Double integrals on regions (Sect. 15.2)

- Review: Fubini’s Theorem on rectangular domains.
- Fubini’s Theorem on non-rectangular domains.
  - Type I: Domain functions $y(x)$.
  - Type II: Domain functions $x(y)$.
- Finding the limits of integration.
Review: Fubini’s Theorem on rectangular domains

Theorem

If \( f : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( R = [a, b] \times [c, d] \), then

\[
\int\int_R f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx,
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\[
= \int_c^d \int_a^b f(x, y) \, dx \, dy.
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**Remark:** Fubini result says that double integrals can be computed doing two one-variable integrals.
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Remark: On a rectangle is simple to switch the order of integration in double integrals of continuous functions.
Review: Fubini’s Theorem on rectangular domains

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Fubini’s Theorem on Type I domains, $y(x)$

**Theorem**

*If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in $D$, then hold (Type I):

If $D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], \ y \in [g_1(x), g_2(x)] \}$, with $g_1, g_2$ continuous functions on $[a, b]$, then

$$\int\int_D f(x, y) \, dx \, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$*
Fubini’s Theorem on Type I domains, \( y(x) \)

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Fubini’s Theorem on Type II domains, $x(y)$

Theorem
If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in $D$, then hold (Type II):
If $D = \{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)], \ y \in [c, d]\}$, with $h_1$, $h_2$ continuous functions on $[c, d]$, then

$$\int \int_D f(x, y) \, dx \, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.$$
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![Diagram of Type II domain with integration boundaries](image)
Fubini’s Theorem on Type II domains, $x(y)$

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Summary: Fubini’s Theorem on non-rectangular domains

Theorem
If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in $D$, then hold:

(a) (Type I) If $D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], \ y \in [g_1(x), g_2(x)]\}$, with $g_1, g_2$ continuous functions on $[a, b]$, then

$$\int\int_D f(x, y) \, dx \, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$

(b) (Type II) If $D = \{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)], \ y \in [c, d]\}$, with $h_1, h_2$ continuous functions on $[c, d]$, then

$$\int\int_D f(x, y) \, dx \, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.$$
Example

Find the integral of $f(x, y) = x^2 + y^2$, on the domain $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}$. 
A double integral on a Type I domain

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Solution:
This is a Type I domain,
A double integral on a Type I domain

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Solution:
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y = g_1(x) = x^2,
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and upper boundary
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I = \int_0^1 \left[ x^2 \left( y \right|_{x^2}^x \right) + \left( \frac{y^3}{3} \right|_{x^2}^x \right] \, dx.
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Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain \( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\} \).

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I = \int_0^1 \left[ x^2 (x - x^2) + \frac{1}{3} (x^3 - x^6) \right] \, dx.
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Find the integral of $f(x, y) = x^2 + y^2$, on the domain $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}$.

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Solution: Recall: \( I = \int_0^1 \left[ x^2(x - x^2) + \frac{1}{3}(x^3 - x^6) \right] dx \).

\[
I = \int_0^1 \left[ x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6 \right] dx
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I = \frac{1}{3} - \frac{1}{5} - \frac{1}{21}
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A double integral on a Type I domain

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I = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{(3)(5)(7)}.
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A double integral on a Type I domain

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\[
I = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{35} = \frac{9}{(3)(5)(7)}.
\]

We conclude: 
\[
\iint_D f(x, y) \, dx \, dy = \frac{3}{35}.
\]
Summary: Fubini’s Theorem on non-rectangular domains

Theorem
If \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( D \), then hold:

(a) **(Type I)** If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y \in [g_1(x), g_2(x)]\} \), with \( g_1, g_2 \) continuous functions on \([a, b]\), then

\[
\int \int_D f(x, y) \, dx \, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
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(b) **(Type II)** If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)], y \in [c, d]\} \), with \( h_1, h_2 \) continuous functions on \([c, d]\), then

\[
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\]
A double integral on a Type II domain

Example

Find the integral of $f(x, y) = x^2 + y^2$ on the domain $D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \ 0 \leq y \leq 1\}$. 
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Example

Find the integral of \( f(x, y) = x^2 + y^2 \) on the domain
\[ D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1\}. \]

Solution:

This is a Type II domain,
A double integral on a Type II domain

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Solution:
This is a Type II domain, with left boundary

$$x = h_1(y) = y,$$
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\[ x = h_1(y) = y, \]

and right boundary

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Remark:
This domain is both Type I and Type II:
A double integral on a Type II domain

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Remark:

This domain is both Type I and Type II: \( y = x^2 \iff x = \sqrt{y} \).
A double integral on a Type I domain

Example

Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain
\( D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \ 0 \leq y \leq 1\} \).

Solution: \( I = \iint_D f(x, y) \, dx \, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \)
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\[
I = \int_0^1 \int_0^{\sqrt{y}} (x^2 + y^2) \, dx \, dy,
\]

\[
I = \int_0^1 \left[ \left( \frac{x^3}{3} \right)_0^{\sqrt{y}} \right] + y^2 \left( x \right)_0^{\sqrt{y}} \right] dy,
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\]

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I = \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] \, dy.
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I = \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] dy.
\]

\[
I = \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy,
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Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain \( D = \{ (x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \ 0 \leq y \leq 1 \} \).

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\]
\[
I = \left. \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy \right|_0^1,
\]
\[
I = \left[ \frac{1}{3} \frac{2}{5} y^{5/2} - \frac{1}{3} \frac{4}{4} + \frac{2}{7} y^{7/2} - \frac{4}{4} \right]_0^1,
\]
\[
I = \frac{3}{35}.
\]
A double integral on a Type I domain

Example
Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain
\( D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1\} \).

Solution: 
\[
I = \int_0^1 \left[ \frac{1}{3} \left( y^{3/2} - y^3 \right) + y^2 \left( y^{1/2} - y \right) \right] dy.
\]
\[
I = \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy,
\]
\[
I = \left[ \frac{1}{3} \frac{2}{5} y^{5/2} - \frac{1}{3} \frac{y^4}{4} + \frac{2}{7} y^{7/2} - \frac{y^4}{4} \right]_0^1,
\]
\[
I = \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4}
\]

We conclude
\[
\int\int_D f(x, y) \, dx \, dy = \frac{3}{35}.
\]
A double integral on a Type I domain

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Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain 
\( D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1\} \).

Solution: 
\[
I = \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] dy.
\]
\[
I = \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy,
\]
\[
I = \left[ \frac{1}{3} \cdot \frac{2}{5} y^{5/2} - \frac{1}{3} \cdot \frac{4}{4} + 2 \cdot \frac{2}{7} y^{7/2} - \frac{4}{4} \right]_0^1,
\]
\[
I = \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{(3)(5)(7)}.
\]
A double integral on a Type I domain

Example

Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain
\[ D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1 \}. \]

Solution: 
\[
I = \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] dy.
\]
\[
I = \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] dy,
\]
\[
I = \left[ \frac{1}{3} \frac{2}{5} y^{5/2} - \frac{1}{3} \frac{4}{4} y^4 + \frac{2}{7} y^{7/2} - \frac{4}{4} y^4 \right]_0^1,
\]
\[
I = \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{(3)(5)(7)}.
\]

We conclude 
\[
\iint_D f(x, y) \, dx \, dy = \frac{3}{35}.
\]
Domains Type I and Type II

Summary: We have shown that a double integral of a function $f$ on the domain $D$ given in the pictures below holds,

$$
\int \int_D f(x, y) \, dx \, dy = \int_0^1 \int_{x^2}^x f(x, y) \, dy \, dx = \int_0^1 \int_y \sqrt{y} f(x, y) \, dx \, dy.
$$
Double integrals on regions (Sect. 15.2)

- Review: Fubini’s Theorem on rectangular domains.
- Fubini’s Theorem on non-rectangular domains.
  - Type I: Domain functions $y(x)$.
  - Type II: Domain functions $x(y)$.
- Finding the limits of integration.
Domains Type I and Type II

Example

Find the limits of integration of $\int\int_{D} f(x, y) \, dx \, dy$ in the domain $D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ when $D$ is considered first as Type I and then as Type II.
Domains Type I and Type II

Example

Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain \( D = \{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \} \) when \( D \) is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \).
Domains Type I and Type II

Example

Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain

\[ D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\} \]

when \( D \) is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \).

So, the boundary as Type I is given by

\[ y = -2 \sqrt{1 - \frac{x^2}{9}} \]
Domains Type I and Type II

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Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain

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Solution: The boundary is the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \).
So, the boundary as Type I is given by

\[ y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \]
Domains Type I and Type II

Example

Find the limits of integration of \( \int \int_{D} f(x, y) \, dx \, dy \) in the domain

\[ D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \} \]

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So, the boundary as Type I is given by

\[
y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}}
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Domains Type I and Type II

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Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain

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  y = -2 \sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2 \sqrt{1 - \frac{x^2}{9}} = g_2(x).
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Domains Type I and Type II

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$D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ when $D$ is considered first as Type I and then as Type II.

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$$y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).$$

The boundary as Type II is given by

$$x = -3\sqrt{1 - \frac{y^2}{4}}.$$
Domains Type I and Type II

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Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain

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\]

The boundary as Type II is given by

\[
x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y),
\]
Domains Type I and Type II

Example

Find the limits of integration of $\int \int_D f(x, y) \, dx \, dy$ in the domain $D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ when $D$ is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
So, the boundary as Type I is given by

$$y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).$$

The boundary as Type II is given by

$$x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y), \quad x = 3\sqrt{1 - \frac{y^2}{4}}.$$
Domains Type I and Type II

Example

Find the limits of integration of \( \int \int_D f(x, y) \, dx \, dy \) in the domain 

\[
D = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}
\]

when \( D \) is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \).

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y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).
\]

The boundary as Type II is given by

\[
x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y), \quad x = 3\sqrt{1 - \frac{y^2}{4}} = h_2(y).
\]
Domains Type I and Type II

Example

Reverse the order of integration in \( \int_0^1 \int_1^{e^x} dy \, dx \).

\[ \int_0^1 \int_1^{e^x} dy \, dx = \int_1^{e^1} \int_{\ln(y)}^1 dx \, dy. \]
Domains Type I and Type II

Example

Reverse the order of integration in \[ \int_0^1 \int_1^{e^x} dy \, dx. \]

Solution:
This integral is written as Type I,
Example
Reverse the order of integration in \( \int_0^1 \int_1^{e^x} dy \, dx \).

Solution:
This integral is written as Type I, since we first integrate on vertical intervals \([1, e^x]\),
Domains Type I and Type II

Example

Reverse the order of integration in \( \int_{0}^{1} \int_{1}^{e^{x}} dy \, dx \).

Solution:
This integral is written as Type I, since we first integrate on vertical intervals \([1, e^{x}]\), with boundaries \( y = e^{x} \),

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Reverse the order of integration in \( \int_0^1 \int_1^{e^x} dy \, dx \).

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Domains Type I and Type II

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Solution:

This integral is written as Type I, since we first integrate on vertical intervals \([1, e^x]\), with boundaries \(y = e^x\), \(y = 1\), while \(x \in [0, 1]\).
Domains Type I and Type II

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Invert the first equation
Domains Type I and Type II

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Solution:

This integral is written as Type I, since we first integrate on vertical intervals \([1, e^x]\), with boundaries \(y = e^x\), \(y = 1\), while \(x \in [0, 1]\).

Invert the first equation and from the figure we get the left and right boundaries:

\[ x = \ln(y), \]
Domains Type I and Type II

Example

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Invert the first equation and from the figure we get the left and right boundaries:

\[
x = \ln(y), \quad x = 1, \quad \text{with} \quad y \in [1, e].
\]
Domains Type I and Type II

Example

Reverse the order of integration in \( \int_{0}^{1} \int_{1}^{e^{x}} dy \, dx \).

Solution:
This integral is written as Type I, since we first integrate on vertical intervals \([1, e^{x}]\), with boundaries \(y = e^{x}, \ y = 1\), while \(x \in [0, 1]\).

Invert the first equation and from the figure we get the left and right boundaries:
\[ x = \ln(y), \quad x = 1, \quad \text{with} \quad y \in [1, e]. \]

Therefore, we conclude that
\[ \int_{0}^{1} \int_{1}^{e^{x}} dy \, dx = \int_{1}^{e} \int_{\ln(y)}^{1} dx \, dy. \]
Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
- Average value of a function.
- More examples of double integrals.
Areas of a region on a plane

Definition

The *area* of a closed, bounded region $R$ on a plane is given by

$$A = \iint_R \, dx \, dy.$$
Areas of a region on a plane

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The area of a closed, bounded region $R$ on a plane is given by

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Remark:
- To compute the area of a region $R$ we integrate the function $f(x, y) = 1$ on that region $R$. 

Areas of a region on a plane

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\[ A = \int\int_R dx \, dy. \]

Remark:
- To compute the area of a region $R$ we integrate the function $f(x, y) = 1$ on that region $R$.
- The area of a region $R$ is computed as the volume of a 3-dimensional region with base $R$ and height equal to 1.
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\}\).
Areas of a region on a plane

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Find the area of $R = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2] \}$.

Solution: We express the region $R$ as an integral Type I, integrating first on vertical directions:
Areas of a region on a plane

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Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\}\).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx.
\]
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx.
\]

\[
A = \int_{-1}^{2} (y \bigg|_{x^2}^{x+2}) \, dx
\]

\[
A = \int_{-1}^{2} (x + 2 - x^2) \, dx
\]

\[
A = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}
\]

\[
A = \left( \frac{4}{2} + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right)
\]

\[
A = \frac{8}{2} - \frac{1}{2} + 4 - 2 - \frac{2}{3} + \frac{1}{3}
\]

\[
A = \frac{9}{2}
\]

\[\text{\( \blacksquare \)}\]
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \ dx.
\]

\[
A = \int_{-1}^{2} \left( y \right|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} (x + 2 - x^2) \ dx
\]
Areas of a region on a plane

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

Solution: We express the region $R$ as an integral Type I, integrating first on vertical directions:

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \ dx.$$  

$$A = \int_{-1}^{2} (y \bigg|_{x^2}^{x+2}) \ dx = \int_{-1}^{2} (x + 2 - x^2) \ dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\bigg|_{-1}^{2}.$$
Areas of a region on a plane

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Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}\).

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\[
A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3}
\]

\[
A = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.
\]
Areas of a region on a plane

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

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A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx.
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A = \int_{-1}^{2} (y \bigg|_{x^2}^{x+2}) \, dx = \int_{-1}^{2} (x + 2 - x^2) \, dx = \left( \frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \bigg|_{-1}^{2}.
\]

\[
A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3
\]
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \; dx.
\]

\[
A = \int_{-1}^{2} \left( y \bigg|_{x^2}^{x+2} \right) \; dx = \int_{-1}^{2} (x + 2 - x^2) \; dx = \left( \frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \bigg|_{-1}^{2}.
\]

\[
A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3 \Rightarrow A = \frac{9}{2}.
\]
Areas of a region on a plane

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}\)
integrating first along horizontal directions.

Solution:

We express the region \( R \) as an integral Type II, integrating first on horizontal directions:

\[
A = \int \int_{R_1} dx \ dy + \int \int_{R_2} dx \ dy.
\]

We must get the same result:

\[
A = \frac{9}{2}.
\]
Areas of a region on a plane

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: We express the region $R$ as an integral Type II, integrating first on horizontal directions:
Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: We express the region $R$ as an integral Type II, integrating first on horizontal directions:

$$A = \int_{R_1} \int dx \ dy + \int_{R_2} \int dx \ dy.$$
Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x+2]\}$ integrating first along horizontal directions.

Solution: We express the region $R$ as an integral Type II, integrating first on horizontal directions:

$$A = \int \int_{R_1} dx \ dy + \int \int_{R_2} dx \ dy.$$

$$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{y} dx \ dy.$$
Areas of a region on a plane

Example

Find the area of \( R = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2] \} \) integrating first along horizontal directions.

Solution: We express the region \( R \) as an integral Type II, integrating first on horizontal directions:

\[
A = \int \int_{R_1} dx \ dy + \int \int_{R_2} dx \ dy.
\]

\[
A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{y} dx \ dy.
\]

We must get the same result: \( A = 9/2 \).
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \)
integrating first along horizontal directions.

Solution: Recall: \( A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{y} dx \ dy. \)
Areas of a region on a plane

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

Solution: Recall: \[ A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx \ dy. \]

\[ A = \int_{0}^{1} 2\sqrt{y} \ dy + \]
Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall: $A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \ dy$.

$$A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 \left( \sqrt{y} - y + 2 \right) dy$$
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \)
integrating first along horizontal directions.

Solution: Recall: 
\[
A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \ dy.
\]

\[
A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 (\sqrt{y} - y + 2) \ dy
\]

\[
A = 2 \left( \frac{2}{3} y^{3/2} \right) \bigg|_0^1 +
\]
Areas of a region on a plane

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall:

$$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx \ dy$$

$$A = \int_{0}^{1} 2\sqrt{y} \ dy + \int_{1}^{4} (\sqrt{y} - y + 2) \ dy$$

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We conclude that $A = \frac{9}{2}$.\[\square\]
Areas of a region on a plane

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

Solution: Recall: \( A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx \, dy \).

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\]

\[
A = 2\left(\frac{2}{3} y^{3/2}\right)\bigg|_{0}^{1} + \left(\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y\right)\bigg|_{1}^{4}
\]

\[
A = \frac{4}{3} +
\]
Areas of a region on a plane

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall: $A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \ dy$.

\[
A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 (\sqrt{y} - y + 2) \ dy \\
A = 2 \left( \frac{2}{3} y^{3/2} \right) \bigg|_0^1 + \left( \frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y \right) \bigg|_1^4 \\
A = \frac{4}{3} + \frac{16}{3} -\]
Areas of a region on a plane

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall: $A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \ dy$.

\[
A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 (\sqrt{y} - y + 2) \ dy
\]

\[
A = 2\left(\frac{2}{3} y^{3/2}\right)\bigg|_0^1 + \left(\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y\right)\bigg|_1^4
\]

\[
A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - \frac{4}{3} = \frac{2}{3}.
\]
Areas of a region on a plane

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Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

Solution: Recall: \[ A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \ dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} \ dx \ dy. \]

\[
A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 (\sqrt{y} - y + 2) \ dy
\]

\[
A = 2\left(\frac{2}{3} y^{3/2}\right)\bigg|_0^1 + \left(\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y\right)\bigg|_1^4
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Areas of a region on a plane

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Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

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\[
A = \int_0^1 2\sqrt{y} \ dy + \int_1^4 (\sqrt{y} - y + 2) \ dy
\]

\[
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\]

\[
A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} +\]
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \)
inintgrating first along horizontal directions.

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\]

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A = 2 \left( \frac{2}{3} y^{3/2} \right) \bigg|_0^1 + \left( \frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y \right) \bigg|_1^4
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\[
A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} + 8 - \quad
\]
Areas of a region on a plane

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: Recall: $A = \int_0^4 \int_{\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_1^4 \int_{\sqrt{y}-2}^{\sqrt{y}} dx \, dy$.

$$A = \int_0^1 2\sqrt{y} \, dy + \int_1^4 (\sqrt{y} - y + 2) \, dy$$

$$A = 2\left(\frac{2}{3} y^{3/2}\right)|_0^1 + \left(\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y\right)|_1^4$$

$$A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} + 8 - 2$$

We conclude that $A = \frac{9}{2}$. ◯
Areas of a region on a plane

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

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\[ A = \int_0^1 2\sqrt{y} \, dy + \int_1^4 (\sqrt{y} - y + 2) \, dy \]

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\[ A = \frac{4}{3} + \frac{16}{3} - \frac{2}{3} - 8 + \frac{1}{2} + 8 - 2 = 6 - \frac{3}{2}. \]
Areas of a region on a plane

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

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\]

We conclude that \( A = \frac{9}{2} \). \( \triangle \)
Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
- **Average value of a function.**
- More examples of double integrals.
Average value of a function

**Review:** The average of a single variable function.

**Definition**
The *average* of a function \( f : [a, b] \rightarrow \mathbb{R} \) on the interval \([a, b]\), denoted by \( f \), is given by

\[
\bar{f} = \frac{1}{(b-a)} \int_{a}^{b} f(x) \, dx.
\]
Average value of a function

**Review:** The average of a single variable function.

**Definition**

The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\bar{f}$, is given by

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$$
Average value of a function

**Review:** The average of a single variable function.

**Definition**
The *average* of a function \( f : [a, b] \to \mathbb{R} \) on the interval \([a, b]\), denoted by \( \bar{f} \), is given by

\[
\bar{f} = \frac{1}{(b - a)} \int_a^b f(x) \, dx.
\]

**Definition**
The *average* of a function \( f : R \subset \mathbb{R}^2 \to \mathbb{R} \) on the region \( R \) with area \( A(R) \), denoted by \( \bar{f} \), is given by

\[
\bar{f} = \frac{1}{A(R)} \iint_R f(x, y) \, dx \, dy.
\]
Average value of a function

Example

Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).
Average value of a function

Example
Find the average of \( f(x, y) = xy \) on the region \( R = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3] \} \).

Solution: The area of the rectangle \( R \) is
Average value of a function

Example
Find the average of \( f(x, y) = xy \) on the region 
\[ R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\}. \]

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).
Average value of a function

Example

Find the average of $f(x, y) = xy$ on the region $R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\}$.

Solution: The area of the rectangle $R$ is $A(R) = 6$. We only need to compute $I = \iint_R f(x, y) \, dx \, dy$. 

Average value of a function

Example
Find the average of $f(x, y) = xy$ on the region $R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\}$. 

Solution: The area of the rectangle $R$ is $A(R) = 6$. We only need to compute $I = \iint_R f(x, y) \, dx \, dy$.

\[
I = \int_0^2 \int_0^3 xy \, dy \, dx
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Average value of a function

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Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\} \).

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We only need to compute \( I = \iint_R f(x, y) \, dx \, dy \).

\[
I = \int_0^2 \int_0^3 xy \, dy \, dx = \int_0^2 x \left( \frac{y^2}{2} \right)_0^3 \, dx
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Average value of a function

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Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).

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I = \int_0^2 \int_0^3 xy \, dy \, dx = \int_0^2 x \left( \frac{y^2}{2} \bigg|_0^3 \right) \, dx = \int_0^2 \frac{9}{2} x \, dx.
\]
Average value of a function

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\( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).
We only need to compute
\[
I = \int \int_R f(x, y) \, dx \, dy.
\]

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\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \bigg|_0^2 \right)
\]
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\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \bigg|_0^2 \right) \Rightarrow I = 9.
\]
Average value of a function

Example
Find the average of $f(x, y) = xy$ on the region $R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\}$.

Solution: The area of the rectangle $R$ is $A(R) = 6$. We only need to compute $I = \int \int_R f(x, y) \, dx \, dy$.

\[
I = \int_0^2 \int_0^3 xy \, dy \, dx = \int_0^2 x \left( \frac{y^2}{2} \right)_0^3 \, dx = \int_0^2 \frac{9}{2} x \, dx.
\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \right)_0^2 \Rightarrow I = 9.
\]

Since $\bar{f} = I / A(R)$
Average value of a function

Example

Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).

We only need to compute \( I = \int \int_R f(x, y) \, dx \, dy \).

\[
I = \int_0^2 \int_0^3 xy \, dy \, dx = \int_0^2 x \left( \frac{y^2}{2} \bigg|_0^3 \right) \, dx = \int_0^2 \frac{9}{2}x \, dx.
\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \bigg|_0^2 \right) \Rightarrow I = 9.
\]

Since \( \bar{f} = I / A(R) = 9 / 6 \),
Average value of a function

Example

Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).

We only need to compute \( I = \int \int_R f(x, y) \, dx \, dy \).

\[
I = \int_0^2 \int_0^3 xy \, dy \, dx = \int_0^2 x \left( \frac{y^2}{2} \bigg|_0^3 \right) \, dx = \int_0^2 \frac{9}{2} x \, dx.
\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \bigg|_0^2 \right) \quad \Rightarrow \quad I = 9.
\]

Since \( \bar{f} = I / A(R) = 9/6 \), we get \( \bar{f} = 3/2 \).\( \triangle \)
Areas and double integrals. (Sect. 15.3)

- Areas of a region on a plane.
- Average value of a function.
- More examples of double integrals.
More examples of double integrals

Example

Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$. 

Solution:

We need to compute $M = \int\int_R \rho(x, y) \, dx \, dy$.

Remark: If $\rho$ is the mass density, then $M$ is the total mass.

$$M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx = \int_0^1 \left[ xy + \frac{y^2}{2} \right]_0^{2x} \, dx = \int_0^1 \left[ 2x^2 + \frac{4x^2}{2} \right] \, dx = 4 \int_0^1 x^3 \, dx = 4 \left[ \frac{x^4}{4} \right]_0^1 = 4.$$ 

$\blacksquare$
More examples of double integrals

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Find the integral of \( \rho(x, y) = x + y \) in the triangle with boundaries \( y = 0, x = 1 \) and \( y = 2x \).

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M = \int \int_{R} \rho(x, y) \, dx\,dy.
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More examples of double integrals

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More examples of double integrals

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Solution: We need to compute

\[
M = \int_{R} \int \rho(x, y) \, dx \, dy.
\]

Remark: If \( \rho \) is the mass density, then \( M \) is the total mass.

\[
M = \int_{0}^{1} \int_{0}^{2x} (x + y) \, dy \, dx
\]

\[
= \int_{0}^{1} \left[ xy + \frac{y^2}{2} \right]_{0}^{2x} \, dx
\]

\[
= \int_{0}^{1} \left( 2x^2 + \frac{4x^2}{2} \right) \, dx
\]

\[
= \int_{0}^{1} 3x^2 \, dx
\]

\[
= \left[ x^3 \right]_{0}^{1}
\]

\[
= 1
\]

\[
\Rightarrow \quad M = 4
\]
More examples of double integrals

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Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

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$$M = \int_{0}^{1} \int_{0}^{2x} (x + y) \, dy \, dx = \int_{0}^{1} \left[ x \left( y \bigg|_{0}^{2x} \right) + \left( \frac{y^2}{2} \bigg|_{0}^{2x} \right) \right] \, dx.$$
More examples of double integrals

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Find the integral of \( \rho(x, y) = x + y \) in the triangle with boundaries \( y = 0, \ x = 1 \) and \( y = 2x \).

Solution: We need to compute

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M = \int \int_R \rho(x, y) \, dxdy.
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Remark: If \( \rho \) is the mass density, then \( M \) is the total mass.

\[
M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx = \int_0^1 \left[ x \left( y \bigg|_0^{2x} \right) + \left( \frac{y^2}{2} \bigg|_0^{2x} \right) \right] \, dx.
\]

\[
M = \int_0^1 [2x^2 + 2x^2] \, dx
\]
More examples of double integrals

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Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

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$$M = \int_0^1 \left[ 2x^2 + 2x^2 \right] \, dx = 4 \frac{x^3}{3} \bigg|_0^1 = \frac{4}{3}.$$
More examples of double integrals

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Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

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$$M = \iint_R \rho(x, y) \, dx \, dy.$$  

Remark: If $\rho$ is the mass density, then $M$ is the total mass.

$$M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx = \int_0^1 \left[ x \left( y \bigg|_0^{2x} \right) + \frac{y^2}{2} \bigg|_0^{2x} \right] \, dx.$$

$$M = \int_0^1 \left[ 2x^2 + 2x^2 \right] \, dx = 4 \frac{x^3}{3} \bigg|_0^1 \Rightarrow M = \frac{4}{3}.$$  

$\triangleleft$
More examples of double integrals

Example
Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

$$
\bar{r}_x = \frac{1}{M} \int_R x\rho(x, y) \, dy \, dx, \quad \bar{r}_y = \frac{1}{M} \iint_R y\rho(x, y) \, dy \, dx.
$$
More examples of double integrals

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Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

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$$

Remark: $\mathbf{r} = \langle \overline{r}_x, \overline{r}_y \rangle$ is the center of mass of the body.
More examples of double integrals

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Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

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Remark: $\mathbf{r} = \langle \bar{r}_x, \bar{r}_y \rangle$ is the center of mass of the body.

Solution: Recall: $M = \frac{4}{3}$. 

More examples of double integrals

Example

Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

$$
\bar{r}_x = \frac{1}{M} \int_{R} x\rho(x, y) \, dy \, dx, \quad \bar{r}_y = \frac{1}{M} \int_{R} \int y\rho(x, y) \, dy \, dx.
$$

Remark: $\mathbf{r} = \langle \bar{r}_x, \bar{r}_y \rangle$ is the center of mass of the body.

Solution: Recall: $M = \frac{4}{3}$. We need to compute

$$
\bar{r}_x = \frac{1}{M} \int_{0}^{1} \int_{0}^{2x} (x + y)x \, dy \, dx
$$
More examples of double integrals

Example
Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers

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\bar{r}_x = \frac{1}{M} \int_R x \rho(x, y) \, dy \, dx, \quad \bar{r}_y = \frac{1}{M} \iint_R y \rho(x, y) \, dy \, dx.
$$

Remark: $\mathbf{r} = \langle \bar{r}_x, \bar{r}_y \rangle$ is the center of mass of the body.

Solution: Recall: $M = \frac{4}{3}$. We need to compute

$$
\bar{r}_x = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)x \, dy \, dx = \frac{3}{4} \int_0^1 \left[ x^2 \left( y \bigg|_0^{2x} \right) + x \left( \frac{y^2}{2} \bigg|_0^{2x} \right) \right] \, dx
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More examples of double integrals

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Given the function \( \rho(x, y) = x + y \), the number \( M \) computed in the previous example, and the triangle with boundaries \( y = 0 \), \( x = 1 \) and \( y = 2x \), find the numbers

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More examples of double integrals

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\[
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\]
More examples of double integrals

Definition

The centroid of a region $R$ in the plane is the vector $\mathbf{c}$ given by

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\mathbf{c} = \frac{1}{A(R)} \iiint_{R} \langle x, y \rangle \, dx \, dy,
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where $A(R) = \iiint_{R} \, dx \, dy$.

Remark:

$\mathbf{c}$ can be seen as the center of mass vector of region $R$ when the mass density is constant.

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More examples of double integrals

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More examples of double integrals

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More examples of double integrals

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Find the centroid of the triangle inside \( y = 0, \ x = 1 \) and \( y = 2x \).
More examples of double integrals

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so $c_y = \frac{2}{3}$. We conclude, $c = \frac{2}{3} \langle 1, 1 \rangle$. △
Remark: The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.
More examples of double integrals

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Definition
The moment of inertia about the $x$-axis and the $y$-axis of a region $R$ in the plane having mass density $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$ are given by, respectively,

\[
I_x = \iint_R y^2 \rho(x, y) \, dx \, dy, \quad I_y = \iint_R x^2 \rho(x, y) \, dx \, dy.
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More examples of double integrals

**Remark:** The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.

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The *moment of inertia* about the $x$-axis and the $y$-axis of a region $R$ in the plane having mass density $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$ are given by, respectively,

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If $M$ denotes the total mass of the region, then the *radii of gyration* about the $x$-axis and the $y$-axis are given by

$$R_x = \sqrt{I_x/M}, \quad R_y = \sqrt{I_y/M}.$$
The moment of inertia of an object.

Example
Find the moment of inertia and the radius of gyration about the $x$-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution:
The moment of inertia $I_x$ is given by

$$I_x = \int_0^1 \int_0^{2x} x^2 (x + y) \, dy \, dx$$

$$= \int_0^1 \left[ x^3 \left|_0^{2x} \right. \right] \, dx$$

$$= \int_0^1 4x^4 \, dx$$

$$= 4 \left[ \frac{x^5}{5} \right]_0^1$$

$$= \frac{4}{5}.$$ 

Since the mass of the region is $M = \frac{4}{3}$, the radius of gyration along the $x$-axis is $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{4}{3 \cdot 5}}$, that is,$R_x = \sqrt{\frac{3}{5}}$.
The moment of inertia of an object.

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Find the moment of inertia and the radius of gyration about the $x$-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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**Solution:** The moment of inertia \( I_x \) is given by

\[
I_x = \int_0^1 \int_0^{2x} x^2(x + y) \, dy \, dx = \int_0^1 \left[ x^3 \left( y \bigg|_{0}^{2x} \right) + x^2 \left( \frac{y^2}{2} \bigg|_{0}^{2x} \right) \right] \, dx
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The moment of inertia of an object.

Example
Find the moment of inertia and the radius of gyration about the $x$-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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$\Rightarrow R_x = \frac{\sqrt{15}}{10}$. \hspace{1cm} \Box$
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Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.
Review: Polar coordinates

Definition
The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair $(r, \theta)$ defined by the picture.
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The Cartesian coordinates of a point $P = (r, \theta)$ are given by

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Double integrals on disk sections

Theorem

If \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in the region

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R = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\}
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where \( 0 \leq \theta_0 \leq \theta_1 \leq 2\pi \), then the double integral of function \( f \) in that region can be expressed in polar coordinates as follows,

\[
\iint_R f \, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r, \theta) \, r \, dr \, d\theta.
\]

Remark:

- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
- The boundaries of each domain, a rectangle in Cartesian and a disk section in polar coordinates, are defined by a constant value of a coordinate.
- Notice the extra factor \( r \) on the right-hand side above.
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Find the area of an arbitrary circular section

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Evaluate that area in the particular case of a disk with radius \( R \).
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Double integrals on disk sections

Example

Find the integral of \( f(r, \theta) = r^2 \cos(\theta) \) in the disk
\[ R = \{ (r, \theta) \in \mathbb{R}^2 : r \in [0, 1], \ \theta \in [0, \pi/4] \}. \]
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Solution:
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Solution:

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Solution:
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We conclude that $\int \int_R f \, dA = \sqrt{2}/8$. \(\triangleleft\)
Double integrals in polar coordinates (Sect. 15.4)

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Double integrals in arbitrary regions

**Theorem**

If the function \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in the region

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R = \{(r, \theta) \in \mathbb{R}^2 : r \in [h_0(\theta), h_1(\theta)], \ \theta \in [\theta_0, \theta_1]\}.
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Double integrals in arbitrary regions

Example
Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$. 

Solution:
We first show that these curves are actually circles.

$r = \cos(\theta) \iff r^2 = r \cos(\theta) \iff x^2 + y^2 = x$.

Completing the square in $x$ we obtain $(x - \frac{1}{2})^2 + y^2 = \left(\frac{1}{2}\right)^2$.

Analogously, $r = \sin(\theta)$ is the circle $x^2 + (y - \frac{1}{2})^2 = \left(\frac{1}{2}\right)^2$. 

$\sqrt{1/2} \quad \sqrt{1/2}$

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Double integrals in arbitrary regions

Example

Find the area of the region bounded by the curves \( r = \cos(\theta) \) and \( r = \sin(\theta) \).

Solution: We first show that these curves are actually circles.

\[
    r = \cos(\theta) \quad \Leftrightarrow \quad r^2 = r \cos(\theta) \quad \Leftrightarrow \quad x^2 + y^2 = x.
\]

Completing the square in \( x \) we obtain

\[
    \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.
\]

Analogously, \( r = \sin(\theta) \) is the circle

\[
    x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.
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Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first show that these curves are actually circles.

$$r = \cos(\theta) \iff r^2 = r \cos(\theta) \iff x^2 + y^2 = x.$$  

Completing the square in $x$ we obtain

$$(x - \frac{1}{2})^2 + y^2 = \left(\frac{1}{2}\right)^2.$$  

Analogously, $r = \sin(\theta)$ is the circle

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$
Double integrals in arbitrary regions.

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: $A = 2 \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta$
Double integrals in arbitrary regions.

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Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

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Double integrals in arbitrary regions.

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$$A = \frac{1}{2} \left[ \frac{\pi}{4} - \left( \frac{1}{2} - 0 \right) \right] = \frac{\pi}{8} - \frac{1}{4}$$
Double integrals in arbitrary regions.

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Double integrals in arbitrary regions.

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A &= 2 \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta = 2 \int_0^{\pi/4} \frac{1}{2} \sin^2(\theta) \, d\theta; \\
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\end{align*} \)

Also works: \( \begin{align*}
A &= \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\cos(\theta)} r \, dr \, d\theta.
\end{align*} \)
Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- **Changing Cartesian integrals into polar integrals.**
- Computing volumes using double integrals.
Changing Cartesian integrals into polar integrals

Theorem
If \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is a continuous function, and \( f(x, y) \) represents the function values in Cartesian coordinates, then holds

\[
\int\int_D f(x, y) \, dx \, dy = \int\int_D f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta.
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Compute the integral of $f(x, y) = x^2 + 2y^2$ on $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \quad 0 \leq x, \quad 1 \leq x^2 + y^2 \leq 2\}$. 
Changing Cartesian integrals into polar integrals

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Changing Cartesian integrals into polar integrals

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Solution: First, transform Cartesian into polar coordinates: \( x = r \cos(\theta), \ y = r \sin(\theta) \). Since \( f(x, y) = (x^2 + y^2) + y^2 \),
Changing Cartesian integrals into polar integrals

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\( x = r \cos(\theta), \ y = r \sin(\theta) \). Since \( f(x, y) = (x^2 + y^2) + y^2 \),

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f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta).
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Changing Cartesian integrals into polar integrals

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Changing Cartesian integrals into polar integrals

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$$D = \left\{ (r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \frac{\pi}{2}, \ 1 \leq r \leq \sqrt{2} \right\}$$
Changing Cartesian integrals into polar integrals

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\int \int_D f(r, \theta) \, dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2(1 + \sin^2(\theta)) \, r \, dr \, d\theta,
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Changing Cartesian integrals into polar integrals

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\iint_D f(r, \theta) \, dA = \left[ \int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_1^{\sqrt{2}} r^3 \, dr \right].
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Changing Cartesian integrals into polar integrals

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\int \int_D f(r, \theta) \, dA = \left[ (\theta \bigg|_0^{\pi/2}) + \int_0^{\pi/2} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \right] \frac{1}{4} (r^4 \bigg|_1^{\sqrt{2}}) .
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Solution:

\[
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\]

\[
\begin{align*}
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\int \int_D f(r, \theta) dA &= \left[ \frac{\pi}{2} + \frac{1}{2} \theta \bigg|_0^{\pi/2} - \frac{1}{4} \sin(2\theta) \bigg|_0^{\pi/2} \right] \frac{3}{4}
\end{align*}
\]

We conclude:

\[
\int \int_D f(r, \theta) dA = \frac{9}{16} \pi.
\]
Changing Cartesian integrals into polar integrals

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\]

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\]

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Changing Cartesian integrals into polar integrals

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We conclude: \[
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Changing Cartesian integrals into polar integrals

Example

Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain $D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\}$. 
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Integrate \( f(x, y) = e^{-(x^2+y^2)} \) on the domain
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Solution: Since \( f(r \cos(\theta), r \sin(\theta)) = e^{-r^2} \), the double integral is
Changing Cartesian integrals into polar integrals

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Solution: Since \( f(r \cos(\theta), r \sin(\theta)) = e^{-r^2} \), the double integral is

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\iint_D f(x, y) \, dx \, dy = \int_0^\pi \int_0^2 e^{-r^2} \, r \, dr \, d\theta.
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Changing Cartesian integrals into polar integrals

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Substitute \( u = r^2 \),
Changing Cartesian integrals into polar integrals

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\[
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Changing Cartesian integrals into polar integrals

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\]

We conclude:

\[
\iint_D f(x, y) \, dx \, dy = \frac{\pi}{2} \left( 1 - \frac{1}{e^4} \right).
\]

\( \triangle \)
Double integrals in polar coordinates (Sect. 15.4)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- **Computing volumes using double integrals.**
Computing volumes using double integrals

**Example**

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$. 
Computing volumes using double integrals

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Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: Let us first draw the sets that form the volume we are interested to compute.
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$$z = \pm \sqrt{1 - r^2},$$
Computing volumes using double integrals

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\[ z = \pm \sqrt{1 - r^2}, \]
\[ z = r. \]
Computing volumes using double integrals

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Computing volumes using double integrals

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The volume we are interested to compute is:

$$V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2} (rdr) d\theta - \int_0^{2\pi} \int_0^{r_0} r (rdr) d\theta.$$
Computing volumes using double integrals

Example

Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: The integration region can be decomposed as follows:

![Diagram of sphere and cone](image)

The volume we are interested to compute is:

\[
V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2} (rdr)d\theta - \int_0^{2\pi} \int_0^{r_0} r (rdr)d\theta.
\]

We need to find \( r_0 \), the intersection of the cone and the sphere.
Computing volumes using double integrals

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Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

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We conclude: \(V = \frac{\pi}{3} \left( 2 - \sqrt{2} \right).\) \(\triangle\)