rangle \).

Second: Compute \( \mathbf{r}'(t) \). This is: \( \mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle \).

Third: Integrate the dot product \( \mathbf{F}(t) \cdot \mathbf{r}'(t) \).
Work done by a force on a particle

Example
Find the work done by the force \( \mathbf{F}(x, y, z) = \langle 3x^2 - 3x, 3z, 1 \rangle \) on a particle moving along the curve with \( \mathbf{r}(t) = \langle t, t^2, t^4 \rangle, \ t \in [0, 1] \).

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Third: Integrate the dot product \( \mathbf{F}(t) \cdot \mathbf{r}'(t) \).

\[
W = \int_{0}^{1} \left[ (3t^2 - 3t) + (6t^5) + (4t^3) \right] \, dt \\
= \left( t^3 - \frac{3}{2} t^2 + t^6 + t^4 \right) \Bigg|_{0}^{1} = 1 - \frac{3}{2} + 1 + 1.
\]
Work done by a force on a particle

Example
Find the work done by the force \( \mathbf{F}(x, y, z) = \langle 3x^2 - 3x, 3z, 1 \rangle \) on a particle moving along the curve with \( \mathbf{r}(t) = \langle t, t^2, t^4 \rangle, \ t \in [0, 1] \).

Solution:
First: Evaluate \( \mathbf{F} \) along \( \mathbf{r} \). This is: \( \mathbf{F}(t) = \langle 3t^2 - 3t, 3t^4, 1 \rangle \).

Second: Compute \( \mathbf{r}'(t) \). This is: \( \mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle \).

Third: Integrate the dot product \( \mathbf{F}(t) \cdot \mathbf{r}'(t) \).

\[
W = \int_{0}^{1} [(3t^2 - 3t) + (6t^5) + (4t^3)] \, dt \\
= \left[ t^3 - \frac{3}{2}t^2 + t^6 + t^4 \right]_{0}^{1} = 1 - \frac{3}{2} + 1 + 1.
\]

So, \( W = 3 - \frac{3}{2} \).
Work done by a force on a particle

Example

Find the work done by the force \( \mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle \) on a particle moving along the curve with \( \mathbf{r}(t) = \langle t, t^2, t^4 \rangle \), \( t \in [0, 1] \).

Solution:

First: Evaluate \( \mathbf{F} \) along \( \mathbf{r} \). This is: \( \mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle \).

Second: Compute \( \mathbf{r}'(t) \). This is: \( \mathbf{r}'(t) = \langle 1, 2t, 4t^3 \rangle \).

Third: Integrate the dot product \( \mathbf{F}(t) \cdot \mathbf{r}'(t) \).

\[
\mathcal{W} = \int_0^1 \left[ (3t^2 - 3t) + (6t^5) + (4t^3) \right] \, dt
= \left( t^3 - \frac{3}{2} t^2 + t^6 + t^4 \right)
\bigg|_0^1 = 1 - \frac{3}{2} + 1 + 1.
\]

So, \( \mathcal{W} = 3 - \frac{3}{2} \). We conclude: The work done is \( \mathcal{W} = \frac{3}{2} \). \( \triangle \)
Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - **The flow of a fluid along a curve.**
- The flux across a plane curve.
The flow of a fluid along a curve

Definition

In the case that the vector field \( \mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is the velocity field of a flow and \( r : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) is any smooth curve, then the line integral

\[
F = \int_C \mathbf{v} \cdot d\mathbf{r},
\]

is called a flow integral. If the curve is a closed loop, the flow integral is called the circulation of the fluid around the loop.
The flow of a fluid along a curve

**Definition**

In the case that the vector field \( \mathbf{v} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is the velocity field of a flow and \( r : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \) is any smooth curve, then the line integral

\[
F = \int_C \mathbf{v} \cdot d\mathbf{r},
\]

is called a *flow integral*. If the curve is a closed loop, the flow integral is called the *circulation* of the fluid around the loop.

**Example**

![Viscous fluid in a pipe.](image)
The flow of a fluid along a curve

Definition
In the case that the vector field \( \mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is the velocity field of a flow and \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3 \) is any smooth curve, then the line integral

\[
F = \int_C \mathbf{v} \cdot d\mathbf{r},
\]

is called a flow integral. If the curve is a closed loop, the flow integral is called the circulation of the fluid around the loop.

Example
- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
The flow of a fluid along a curve

**Definition**

In the case that the vector field \( \mathbf{v} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is the velocity field of a flow and \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \) is any smooth curve, then the line integral

\[
F = \int_C \mathbf{v} \cdot d\mathbf{r},
\]

is called a *flow integral*. If the curve is a closed loop, the flow integral is called the *circulation* of the fluid around the loop.

**Example**

- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
- The flow vanishes on any curve perpendicular to the section of the pipe.
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \( F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2. \)
Example
Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along
the closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, \pi]$, and
$\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$. 
The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field \( \mathbf{v} = (-y, x) \) along the closed loop given by \( \mathbf{r}_1 = (a \cos(t), a \sin(t)) \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = (t, 0) \) for \( t \in [-a, a] \).

Solution: The circulation is: \( F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 \).

The first term is given by:

\[
\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^\pi \mathbf{v}(t) \cdot \mathbf{r}'_1(t) \, dt.
\]
The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: 
\[ \mathbf{F} = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2. \]

The first term is given by:
\[ \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^\pi \mathbf{v}(t) \cdot \mathbf{r}'_1(t) \, dt. \]
\[ \mathbf{v}(t) = \langle -a \sin(t), a \cos(t) \rangle, \]
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \( F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 \).

The first term is given by:

\[
\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_{0}^{\pi} \mathbf{v}(t) \cdot \mathbf{r}_1'(t) \, dt.
\]

\( \mathbf{v}(t) = \langle -a \sin(t), a \cos(t) \rangle, \)

\( \mathbf{r}_1'(t) = \langle -a \sin(t), a \cos(t) \rangle. \)
The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field $v = \langle -y, x \rangle$ along the closed loop given by $r_1 = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, \pi]$, and $r_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{c_1} v \cdot dr_1 + \int_{c_2} v \cdot dr_2$.

The first term is given by:

$$\int_{c_1} v \cdot dr_1 = \int_0^\pi v(t) \cdot r_1'(t) \, dt.$$ 

$v(t) = \langle -a \sin(t), a \cos(t) \rangle$,

$r_1'(t) = \langle -a \sin(t), a \cos(t) \rangle$.

$$\int_{c_1} v \cdot dr_1 = \int_0^\pi a^2 [\sin^2(t) + \cos^2(t)] \, dt$$
The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \[ F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2. \]

The first term is given by:

\[ \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_{0}^{\pi} \mathbf{v}(t) \cdot \mathbf{r}_1'(t) \, dt. \]

\( \mathbf{v}(t) = \langle -a \sin(t), a \cos(t) \rangle, \)

\( \mathbf{r}_1'(t) = \langle -a \sin(t), a \cos(t) \rangle. \)

\[ \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_{0}^{\pi} a^2 [\sin^2(t) + \cos^2(t)] \, dt \quad \Rightarrow \quad \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2. \]
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$. 
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \[ F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2. \]

The second term is given by:

\[ \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}_2'(t) \, dt, \]

\[ \mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}_2'(t) = \langle 1, 0 \rangle. \]
The flow of a fluid along a curve

Example

Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: 

\[
F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2.
\]

The second term is given by:

\[
\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}'_2(t) \, dt,
\]

\[
\mathbf{v}(t) = \langle 0, t \rangle,
\]
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

Solution: The circulation is: \( F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 \).

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\[
\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}'_2(t) \, dt,
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\mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}'_2(t) = \langle 1, 0 \rangle.
\]
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

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F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2.
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The second term is given by:
\[
\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}'_2(t) \, dt,
\]
where \( \mathbf{v}(t) = \langle 0, t \rangle \), \( \mathbf{r}'_2(t) = \langle 1, 0 \rangle \).

\[
\mathbf{v}(t) \cdot \mathbf{r}'_2(t) = 0
\]
The flow of a fluid along a curve

**Example**

Find the circulation of a fluid with velocity field \( \mathbf{v} = \langle -y, x \rangle \) along the closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0 \rangle \) for \( t \in [-a, a] \).

**Solution:** The circulation is:

\[
F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2.
\]

The second term is given by:

\[
\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}_2'(t) \, dt,
\]

\[
\mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}_2'(t) = \langle 1, 0 \rangle.
\]

\[
\mathbf{v}(t) \cdot \mathbf{r}_2'(t) = 0 \quad \Rightarrow \quad \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.
\]
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$.

The second term is given by:

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot r'_2(t) \, dt,$$

where $\mathbf{v}(t) = \langle 0, t \rangle$, $r'_2(t) = \langle 1, 0 \rangle$.

$$\mathbf{v}(t) \cdot r'_2(t) = 0 \quad \Rightarrow \quad \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.$$

Since $\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2$,
The flow of a fluid along a curve

Example
Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$.

The second term is given by:

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}'_2(t) \, dt,$$

$$\mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}'_2(t) = \langle 1, 0 \rangle.$$

$$\mathbf{v}(t) \cdot \mathbf{r}'_2(t) = 0 \quad \Rightarrow \quad \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.$$

Since $\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2$, we conclude: $F = \pi a^2$. \(\triangle\)
Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- The flux across a plane curve.
The flux across a plane curve

Definition
The flux of a vector field \( \mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \rightarrow \{z = 0\} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{z = 0\} \subset \mathbb{R}^3 \) is given by

\[
F = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the curve outer unit normal vector in the plane \( \{z = 0\} \).
The flux across a plane curve

Definition
The flux of a vector field $F : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $r : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$F = \oint_C F \cdot n \, ds,$$

where $n$ is the curve outer unit normal vector in the plane $\{z = 0\}$.

Example

![Diagram showing the flux across a plane curve](image)
The flux across a plane curve

Definition
The flux of a vector field \( F : \{ z = 0 \} \subset \mathbb{R}^3 \to \{ z = 0 \} \subset \mathbb{R}^3 \) along a closed plane loop \( r : [t_0, t_1] \subset \mathbb{R} \to \{ z = 0 \} \subset \mathbb{R}^3 \) is given by

\[
F = \oint_C F \cdot n \, ds,
\]

where \( n \) is the curve outer unit normal vector in the plane \( \{ z = 0 \} \).

Example

Remarks:
- \( F \) is defined on \( \{ z = 0 \} \).
The flux across a plane curve

Definition
The flux of a vector field \( \mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3 \) is given by

\[
\Phi = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the curve outer unit normal vector in the plane \( \{z = 0\} \).

Example

Remarks:
- \( \mathbf{F} \) is defined on \( \{z = 0\} \).
- The loop \( C \) lies on \( \{z = 0\} \).
The flux across a plane curve

Definition
The flux of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \rightarrow \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbf{F} = \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the curve outer unit normal vector in the plane $\{z = 0\}$.

Example

Remarks:
- $\mathbf{F}$ is defined on $\{z = 0\}$.
- The loop $C$ lies on $\{z = 0\}$.
- Simple formula for $\mathbf{n}$?
The flux across a plane curve

Definition
The flux of a vector field \( \mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3 \) is given by

\[
\mathbf{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the curve outer unit normal vector in the plane \( \{z = 0\} \).

Example

Remarks:
- \( \mathbf{F} \) is defined on \( \{z = 0\} \).
- The loop \( C \) lies on \( \{z = 0\} \).
- Simple formula for \( \mathbf{n} \)? Yes.

\[
\mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle.
\]
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x y'(t) - F_y x'(t) \right] \, dt.
\]
The flux across a plane curve

**Theorem (Counterclockwise loops.)**

The flux of a vector field $\mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle$ along a closed, counterclockwise plane loop $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$ for $t \in [t_0, t_1]$ is given by

$$\int_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x y'(t) - F_y x'(t)] \, dt.$$

**Proof:**

![Diagram of a plane curve with vectors and normal vector]
The flux across a plane curve

**Theorem (Counterclockwise loops.)**

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x y'(t) - F_y x'(t) \right] \, dt.
\]

**Proof:**

**Remarks:** Since \( C \) is counterclockwise traversed, \( \mathbf{n} = \mathbf{u} \times \mathbf{k} \), where \( \mathbf{u} = \mathbf{r}'/|\mathbf{r}'| \).
The flux across a plane curve

**Theorem (Counterclockwise loops.)**

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x y'(t) - F_y x'(t)] \, dt.
\]

**Proof:**

**Remarks:** Since \( C \) is counterclockwise traversed, \( \mathbf{n} = \mathbf{u} \times \mathbf{k} \), where \( \mathbf{u} = \mathbf{r}'/|\mathbf{r}'| \).

\[
\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|} \langle x'(t), y'(t), 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.
\]
**The flux across a plane curve**

**Theorem (Counterclockwise loops.)**

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x y'(t) - F_y x'(t) \right] \, dt.
\]

**Proof:**

**Remarks:** Since \( C \) is counterclockwise traversed, \( \mathbf{n} = \mathbf{u} \times \mathbf{k} \), where \( \mathbf{u} = \mathbf{r}'/|\mathbf{r}'| \).

\[
\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|} \langle x'(t), y'(t), 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.
\]

\[
\mathbf{n} = \frac{1}{|\mathbf{r}'|} \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
 i' & j' & k' \\
x' & y' & 0 \\
0 & 0 & 1 \\
\end{vmatrix}
\]
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x y'(t) - F_y x'(t)] \, dt.
\]

Proof:

Remarks: Since \( C \) is counterclockwise traversed, \( \mathbf{n} = \mathbf{u} \times \mathbf{k} \), where \( \mathbf{u} = \mathbf{r}'/|\mathbf{r}'| \).

\[
\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|} \langle x'(t), y'(t), 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.
\]

\[
\mathbf{n} = \frac{1}{|\mathbf{r}'|} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \Rightarrow \quad \mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle.
\]
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

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\]

Proof: Recall: \( \mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle. \)
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

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\]

Proof: Recall: \( \mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle. \)

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \langle F_x, F_y, 0 \rangle \cdot \langle y'(t), -x'(t), 0 \rangle \frac{1}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| \, dt
\]
The flux across a plane curve

Theorem (Counterclockwise loops.)

The flux of a vector field \( \mathbf{F} = \langle F_x(x, y), F_y(x, y), 0 \rangle \) along a closed, counterclockwise plane loop \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \) for \( t \in [t_0, t_1] \) is given by

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\]

Proof: Recall: \( \mathbf{n} = \frac{1}{|\mathbf{r}'(t)|} \langle y'(t), -x'(t), 0 \rangle. \)

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \langle F_x, F_y, 0 \rangle \cdot \langle y'(t), -x'(t), 0 \rangle \frac{1}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| \, dt
\]

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} [F_x y'(t) - F_y x'(t)] \, dt. \quad \square
\]
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$. 
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\int_{c} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$
The flux across a plane curve

Example

Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)

Along \( C_1 \) we have: \( \mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle \)
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)
Along \( C_1 \) we have: \( \mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle \) and
\[
\begin{align*}
x'(t) &= -a \sin(t), & y'(t) &= a \cos(t).
\end{align*}
\]
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$

Along $C_1$ we have: $\mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle$ and

$$x'(t) = -a \sin(t), \quad y'(t) = a \cos(t).$$

Therefore,

$$F_{1x}(t) y'(t) - F_{1y}(t) x'(t)$$
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_c \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)

Along \( C_1 \) we have: \( \mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle \) and \( x'(t) = -a \sin(t), \quad y'(t) = a \cos(t). \)

Therefore,

\[
F_{1x}(t) y'(t) - F_{1y}(t) x'(t) = -a^2 \sin(t) \cos(t) + a^2 \sin(t) \cos(t) = 0.
\]
The flux across a plane curve

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$

Along $C_1$ we have: $\mathbf{F}_1(t) = \langle -a \sin(t), a \cos(t), 0 \rangle$ and

$$x'(t) = -a \sin(t), \quad y'(t) = a \cos(t).$$

Therefore,

$$F_{1x}(t) y'(t) - F_{1y}(t) x'(t) = -a^2 \sin(t) \cos(t) + a^2 \sin(t) \cos(t) = 0.$$

Hence: $\int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = 0.$
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).
The flux across a plane curve

Example
Find the flux of a field $F = \langle -y, x, 0 \rangle$ across the plane closed loop given by $r_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $r_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\int_c F \cdot n \, ds = \int_{c_1} F_1 \cdot n_1 \, ds + \int_{c_2} F_2 \cdot n_2 \, ds$
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\int_c \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$

Along $C_2$ we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$
The flux across a plane curve

Example

Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_c \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)

Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, \ y'(t) = 0 \).
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_c \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)
Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, y'(t) = 0 \). So,
\[ F_{2x}(t) y'(t) - F_{2y}(t) x'(t) \]
The flux across a plane curve

Example

Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \[ \int_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \]

Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, y'(t) = 0 \). So,

\[ F_{2x}(t) y'(t) - F_{2y}(t) x'(t) = 0 - t \]

\[ \Rightarrow \int_{C_2} F_2 \cdot n_2 \, ds = 0 - t \]
The flux across a plane curve

Example
Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)
Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, \ y'(t) = 0 \). So, \( F_{2x}(t) \ y'(t) - F_{2y}(t) \ x'(t) = 0 - t \) \( \Rightarrow \) \( \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-a}^{a} -t \, dt \),
The flux across a plane curve

Example

Find the flux of a field \( \mathbf{F} = \langle -y, x, 0 \rangle \) across the plane closed loop given by \( \mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle \) for \( t \in [0, \pi] \), and \( \mathbf{r}_2 = \langle t, 0, 0 \rangle \) for \( t \in [-a, a] \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds \)

Along \( C_2 \) we have: \( \mathbf{F}_2(t) = \langle 0, t, 0 \rangle \) and \( x'(t) = 1, \ y'(t) = 0 \). So,

\[
F_{2x}(t) y'(t) - F_{2y}(t) x'(t) = 0 - t \quad \Rightarrow \quad \int_{c_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-a}^{a} -t \, dt,
\]

\[
\int_{c_2} \mathbf{F} \cdot \mathbf{n} \, ds = -\left( \frac{t^2}{2} \right)_{-a}^{a}.
\]
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{c_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{c_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$

Along $C_2$ we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$ and $x'(t) = 1$, $y'(t) = 0$. So,

$F_{2x}(t) y'(t) - F_{2y}(t) x'(t) = 0 - t \quad \Rightarrow \quad \int_{c_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-a}^{a} -t \, dt,$

$\int_{c_2} \mathbf{F} \cdot \mathbf{n} \, ds = -\left(\frac{t^2}{2}\right)_{-a}^{a} \quad \Rightarrow \quad \int_{c_2} \mathbf{F} \cdot \mathbf{n} \, ds = 0.$
The flux across a plane curve

Example
Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a \cos(t), a \sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1 \, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2 \, ds$

Along $C_2$ we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$ and $x'(t) = 1$, $y'(t) = 0$. So,

$F_{2x}(t)y'(t) - F_{2y}(t)x'(t) = 0 - t \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-a}^{a} -t \, dt,$

$\int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = -\left( \frac{t^2}{2} \right|_{-a}^{a} \right) \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = 0.$

We conclude: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = 0.$
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- Conservative fields.
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.
The line integral of a vector field along a curve

Recall: The *line integral* of $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n = 2, 3$, along $r : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3$ is given by

$$\int_C F \cdot dr = \int_{s_0}^{s_1} F(\hat{r}(s)) \cdot \hat{r}'(s) \, ds = \int_{t_0}^{t_1} F(r(t)) \cdot r'(t) \, dt,$$

where $\hat{r}(s)$ is the arc length parametrization of the function $r$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points $t_0, t_1$. 

Remark: It is common the notation $\hat{r}' = T$, since $T$ is tangent to the curve and unit, since $s$ is the curve arc-length parameter.
The line integral of a vector field along a curve

Recall: The line integral of \( \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), along \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \hat{\mathbf{r}}(s) \) is the arc length parametrization of the function \( \mathbf{r} \), and \( s(t_0) = s_0, s(t_1) = s_1 \) are the arc lengths at the points \( t_0, t_1 \).

Example
The line integral of a vector field along a curve

**Recall:** The *line integral* of $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0, \ s(t_1) = s_1$ are the arc lengths at the points $t_0, t_1$.

**Example**

**Remark:** It is common the notation

$$\hat{\mathbf{r}}' = \mathbf{T},$$
The line integral of a vector field along a curve

**Recall:** The line integral of \( \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), along \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F}(\hat{\mathbf{r}}(s)) \cdot \hat{\mathbf{r}}'(s) \, ds = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \hat{\mathbf{r}}(s) \) is the arc length parametrization of the function \( \mathbf{r} \), and \( s(t_0) = s_0, s(t_1) = s_1 \) are the arc lengths at the points \( t_0, t_1 \).

**Example**

**Remark:** It is common the notation

\( \hat{\mathbf{r}}' = \mathbf{T} \),

since \( \mathbf{T} \) is tangent to the curve and unit, since \( s \) is the curve arc-length parameter.
Work done by a force on a particle

Definition
In the case that \( \mathbf{F} \) is a force on a particle with position function \( \mathbf{r} \) then the line integral
\[
W = \int_{C} \mathbf{F} \cdot d\mathbf{r},
\]
is called the \textit{work} done by the force on the particle.

Example
A projectile of mass \( m \) moving on the surface of Earth.
Work done by a force on a particle

Definition
In the case that $\mathbf{F}$ is a force on a particle with position function $\mathbf{r}$ then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the work done by the force on the particle.

Example

A projectile of mass $m$ moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$. 
Work done by a force on a particle

Definition
In the case that $\mathbf{F}$ is a force on a particle with position function $\mathbf{r}$ then the line integral
\[ W = \int_C \mathbf{F} \cdot d\mathbf{r}, \]
is called the work done by the force on the particle.

Example
A projectile of mass $m$ moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leq 0$ in the first half of the trajectory, and $W \geq 0$ on the second half.
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- Conservative fields.
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.
Gradient fields

Definition
A vector field $\mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n = 2, 3$, is called a gradient field iff there exists a scalar function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, called potential function, such that

$$\mathbf{F} = \nabla f.$$
Gradient fields

Definition
A vector field $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called a *gradient field* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$F = \nabla f.$$ 

Example
A projectile of mass $m$ moving on the surface of Earth.
Gradient fields

Definition
A vector field \( F : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is called a gradient field iff there exists a scalar function \( f : D \subset \mathbb{R}^n \to \mathbb{R} \), called potential function, such that

\[
F = \nabla f.
\]

Example
A projectile of mass \( m \) moving on the surface of Earth.

▶ The movement takes place on a plane, and \( F = \langle 0, -mg \rangle \).
Gradient fields

**Definition**
A vector field \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is called a **gradient field** iff there exists a scalar function \( f : D \subset \mathbb{R}^n \to \mathbb{R} \), called **potential function**, such that

\[
\mathbf{F} = \nabla f.
\]

**Example**

A projectile of mass \( m \) moving on the surface of Earth.

- The movement takes place on a plane, and \( \mathbf{F} = \langle 0, -mg \rangle \).
- \( \mathbf{F} = \nabla f \), with \( f = -mgy \).
Gradient fields

Example

Show that the vector field $\mathbf{F} = \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is a gradient field and find the potential function.

Solution:
The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a gradient field iff there exists a potential function $f$ such that $\mathbf{F} = \nabla f$, that is, $F_1 = \frac{\partial f}{\partial x_1}$, $F_2 = \frac{\partial f}{\partial x_2}$, $F_3 = \frac{\partial f}{\partial x_3}$.

Since $x_i^2 (x_1^2 + x_2^2 + x_3^2)^{3/2} = -\frac{\partial}{\partial x_i} [(x_1^2 + x_2^2 + x_3^2)^{-1/2}]$, then we conclude that $\mathbf{F} = \nabla f$, with $f = -\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$. ◀
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F_1 = \partial_{x_1} f, \quad F_2 = \partial_{x_2} f, \quad F_3 = \partial_{x_3} f.
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Since

\[
\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \left[ (x_1^2 + x_2^2 + x_3^2)^{-1/2} \right], \quad i = 1, 2, 3,
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Gradient fields

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Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- **Conservative fields.**
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.
The line integral of conservative fields

Definition
A vector field \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), is called a **conservative field** iff the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) depend only on the initial and end points of the path.
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A vector field $\mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n = 2, 3$, is called a \textit{conservative field} iff the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depend only on the initial and end points of the path.

Remark: For conservative fields is useful the following notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0, \mathbf{r}_1$, then denote the line integral of a conservative field $\mathbf{F}$ along $C$ as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$
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Remarks:
- This notation emphasizes the end points, not the path.
- This notation is useful only for conservative fields.
- A field \( \mathbf{F} \) is conservative iff \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independent.
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
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- **Equivalence of Gradient and Conservative fields.**
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.
Equivalence of Gradient and Conservative fields

Theorem (Equivalence of gradient and conservative fields)

- A smooth vector field \( \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( n = 2, 3 \), defined on a simply connected domain \( D \subset \mathbb{R}^n \), is a gradient field iff it is a conservative field.

Remarks:
- This is a Fundamental Theorem of Calculus for vector fields.
- A set is simply connected iff it consists of one piece and it contains no holes.
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- Furthermore, if $\mathbf{F} = \nabla f$ and the curve $C \subset D$ starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$, then holds

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$
Equivalent to Gradient and Conservative Fields

**Theorem (Equivalence of gradient and conservative fields)**

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$$\int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0).$$

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Recall: A field $\mathbf{F}$ on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

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$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{r_0}^{r_1} \nabla f \cdot dr$$
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$$\int_C F \cdot dr = \int_{r_0}^{r_1} \nabla f \cdot dr = \int_{t_0}^{t_1} (\nabla f)_{r(t)} \cdot r'(t) dt,$$
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Recall: A field $\mathbf{F}$ on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

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where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. 

(The statement ($\Leftarrow$) is more complicated to prove.)
Equivalence of Gradient and Conservative fields

Recall: A field $F$ on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

$$\int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0).$$

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where $r(t_0) = r_0$ and $r(t_1) = r_1$. Therefore,

$$\int_{r_0}^{r_1} F \cdot dr = \int_{t_0}^{t_1} \frac{d}{dt} [f(r(t))] \ dt.$$
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**Recall:** A field $\mathbf{F}$ on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

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**Proof:** Only ($\Rightarrow$).

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Equivalence of Gradient and Conservative fields

Recall: A field $\mathbf{F}$ on a simply connected domain is a gradient field \textit{iff} it is a conservative field. Furthermore,

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We conclude that $\int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0)$. \qed
Equivalence of Gradient and Conservative fields

Recall: A field \( \mathbf{F} \) on a simply connected domain is a gradient field iff it is a conservative field. Furthermore,

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\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).
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Proof: Only (\( \Rightarrow \)).

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\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f) \bigg|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t) \, dt,
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where \( \mathbf{r}(t_0) = \mathbf{r}_0 \) and \( \mathbf{r}(t_1) = \mathbf{r}_1 \). Therefore,

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We conclude that \( \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0). \)

(The statement (\( \Leftarrow \)) is more complicated to prove.)
Conservative fields and potential functions. (Sect. 16.3)

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- **The line integral conservative fields.**
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The line integral of conservative fields

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \).

Solution: \( I \) is a line integral for a field in \( \mathbb{R}^3 \), since

\( I = \int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r} \),

where \( \mathbf{F} = \langle 2x, 2y, 2z \rangle \) and \( \mathbf{r}_0 = (0,0,0) \) and \( \mathbf{r}_1 = (1,2,3) \).

The field \( \mathbf{F} \) is a gradient field, since \( \mathbf{F} = \nabla f \) with

\( f(x,y,z) = x^2 + y^2 + z^2 \).

Therefore, \( I = \int_{(0,0,0)}^{(1,2,3)} \nabla f \cdot d\mathbf{r} = f(1,2,3) - f(0,0,0) = 1 + 4 + 9 \).

We conclude that \( I = 14 \).
The line integral of conservative fields

Example

Evaluate \[ I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz. \]

Solution: \( I \) is a line integral for a field in \( \mathbb{R}^3 \),
The line integral of conservative fields

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I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.
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Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz. \)

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Introduce \( F = \langle 2x, 2y, 2z \rangle \), \( r_0 = (0, 0, 0) \) and \( r_1 = (1, 2, 3) \),
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I = \int_{r_0}^{r_1} F \cdot dr.
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The field \( \mathbf{F} \) is a gradient field, since \( \mathbf{F} = \nabla f \).
The line integral of conservative fields

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} (2x \, dx + 2y \, dy + 2z \, dz) \).

Solution: \( I \) is a line integral for a field in \( \mathbb{R}^3 \), since

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The field \( \mathbf{F} \) is a gradient field, since \( \mathbf{F} = \nabla f \) with potential \( f(x, y, z) = x^2 + y^2 + z^2 \).
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\[
I = \int_{r_0}^{r_1} F \cdot dr. \quad \text{The field } F \text{ is a gradient field, since } F = \nabla f \text{ with potential } f(x, y, z) = x^2 + y^2 + z^2. \quad \text{That is } f(r) = |r|^2. \quad \text{Therefore,}
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I = \int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0) = |r_1|^2 - |r_0|^2
\]

We conclude that \( I = 14 \). \( \Box \)
The line integral of conservative fields

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} (2x \, dx + 2y \, dy + 2z \, dz). \)

Solution: \( I \) is a line integral for a field in \( \mathbb{R}^3 \), since

\[
I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.
\]

Introduce \( \mathbf{F} = \langle 2x, 2y, 2z \rangle \), \( \mathbf{r}_0 = (0,0,0) \) and \( \mathbf{r}_1 = (1,2,3) \), then

\[
I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.
\]

The field \( \mathbf{F} \) is a gradient field, since \( \mathbf{F} = \nabla f \) with potential \( f(x, y, z) = x^2 + y^2 + z^2 \). That is \( f(\mathbf{r}) = |\mathbf{r}|^2 \). Therefore,

\[
I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2 = (1 + 4 + 9).
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Introduce \( F = \langle 2x, 2y, 2z \rangle \), \( r_0 = (0, 0, 0) \) and \( r_1 = (1, 2, 3) \), then \( I = \int_{r_0}^{r_1} F \cdot dr \). The field \( F \) is a gradient field, since \( F = \nabla f \) with potential \( f(x, y, z) = x^2 + y^2 + z^2 \). That is \( f(r) = |r|^2 \). Therefore,

\[
I = \int_{r_0}^{r_1} \nabla f \cdot dr = f(r_1) - f(r_0) = |r_1|^2 - |r_0|^2 = (1 + 4 + 9).
\]

We conclude that \( I = 14. \) \( \triangle \)
The line integral of conservative fields (Along a path.)

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

Solution:
Consider the path \( C \) given by \( r(t) = \langle 1, 2, 3 \rangle \). Then \( r(0) = \langle 0, 0, 0 \rangle \), and \( r(1) = \langle 1, 2, 3 \rangle \).

We now evaluate \( F = \langle 2x, 2y, 2z \rangle \) along \( r(t) \), that is, \( F(t) = \langle 2t, 4t, 6t \rangle \).

Therefore,
\[
I = \int_0^1 F(t) \cdot r'(t) \, dt = \int_0^1 (2t + 8t + 18t) \, dt = \int_0^1 28t \, dt = 28 \left( t^2 \right) \Big|_0^1 = 14.
\]

We conclude that \( I = 14 \).
The line integral of conservative fields (Along a path.)

Example
Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

Solution: Consider the path \( C \) given by \( \mathbf{r}(t) = \langle 1, 2, 3 \rangle \, t \).
The line integral of conservative fields (Along a path.)

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The line integral of conservative fields (Along a path.)

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Then \( r(0) = \langle 0, 0, 0 \rangle \), and \( r(1) = \langle 1, 2, 3 \rangle \). We now evaluate \( \mathbf{F} = \langle 2x, 2y, 2z \rangle \) along \( r(t) \),
The line integral of conservative fields (Along a path.)

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Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

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\( \mathbf{F} = \langle 2x, 2y, 2z \rangle \) along \( \mathbf{r}(t) \), that is, \( \mathbf{F}(t) = \langle 2t, 4t, 6t \rangle \).
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\[
I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt
\]
The line integral of conservative fields (Along a path.)

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\[
I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt = \int_0^1 \langle 2t, 4t, 6t \rangle \cdot \langle 1, 2, 3 \rangle \, dt
\]
The line integral of conservative fields (Along a path.)

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

Solution: Consider the path \( C \) given by \( \mathbf{r}(t) = \langle 1, 2, 3 \rangle \, t \).
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\[
I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt = \int_{0}^{1} \langle 2t, 4t, 6t \rangle \cdot \langle 1, 2, 3 \rangle \, dt \\
I = \int_{0}^{1} (2t + 8t + 18t) \, dt
\]
The line integral of conservative fields (Along a path.)

Example

Evaluate

\[
I = \int_{(0,0,0)}^{(1,2,3)} (2x \, dx + 2y \, dy + 2z \, dz)
\]

along a straight line.

Solution: Consider the path \( C \) given by \( \mathbf{r}(t) = \langle 1, 2, 3 \rangle \, t \).

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\]

\[
I = \int_{0}^{1} (2t + 8t + 18t) \, dt = \int_{0}^{1} 28t \, dt
\]

We conclude that \( I = 14 \). ▷
The line integral of conservative fields (Along a path.)

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

Solution: Consider the path \( C \) given by \( \mathbf{r}(t) = \langle 1, 2, 3 \rangle t \).
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\]

\[
I = \int_{0}^{1} (2t + 8t + 18t) \, dt = \int_{0}^{1} 28t \, dt = 28 \left( \frac{t^2}{2} \right|_{0}^{1}.
\]

We conclude that \( I = 14 \).
The line integral of conservative fields (Along a path.)

Example

Evaluate \( I = \int_{(0,0,0)}^{(1,2,3)} 2x \, dx + 2y \, dy + 2z \, dz \) along a straight line.

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We conclude that \( I = 14 \).
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- Conservative fields.
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- **Finding the potential of a gradient field.**
- Comments on exact differential forms.
Finding the potential of a gradient field

Theorem (Characterization of gradient fields)

A smooth field \( \mathbf{F} = \langle F_1, F_2, F_3 \rangle \) on a simply connected domain \( D \subset \mathbb{R}^3 \) is a gradient field iff hold

\[
\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
\]
Finding the potential of a gradient field

Theorem (Characterization of gradient fields)

A smooth field \( \mathbf{F} = (F_1, F_2, F_3) \) on a simply connected domain \( D \subset \mathbb{R}^3 \) is a gradient field iff hold

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\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
\]

Proof: Only (\( \Rightarrow \)).
Finding the potential of a gradient field

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Proof: Only \( \Rightarrow \).
Since the vector field \( \mathbf{F} \) is a gradient field,
Finding the potential of a gradient field

**Theorem (Characterization of gradient fields)**

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ on a simply connected domain $D \subset \mathbb{R}^3$ is a gradient field iff hold

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**Proof:** Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is a gradient field, there exists a scalar field $f$ such that $\mathbf{F} = \nabla f$. 
Finding the potential of a gradient field

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A smooth field \( \mathbf{F} = \langle F_1, F_2, F_3 \rangle \) on a simply connected domain \( D \subset \mathbb{R}^3 \) is a gradient field iff hold

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Proof: Only \( \Rightarrow \).

Since the vector field \( \mathbf{F} \) is a gradient field, there exists a scalar field \( f \) such that \( \mathbf{F} = \nabla f \). Then the equations above are satisfied,
Finding the potential of a gradient field

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A smooth field $F = \langle F_1, F_2, F_3 \rangle$ on a simply connected domain $D \subset \mathbb{R}^3$ is a gradient field iff hold

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Proof: Only ($\Rightarrow$).

Since the vector field $F$ is a gradient field, there exists a scalar field $f$ such that $F = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f$$
Finding the potential of a gradient field

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\[
F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f
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Theorem (Characterization of gradient fields)

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Finding the potential of a gradient field

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\[
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\]

(The statement (\( \Leftarrow \)) is more complicated to prove.)
Finding the potential of a gradient field

Example
Show that the field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \) is a gradient field.
Finding the potential of a gradient field

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Show that the field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \) is a gradient field.

Solution: We need to show that the equations in the Theorem above hold,
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Show that the field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \) is a gradient field.

Solution: We need to show that the equations in the Theorem above hold, that is

\[
\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
\]

with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \)
Finding the potential of a gradient field

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\partial_1 F_2
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Finding the potential of a gradient field

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with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z \). This is the case, since

\[
\partial_1 F_2 = 2x,
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Finding the potential of a gradient field

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Finding the potential of a gradient field

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with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \) This is the case, since

\[
\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \quad \partial_2 F_3 = -2z, \quad \partial_3 F_2
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Finding the potential of a gradient field

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\[
\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \\
\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z,
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Finding the potential of a gradient field

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\[
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\]
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\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z,
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\]

with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \) This is the case, since

\[
\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x,
\]

\[
\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z,
\]

\[
\partial_3 F_1 = 0,
\]
Finding the potential of a gradient field

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Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is a gradient field.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.$$ 

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0, \quad \partial_1 F_3.$$
Finding the potential of a gradient field

Example
Show that the field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \) is a gradient field.

Solution: We need to show that the equations in the Theorem above hold, that is

\[
\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
\]

with \( x_1 = x, \ x_2 = y, \) and \( x_3 = z. \) This is the case, since

\[
\partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \\
\partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z, \\
\partial_3 F_1 = 0, \quad \partial_1 F_3 = 0.
\]

\( \triangleright \)
Finding the potential of a gradient field

Example

Find the potential of the gradient field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$. 

Solution:

We know there exists a scalar function $f$ solution of $\mathbf{F} = \nabla f$ implies

$$
\partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.
$$

So,

$$
f = \int 2xy \, dx + g(y, z) \Rightarrow f = x^2y + g(y, z).
$$

Also,

$$
\partial_y f = x^2 + \partial_y g(y, z) = x^2 - z^2 \Rightarrow \partial_y g(y, z) = -z^2.
$$

So,

$$
g(y, z) = -\int z^2 \, dy + h(z) = -z^2y + h(z) \Rightarrow f = x^2y - z^2y + h(z).
$$

Lastly,

$$
\partial_z f = -2yz + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0 \Rightarrow h(z).
$$

Therefore,

$$
f = (x^2 - z^2)y + c_0.
$$
Finding the potential of a gradient field

Example
Find the potential of the gradient field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \).

Solution: We know there exists a scalar function \( f \) solution of

\[
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Finding the potential of a gradient field

Example
Find the potential of the gradient field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

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$$\mathbf{F} = \nabla f \quad \iff \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$
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Finding the potential of a gradient field

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Finding the potential of a gradient field

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Find the potential of the gradient field $F = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

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$$F = \nabla f \iff \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$  

$$f = \int 2xy \, dx + g(y, z) \quad \Rightarrow \quad f = x^2y + g(y, z).$$

$$\partial_y f = x^2 + \partial_y g(y, z) = x^2 - z^2.$$
Finding the potential of a gradient field

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Find the potential of the gradient field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \).

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g(y, z) = -\int z^2 \, dy + h(z)
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Finding the potential of a gradient field

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g(y, z) = -\int z^2 \, dy + h(z) = -z^2y + h(z) \Rightarrow f = x^2y - z^2y + h(z).
\]

\[
\partial_z f = -2zy + \partial_z h(z)
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Finding the potential of a gradient field

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\partial_y f = x^2 + \partial_y g(y, z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y, z) = -z^2.
\]

\[
g(y, z) = - \int z^2 \, dy + h(z) = -z^2y + h(z) \quad \Rightarrow \quad f = x^2y - z^2y + h(z).
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\partial_z f = -2zy + \partial_z h(z) = -2yz \implies \partial_z h(z) = 0
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Finding the potential of a gradient field

Example

Find the potential of the gradient field \( \mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle \).

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g(y, z) = -\int z^2 \, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).
\]

\[
\partial_z f = -2zy + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0 \Rightarrow f = (x^2 - z^2)y + c_0.
\]
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Gradient fields.
- Conservative fields.
- Equivalence of Gradient and Conservative fields.
- The line integral conservative fields.
- Finding the potential of a gradient field.
- Comments on exact differential forms.
Comments on exact differential forms

**Notation:** We call a *differential form* to the integrand in a line integral for a smooth field \( F \),

\[
F \cdot dr = F_x \, dx + F_y \, dy + F_z \, dz.
\]
Comments on exact differential forms

**Notation:** We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$F \cdot dr$$
Comments on exact differential forms

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $F$, that is,

$$F \cdot dr = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle$$
Comments on exact differential forms

**Notation:** We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$
Comments on exact differential forms

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Remark: A differential form is a quantity that can be integrated along a path.
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Remark: A differential form is a quantity that can be integrated along a path.

Definition
A differential form $F \cdot dr = F_x dx + F_y dy + F_z dz$ is called exact iff there exists a scalar function $f$ such that

$$F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$
Comments on exact differential forms

**Notation:** We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

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**Definition**

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function $f$ such that

$$F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$

**Remarks:**

- A differential form $\mathbf{F} \cdot d\mathbf{r}$ is exact iff $\mathbf{F} = \nabla f$. 

▶ In this context an exact differential form is nothing else than another name for a gradient field.
Comments on exact differential forms

Notation: We call a *differential form* to the integrand in a line integral for a smooth field \( \mathbf{F} \), that is,

\[
\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.
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A differential form \( \mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz \) is called *exact* iff there exists a scalar function \( f \) such that

\[
F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.
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Remarks:

- A differential form \( \mathbf{F} \cdot d\mathbf{r} \) is exact iff \( \mathbf{F} = \nabla f \).
- In this context an exact differential form is nothing else than another name for a gradient field.
Comments on exact differential forms

Example

Show that the differential form given below is exact, where
\[ \mathbf{F} \cdot d\mathbf{r} = 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz. \]
Comments on exact differential forms

Example
Show that the differential form given below is exact, where
\[ F \cdot dr = 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz. \]

Solution: We need to do the same calculation we did above:
Comments on exact differential forms

Example
Show that the differential form given below is exact, where
\[ \mathbf{F} \cdot d\mathbf{r} = 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz . \]

Solution: We need to do the same calculation we did above:
Writing \( \mathbf{F} \cdot d\mathbf{r} = F_1 \, dx_1 + F_2 \, dx_2 + F_3 \, dx_3 \), show that
\[ \partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1. \]
with \( x_1 = x, \) \( x_2 = y, \) and \( x_3 = z. \)
Comments on exact differential forms

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Show that the differential form given below is exact, where
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\[ \partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1. \]

with \( x_1 = x \), \( x_2 = y \), and \( x_3 = z \). We showed that this is the case, since
\[ \partial_1 F_2 = 2x, \quad \partial_2 F_1 = 2x, \]
\[ \partial_2 F_3 = -2z, \quad \partial_3 F_2 = -2z, \]
\[ \partial_3 F_1 = 0, \quad \partial_1 F_3 = 0. \]
Comments on exact differential forms

Example
Show that the differential form given below is exact, where
\( \mathbf{F} \cdot d\mathbf{r} = 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz \).

Solution: We need to do the same calculation we did above:
Writing \( \mathbf{F} \cdot d\mathbf{r} = F_1 \, dx_1 + F_2 \, dx_2 + F_3 \, dx_3 \), show that
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\partial_2 F_3 = \partial_3 F_2, \quad \partial_3 F_1 = \partial_1 F_3, \quad \partial_1 F_2 = \partial_2 F_1.
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\partial_3 F_1 = 0, \quad \partial_1 F_3 = 0.
\]
So, there exists \( f \) such that \( \mathbf{F} \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r} \).
Green’s Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green’s Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.
Review: The line integral of a vector field along a curve

Definition

The line integral of a vector-valued function \( \mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), along the curve \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3 \), with arc length function \( s \), is given by

\[
\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,
\]

where \( \mathbf{u} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \), and \( s_0 = s(t_0) \), \( s_1 = s(t_1) \).
Review: The line integral of a vector field along a curve

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The line integral of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow D \subset \mathbb{R}^3$, with arc length function $s$, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt,$$

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Review: The line integral of a vector field along a curve

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where \( \mathbf{u} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \), and \( s_0 = s(t_0), \ s_1 = s(t_1) \).

Example

Remark: Since \( \mathbf{F} = \langle F_x, F_y \rangle \) and \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \), in components,

\[
\int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt
= \int_{t_0}^{t_1} [F_x(t)x'(t) + F_y(t)y'(t)] \, dt.
\]
Review: The line integral of a vector field along a curve

Example
Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$. 

Solution:
Evaluate $\mathbf{F}$ along the curve:
$\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.

Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.

Then evaluate the line integral in components,
$\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} [\mathbf{F}_x(t)x'(t) + \mathbf{F}_y(t)y'(t)] \, dt$,

$\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} \left[ (-\sin(t))(-\sin(t)) + \cos(t) \cos(t) \right] \, dt$,

$\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} \left[ \sin^2(t) + \cos^2(t) \right] \, dt$,

$\Rightarrow \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi$. 

$\triangleleft$
Example
Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

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Example
Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$. 
Review: The line integral of a vector field along a curve

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Evaluate the line integral of \( \mathbf{F} = \langle -y, x \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle \). Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle \). Then evaluate the line integral in components,

\[
\int_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)x'(t) + F_y(t)y'(t) \right] \, dt,
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Review: The line integral of a vector field along a curve

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Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$. Then evaluate the line integral in components,

$$\int_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)x'(t) + F_y(t)y'(t) \right] \, dt,$$

$$\int_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_{0}^{2\pi} \left[ ( - \sin(t) ) ( - \sin(t) ) + \cos(t) \cos(t) \right] \, dt,$$

$\Rightarrow \int_C \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi$. \hfill $\square$
Example
Evaluate the line integral of \( F = \langle -y, x \rangle \) along the loop \( r(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( F \) along the curve: \( F(t) = \langle -\sin(t), \cos(t) \rangle \).
Now compute the derivative vector \( r'(t) = \langle -\sin(t), \cos(t) \rangle \).
Then evaluate the line integral in components,

\[
\oint_C F \cdot u \, ds = \int_{t_0}^{t_1} \left[ F_x(t)x'(t) + F_y(t)y'(t) \right] \, dt,
\]

\[
\oint_C F \cdot u \, ds = \int_{0}^{2\pi} \left[ (-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \right] \, dt,
\]

\[
\oint_C F \cdot u \, ds = \int_{0}^{2\pi} \left[ \sin^2(t) + \cos^2(t) \right] \, dt
\]
Review: The line integral of a vector field along a curve

Example
Evaluate the line integral of $F = \langle -y, x \rangle$ along the loop $r(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $F$ along the curve: $F(t) = \langle -\sin(t), \cos(t) \rangle$. Now compute the derivative vector $r'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C F \cdot u \, ds = \int_{t_0}^{t_1} \left[ F_x(t)x'(t) + F_y(t)y'(t) \right] \, dt,$$

$$\oint_C F \cdot u \, ds = \int_0^{2\pi} \left[ (-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \right] \, dt,$$

$$\oint_C F \cdot u \, ds = \int_0^{2\pi} \left[ \sin^2(t) + \cos^2(t) \right] \, dt \quad \Rightarrow \quad \oint_C F \cdot u \, ds = 2\pi.$$
Review: The flux across a plane loop

Definition
The flux of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \rightarrow \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbf{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.
Review: The flux across a plane loop

Definition
The flux of a vector field \( \mathbf{F} : \{ z = 0 \} \subset \mathbb{R}^3 \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) is given by

\[
\mathbf{F} = \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the unit outer normal vector to the curve inside the plane \( \{ z = 0 \} \).

Example

![Diagram of plane loop with vectors and normal vector n]
Review: The flux across a plane loop

Definition
The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbf{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$,

Example
Review: The flux across a plane loop

Definition
The flux of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbf{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$, $\mathbf{n}_{\mathbb{R}^2} = \mathbf{r}'(t)$,

Example
Review: The flux across a plane loop

Definition
The **flux** of a vector field \( \mathbf{F} : \{ z = 0 \} \subset \mathbb{R}^3 \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) is given by

\[
\mathbf{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the unit outer normal vector to the curve inside the plane \( \{ z = 0 \} \).

Remark: Since \( \mathbf{F} = \langle F_x, F_y, 0 \rangle \),
\[
\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle, \quad ds = |\mathbf{r}'(t)| \, dt,
\]

Example
Review: The flux across a plane loop

Definition
The flux of a vector field \( F : \{ z = 0 \} \subset \mathbb{R}^3 \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) along a closed plane loop \( r : [t_0, t_1] \subset \mathbb{R} \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) is given by

\[
F = \oint_C F \cdot n \, ds,
\]

where \( n \) is the unit outer normal vector to the curve inside the plane \( \{ z = 0 \} \).

Example

Remark: Since \( F = \langle F_x, F_y, 0 \rangle \),
\( r(t) = \langle x(t), y(t), 0 \rangle \), \( ds = |r'(t)| \, dt \), and
\[
n = \frac{1}{|r'|} \langle y'(t), -x'(t), 0 \rangle,
\]
Definition
The *flux* of a vector field \( \mathbf{F} : \{ z = 0 \} \subset \mathbb{R}^3 \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) along a closed plane loop \( \mathbf{r} : [t_0, t_1] \subset \mathbb{R} \rightarrow \{ z = 0 \} \subset \mathbb{R}^3 \) is given by

\[
\mathbf{F} = \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is the unit outer normal vector to the curve inside the plane \( \{ z = 0 \} \).

Example

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]

Remark: Since \( \mathbf{F} = \langle F_x, F_y, 0 \rangle \), \( \mathbf{r}(t) = \langle x(t), y(t), 0 \rangle \), \( ds = |\mathbf{r}'(t)| \, dt \), and

\[
\mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle,
\]
in components,

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]
Example
Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$. 
Review: The flux across a plane loop

Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \). Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Example
Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$. 

$\Rightarrow$
Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the normal vector \( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, \( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \).
Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \). Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \). Now compute the normal vector \( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, \( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \). Evaluate the flux integral in components,

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt,
\]
Review: The flux across a plane loop

Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the normal vector \( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, \( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \).
Evaluate the flux integral in components,
\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt,
\]
\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ -\sin(t)\cos(t) - \cos(t)(-\sin(t)) \right] \, dt,
\]
Review: The flux across a plane loop

Example
Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop 
\( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: 
\( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the derivative vector 
\( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the normal vector 
\( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, 
\( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \).
Evaluate the flux integral in components,

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt,
\]

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} \left[ -\sin(t) \cos(t) - \cos(t)(-\sin(t)) \right] \, dt,
\]

\[
\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} 0 \, dt
\]
Review: The flux across a plane loop

Example

Evaluate the flux of \( \mathbf{F} = \langle -y, x, 0 \rangle \) along the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( t \in [0, 2\pi] \).

Solution: Evaluate \( \mathbf{F} \) along the curve: \( \mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the derivative vector \( \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle \).
Now compute the normal vector \( \mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle \), that is, \( \mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle \). Evaluate the flux integral in components,

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\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ -\sin(t)\cos(t) - \cos(t)(-\sin(t)) \right] \, dt,
\]

\[
\oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = \int_{0}^{2\pi} 0 \, dt \quad \Rightarrow \quad \oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = 0.
\]
Green’s Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- **Green’s Theorem on a plane.**
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.
Green’s Theorem on a plane

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F} \cdot \mathbf{u} \, ds$ of the field $\mathbf{F} = \langle F_x, F_y \rangle$ along a loop $C$ enclosing a region $R \subset \mathbb{R}^2$ and given by the function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [t_0, t_1]$ and with unit tangent vector $\mathbf{u}$, satisfies that

$$\int_{t_0}^{t_1} \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] \, dt = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy.$$
Green’s Theorem on a plane

Theorem (Circulation-tangential form)

The counterclockwise line integral \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds \) of the field \( \mathbf{F} = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \subset \mathbb{R}^2 \) and given by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit tangent vector \( \mathbf{u} \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] \, dt = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Theorem (Circulation-tangential form)

The counterclockwise line integral \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds \) of the field \( \mathbf{F} = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \in \mathbb{R}^2 \) and given by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit tangent vector \( \mathbf{u} \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) x'(t) + F_y(t) y'(t) \right] \, dt = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy.
\]

Equivalently,

\[
\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field \( \mathbf{F} = \langle -y, x \rangle \)
and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi \).

Now we compute the double integral \( I = \iint_{R} (\partial_x F_y - \partial_y F_x) \, dx \, dy \)
and we verify that we get the same result, \( 2\pi \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi$.

Now we compute the double integral $I = \iint_{R} (\partial_{x} F_{y} - \partial_{y} F_{x}) \, dx \, dy$ and we verify that we get the same result, $2\pi$.

$$I = \iint_{R} [1 - (-1)] \, dx \, dy$$
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field $F = \langle -y, x \rangle$ and the loop $r(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C F \cdot u \, ds = 2\pi$.

Now we compute the double integral $I = \iint_{R} (\partial_x F_y - \partial_y F_x) \, dx \, dy$
and we verify that we get the same result, $2\pi$.

\[
I = \iint_{R} [1 - (-1)] \, dx \, dy = 2 \iint_{R} \, dx \, dy
\]
Green’s Theorem on a plane

Example

Verify Green’s Theorem tangential form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi \).

Now we compute the double integral \( I = \iint_R \left( \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \right) \, dx \, dy \)
and we verify that we get the same result, \( 2\pi \).

\[
I = \iint_R \left[ 1 - (-1) \right] \, dx \, dy = 2 \iint_R \, dx \, dy = 2 \int_0^{2\pi} \int_0^1 r \, dr \, d\theta
\]
Green’s Theorem on a plane

Example

Verify Green’s Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_{C} \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi$.

Now we compute the double integral $I = \iint_{R} (\partial_{x} F_{y} - \partial_{y} F_{x}) \, dx \, dy$ and we verify that we get the same result, $2\pi$.

$$I = \iint_{R} [1 - (-1)] \, dx \, dy = 2 \iint_{R} \, dx \, dy = 2 \int_{0}^{2\pi} \int_{0}^{1} r \, dr \, d\theta$$

$$I = 2(2\pi) \left( \frac{r^2}{2} \right)_{0}^{1}$$
Green's Theorem on a plane

Example

Verify Green's Theorem tangential form for the field $F = \langle -y, x \rangle$ and the loop $r(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\int_C F \cdot u \, ds = 2\pi$.

Now we compute the double integral $I = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy$ and we verify that we get the same result, $2\pi$.

\[
I = \iint_R [1 - (-1)] \, dx \, dy = 2 \iint_R \, dx \, dy = 2 \int_0^{2\pi} \int_0^1 r \, dr \, d\theta
\]

\[
I = 2(2\pi) \left( \frac{r^2}{2} \bigg|_0^1 \right) \quad \Rightarrow \quad I = 2\pi.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint \mathbf{F} \cdot \mathbf{u} \, ds = 2\pi$.

Now we compute the double integral $I = \iint_{R} (\partial_x F_y - \partial_y F_x) \, dx \, dy$ and we verify that we get the same result, $2\pi$.

$$I = \iint_{R} [1 - (-1)] \, dx \, dy = 2 \iint_{R} \, dx \, dy = 2 \int_{0}^{2\pi} \int_{0}^{1} r \, dr \, d\theta$$

$$I = 2(2\pi) \left( \frac{r^2}{2} \right) \bigg|_{0}^{1} \quad \Rightarrow \quad I = 2\pi.$$

We verified that $\oint \mathbf{F} \cdot \mathbf{u} \, ds = \iint_{R} (\partial_x F_y - \partial_y F_x) \, dx \, dy = 2\pi$. \triangleleft
Green’s Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- **Green’s Theorem on a plane.**
  - Circulation-tangential form.
  - **Flux-normal form.**
- Tangential and normal forms equivalence.
Green’s Theorem on a plane

Theorem (Flux-normal form)

The counterclockwise flux integral \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) of the field \( \mathbf{F} = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \in \mathbb{R}^2 \) and given by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit normal vector \( \mathbf{n} \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] \, dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Theorem (Flux-normal form)

The counterclockwise flux integral \( \oint_C F \cdot n \, ds \) of the field \( F = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \in \mathbb{R}^2 \) and given by the function \( r(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit normal vector \( n \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] \, dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Theorem (Flux-normal form)

The counterclockwise flux integral \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) of the field \( \mathbf{F} = \langle F_x, F_y \rangle \) along a loop \( C \) enclosing a region \( R \in \mathbb{R}^2 \) and given by the function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for \( t \in [t_0, t_1] \) and with unit normal vector \( \mathbf{n} \), satisfies that

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] \, dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]

Equivalently,

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = 0$. 

\[ I = \int_{R} \left[ \partial_{x} \mathbf{F} \right]_{x} + \partial_{y} \mathbf{F} \right]_{y} \, dx \, dy = \int_{R} 0 \, dx \, dy = 0 \]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $F = \langle -y, x \rangle$ and the loop $r(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C F \cdot n \, ds = 0$.

Now we compute the double integral $I = \iint_R (\partial_x F_x + \partial_y F_y) \, dx \, dy$ and we verify that we get the same result, 0.
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle -y, x \rangle \) and the loop \( \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \) for \( t \in [0, 2\pi] \).

Solution: Recall: We found that \( \oint_C \not{\mathbf{F} \cdot n} \not{ds} = 0 \).

Now we compute the double integral \( I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) dx \, dy \) and we verify that we get the same result, 0.

\[
I = \iint_R \left[ \partial_x (-y) + \partial_y (x) \right] dx \, dy
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = 0$.

Now we compute the double integral
$$I = \int \int_{R} \left( \partial_{x} F_{x} + \partial_{y} F_{y} \right) \, dx \, dy$$
and we verify that we get the same result, 0.

$$I = \int \int_{R} \left[ \partial_{x}(-y) + \partial_{y}(x) \right] \, dx \, dy = \int \int_{R} 0 \, dx \, dy = 0.$$
Example
Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$.

Now we compute the double integral $I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \left[ \partial_x (-y) + \partial_y (x) \right] \, dx \, dy = \iint_R 0 \, dx \, dy = 0.$$ 

We verified that $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy = 0$. \triangleleft
Green’s Theorem on a plane

Example

Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).
Green’s Theorem on a plane

Example

Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: We start with the line integral

\[
\oint_{c} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi], \ a > 0. \)

Solution: We start with the line integral

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\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]

It is simple to see that \( \mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle, \)
**Green’s Theorem on a plane**

**Example**
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \)
and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

**Solution:** We start with the line integral
\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]
It is simple to see that \( \mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle \),
and also that \( \mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$
\oint \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
$$

It is simple to see that $\mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle$, and also that $\mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle$.

Therefore,

$$
\oint \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} \left[ 2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] \, dt.
$$
Green’s Theorem on a plane

Example

Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi], \ a > 0 \).

Solution: We start with the line integral

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]

It is simple to see that \( \mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle \), and also that \( \mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle \).

Therefore,

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] \, dt,
\]

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \frac{1}{2} (1 + \cos(2t)) - 3a^2 \frac{1}{2} (1 - \cos(2t)) \right] \, dt.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: We start with the line integral

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.
\]

It is simple to see that \( \mathbf{F}(t) = \langle 2a \cos(t), -3a \sin(t) \rangle \), and also that \( \mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle \).

Therefore, \( \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] \, dt \),

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} \left[ 2a^2 \frac{1}{2} (1 + \cos(2t)) - 3a^2 \frac{1}{2} (1 - \cos(2t)) \right] \, dt.
\]

Since \( \int_{0}^{2\pi} \cos(2t) \, dt = 0 \),
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $F = \langle 2x, -3y \rangle$ and the loop $r(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi], \ a > 0$.

Solution: We start with the line integral

$$\oint_C F \cdot n \, ds = \int_{t_0}^{t_1} \left[ F_x(t)y'(t) - F_y(t)x'(t) \right] \, dt.$$ 

It is simple to see that $F(t) = \langle 2a \cos(t), -3a \sin(t) \rangle$, and also that $r'(t) = \langle -a \sin(t), a \cos(t) \rangle$.

Therefore, $\oint_C F \cdot n \, ds = \int_0^{2\pi} \left[ 2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] \, dt$,

$$\oint_C F \cdot n \, ds = \int_0^{2\pi} \left[ 2a^2 \frac{1}{2} (1 + \cos(2t)) - 3a^2 \frac{1}{2} (1 - \cos(2t)) \right] \, dt.$$ 

Since $\int_0^{2\pi} \cos(2t) \, dt = 0$, we conclude $\oint_C F \cdot n \, ds = -\pi a^2$. 
Example
Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2$. 
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: Recall: \( \int_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2 \).

Now we compute the double integral \( I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy \).
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi], \ a > 0. \)

Solution: Recall: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2. \)

Now we compute the double integral \( I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy. \)

\[ I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy \]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2 \).

Now we compute the double integral \( I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy \).

\[
I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy.
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

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\]

\[
I = -\iint_R \, dx \, dy
\]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field $F = \langle 2x, -3y \rangle$ and the loop $r(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\int_C F \cdot n \, ds = -\pi a^2$.

Now we compute the double integral $I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy$.

\[ I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy. \]

\[ I = -\iint_R dx \, dy = -\int_0^{2\pi} \int_0^a r \, dr \, d\theta \]
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

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\[
I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy.
\]

\[
I = -\int_0^{2\pi} \int_0^a r \, dr \, d\theta = -2\pi \left( \frac{r^2}{2} \right)_0^a
\]
Green’s Theorem on a plane

Example

Verify Green’s Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2$.

Now we compute the double integral $I = \iint_R \left( \partial_x F_x + \partial_y F_y \right) \, dx \, dy$.

$$I = \iint_R \left[ \partial_x (2x) + \partial_y (-3y) \right] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy.$$  

$$I = -\iint_R \, dx \, dy = -\int_0^{2\pi} \int_0^a r \, dr \, d\theta = -2\pi \left( \frac{r^2}{2} \bigg|_0^a \right) = -\pi a^2.$$
Green’s Theorem on a plane

Example
Verify Green’s Theorem normal form for the field \( \mathbf{F} = \langle 2x, -3y \rangle \) and the loop \( \mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle \) for \( t \in [0, 2\pi] \), \( a > 0 \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = -\pi a^2 \).

Now we compute the double integral \( I = \iint_R (\partial_x F_x + \partial_y F_y) \, dx \, dy \).

\[
I = \iint_R [\partial_x(2x) + \partial_y(-3y)] \, dx \, dy = \iint_R (2 - 3) \, dx \, dy.
\]

\[
I = -\int_0^{2\pi} \int_0^a r \, dr \, d\theta = -2\pi \left( \frac{r^2}{2} \right)_{r=0}^{r=a} = -\pi a^2.
\]

Hence, \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (\partial_x F_x + \partial_y F_y) \, dx \, dy = -\pi a^2 \).
Green’s Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green’s Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.
Tangential and normal forms equivalence

Theorem

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.
Tangential and normal forms equivalence

**Theorem**

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

**Proof:** Green’s Theorem in tangential form for \( \mathbf{F} = \langle \hat{F}_x, \hat{F}_y \rangle \) says

\[
\int_{t_0}^{t_1} \left[ \hat{F}_x(t) x'(t) + \hat{F}_y(t) y'(t) \right] dt = \iint_R \left( \partial_x \hat{F}_y - \partial_y \hat{F}_x \right) dx dy.
\]
Tangential and normal forms equivalence

**Theorem**

*The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.*

**Proof:** Green’s Theorem in tangential form for \( \hat{F} = \langle \hat{F}_x, \hat{F}_y \rangle \) says

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\]

If \( \hat{F} = \langle \hat{F}_x, \hat{F}_y \rangle \) and \( F = \langle F_x, F_y \rangle \) are related by \( \hat{F}_x = -F_y \) and \( \hat{F}_y = F_x \),
Tangential and normal forms equivalence

**Theorem**

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

**Proof:** Green’s Theorem in tangential form for \( \mathbf{\hat{F}} = \langle \mathbf{\hat{F}}_x, \mathbf{\hat{F}}_y \rangle \) says

\[
\int_{t_0}^{t_1} \left[ \mathbf{\hat{F}}_x(t)x'(t) + \mathbf{\hat{F}}_y(t)y'(t) \right] \, dt = \iint_R \left( \partial_x \mathbf{\hat{F}}_y - \partial_y \mathbf{\hat{F}}_x \right) \, dx \, dy.
\]

If \( \mathbf{\hat{F}} = \langle \mathbf{\hat{F}}_x, \mathbf{\hat{F}}_y \rangle \) and \( \mathbf{F} = \langle F_x, F_y \rangle \) are related by \( \mathbf{\hat{F}}_x = -F_y \) and \( \mathbf{\hat{F}}_y = F_x \), then the equation above for \( \mathbf{\hat{F}} \) written in terms of \( \mathbf{F} \) is

\[
\int_{t_0}^{t_1} \left[ -F_y(t)x'(t) + F_x(t)y'(t) \right] \, dt = \iint_R \left( \partial_x F_x - \partial_y (-F_y) \right) \, dx \, dy,
\]
Tangential and normal forms equivalence

**Theorem**

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

**Proof:** Green’s Theorem in tangential form for \( \hat{F} = \langle \hat{F}_x, \hat{F}_y \rangle \) says

\[
\int_{t_0}^{t_1} \left[ \hat{F}_x(t) x'(t) + \hat{F}_y(t) y'(t) \right] dt = \iint_R \left( \partial_x \hat{F}_y - \partial_y \hat{F}_x \right) dx \, dy.
\]

If \( \hat{F} = \langle \hat{F}_x, \hat{F}_y \rangle \) and \( F = \langle F_x, F_y \rangle \) are related by \( \hat{F}_x = -F_y \) and \( \hat{F}_y = F_x \), then the equation above for \( \hat{F} \) written in terms of \( F \) is

\[
\int_{t_0}^{t_1} \left[ -F_y(t) x'(t) + F_x(t) y'(t) \right] dt = \iint_R \left( \partial_x F_x - \partial_y (-F_y) \right) dx \, dy,
\]

so,

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) dx \, dy,
\]
Tangential and normal forms equivalence

**Theorem**

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

**Proof:** Green’s Theorem in tangential form for \( \hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle \) says

\[
\int_{t_0}^{t_1} \left[ \hat{F}_x(t) x'(t) + \hat{F}_y(t) y'(t) \right] dt = \iint_R \left( \partial_x \hat{F}_y - \partial_y \hat{F}_x \right) dx \, dy.
\]

If \( \hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle \) and \( \mathbf{F} = \langle F_x, F_y \rangle \) are related by \( \hat{F}_x = -F_y \) and \( \hat{F}_y = F_x \), then the equation above for \( \hat{\mathbf{F}} \) written in terms of \( \mathbf{F} \) is

\[
\int_{t_0}^{t_1} \left[ -F_y(t) x'(t) + F_x(t) y'(t) \right] dt = \iint_R \left( \partial_x F_x - \partial_y (-F_y) \right) dx \, dy,
\]

so,

\[
\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) dx \, dy,
\]

which is Green’s Theorem in normal form for \( \mathbf{F} \).
Theorem
The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green’s Theorem in tangential form for $\hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle$ says
$$\int_{t_0}^{t_1} \left[ \hat{F}_x(t) x'(t) + \hat{F}_y(t) y'(t) \right] dt = \iint_R \left( \partial_x \hat{F}_y - \partial_y \hat{F}_x \right) dx \, dy.$$  

If $\hat{\mathbf{F}} = \langle \hat{F}_x, \hat{F}_y \rangle$ and $\mathbf{F} = \langle F_x, F_y \rangle$ are related by $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$, then the equation above for $\hat{\mathbf{F}}$ written in terms of $\mathbf{F}$ is
$$\int_{t_0}^{t_1} \left[ -F_y(t) x'(t) + F_x(t) y'(t) \right] dt = \iint_R \left( \partial_x F_x - \partial_y (-F_y) \right) dx \, dy,$$

so, $\int_{t_0}^{t_1} \left[ F_x(t) y'(t) - F_y(t) x'(t) \right] dt = \iint_R \left( \partial_x F_x + \partial_y F_y \right) dx \, dy$,

which is Green’s Theorem in normal form for $\mathbf{F}$. The converse implication is proved in the same way.
Using Green’s Theorem

Example
Use Green’s Theorem to find the counterclockwise circulation of the field \( \mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle \) along the curve \( C \) that is the triangle bounded by \( y = 0, \ x = 3 \) and \( y = x \).
Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field \( \mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle \) along the curve \( C \) that is the triangle bounded by \( y = 0, \ x = 3 \) and \( y = x \).

Solution: Recall: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \right) \, dx \, dy. \]
Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve $C$ that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy$.

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y) \, dx \, dy \]

⇒ $\oint_C \mathbf{F} \cdot d\mathbf{r} = 9$.
Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve $C$ that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x) \, dx \, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y) \, dx \, dy = \int_0^3 \int_0^x (2x - 2y) \, dy \, dx,$$
Using Green’s Theorem

Example
Use Green’s Theorem to find the counterclockwise circulation of the field \( \mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle \) along the curve \( C \) that is the triangle bounded by \( y = 0, \ x = 3 \) and \( y = x \).

Solution: Recall: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy \).

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\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y) \, dx \, dy = \int_0^3 \int_0^x (2x - 2y) \, dy \, dx,
\]

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[ 2x \left( y \bigg|_0^x \right) - \left( y^2 \bigg|_0^x \right) \right] \, dx
\]
Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle y^2 - x^2, x^2 + y^2 \rangle$ along the curve $C$ that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \partial_x F_y - \partial_y F_x \right) \, dx \, dy$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y) \, dx \, dy = \int_0^3 \int_0^x (2x - 2y) \, dy \, dx,$$

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\]

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2 \, dx = \frac{x^3}{3} \bigg|_0^3
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Using Green’s Theorem

Example

Use Green’s Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve $C$ that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

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\]

\[
\begin{align*}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \left[ 2x \left( y \bigg|_0^x \right) - \left( y^2 \bigg|_0^x \right) \right] \, dx = \int_0^3 \left( 2x^2 - x^2 \right) \, dx, \\
&= \int_0^3 x^2 \, dx = \frac{x^3}{3} \bigg|_0^3 = 9.
\end{align*}
\]