Local and absolute extrema, saddle points (Sect. 14.7)

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.
Review: Local extrema for functions of one variable

Recall: Main results on local extrema for $f(x)$:

- If $x_0$ is a local maximum or minimum of $f$, then $f'(x_0) = 0$.
- If $f'(x_0) = 0$, then $x_0$ is a critical point of $f$, that is, $x_0$ is a maximum or a minimum or an inflection point.
- The second derivative test determines whether a critical point is a maximum, minimum or an inflection point.

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Remarks: Assume that $f$ is twice continuously differentiable.
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\[
\begin{array}{|c|c|c|}
\hline
\text{at} & f & f' & f'' \\
\hline
a & \text{max.} & 0 & < 0 \\
\hline
b & \text{infl.} & \neq 0 & \pm 0 \mp \\
\hline
c & \text{min.} & 0 & > 0 \\
\hline
d & \text{infl.} & = 0 & \pm 0 \mp \\
\hline
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\]

Remarks: Assume that \( f \) is twice continuously differentiable.

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Review: Local extrema for functions of one variable.

Definition of local extrema.

Characterization of local extrema.
  - First derivative test.
  - Second derivative test.

Absolute extrema of a function in a domain.
Definition of local extrema for functions of two variables

Definition
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a local maximum at the point $(a, b) \in D$ iff holds that $f(a, b) \geq f(x, y)$ for every point $(x, y)$ in a neighborhood of $(a, b)$.
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a local minimum at the point $(a, b) \in D$ iff holds that $f(a, b) \leq f(x, y)$ for every point $(x, y)$ in a neighborhood of $(a, b)$. 
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A function $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ has a *local maximum* at the point $(a, b) \in D$ iff holds that $f(a, b) \geq f(x, y)$ for every point $(x, y)$ in a neighborhood of $(a, b)$.

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Definition of local extrema for functions of two variables

Definition
A differentiable function $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ has a *saddle point* at an interior point $(a, b) \in D$ iff in every open disk in $D$ centered at $(a, b)$ there always exist points $(x, y)$ where $f(a, b) < f(x, y)$ and other points $(x, y)$ where $f(a, b) > f(x, y)$. 
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Characterization of local extrema

Theorem (First Derivative Test)

If a differentiable function \( f \) has a local maximum or minimum at \((a, b)\) then holds \( (\nabla f)_{|(a,b)} = \langle 0, 0 \rangle. \)

Remark: The tangent plane at a local extremum is horizontal, since its normal vector is \( n = \langle f_x, f_y, -1 \rangle = \langle 0, 0, -1 \rangle. \)

Definition

The interior point \((a, b)\) \(\in\) \(D\) of a differentiable function \(f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}\) is a critical point of \(f\) iff \( (\nabla f)_{|(a,b)} = \langle 0, 0 \rangle. \)

Remark: Critical points include local maxima, local minima, and saddle points.
Characterization of local extrema

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Example
Find the critical points of the function \( f(x, y) = -x^2 - y^2 \).

Solution:
The critical points are the points where \( \nabla f \) vanishes. Since \( \nabla f = \langle -2x, -2y \rangle \), the only solution to \( \nabla f = \langle 0, 0 \rangle \) is \( x = 0, y = 0 \). That is, we again obtain \( (a, b) = (0, 0) \).

Remark:
Since \( f(x, y) \leq 0 \) for all \( (x, y) \in \mathbb{R}^2 \) and \( f(0, 0) = 0 \), then the point \( (0, 0) \) must be a local maximum of \( f \).
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Characterization of local extrema

Theorem (Second derivative test)

Let \((a, b)\) be a critical point of \(f : D \subset \mathbb{R}^2 \to \mathbb{R}\), that is, \((\nabla f)(a, b) = \langle 0, 0 \rangle\). Assume that \(f\) has continuous second derivatives in an open disk in \(D\) with center in \((a, b)\) and denote

\[
D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.
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Then, the following statements hold:
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Characterization of local extrema

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Notation: The number \(D\) is called the discriminant of \(f\) at \((a, b)\).
Characterization of local extrema

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Find the local extrema of \( f(x, y) = y^2 - x^2 \) and determine whether they are local maximum, minimum, or saddle points.
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\nabla f = \langle -2x, 2y \rangle \implies (\nabla f)_{(a,b)} = \langle 0, 0 \rangle \text{ iff } (a, b) = (0, 0).
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The only critical point is \((a, b) = (0, 0)\).
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We need to compute \( D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 \).
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Since \( f_{xx}(0, 0) = -2 \), \( f_{yy}(0, 0) = 2 \), and \( f_{xy}(0, 0) = 0 \), we get

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D = (-2)(2) = -4 < 0
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Characterization of local extrema

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Find the local extrema of $f(x, y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

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$$\nabla f = \langle -2x, 2y \rangle \quad \Rightarrow \quad (\nabla f)_{(a,b)} = \langle 0, 0 \rangle \text{ iff } (a, b) = (0, 0).$$

The only critical point is $(a, b) = (0, 0)$.

We need to compute $D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

Since $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = 2$, and $f_{xy}(0, 0) = 0$, we get

$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0, 0). \triangle$$
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Since \(f_{xx}(x, y) = 2y^2, \quad f_{yy}(x, y) = 2x^2, \quad \text{and} \quad f_{xy}(x, y) = 4xy\),
we obtain \(f_{xx}(0, 0) = 0, \quad f_{yy}(0, 0) = 0, \quad \text{and} \quad f_{xy}(0, 0) = 0,\)
hence \(D = 0\) and the test is inconclusive.
\[\triangle\]
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This is confirmed in the graph of \(f\).
Local and absolute extrema, saddle points (Sect. 14.7)

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- **Absolute extrema of a function in a domain.**
Absolute extrema of a function in a domain

Definition
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute maximum at the point $(a, b) \in D$ iff $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$.
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute minimum at the point $(a, b) \in D$ iff $f(a, b) \leq f(x, y)$ for all $(x, y) \in D$. 

Remark: Local extrema need not be the absolute extrema.

Remark: Absolute extrema may not be defined on open intervals.
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Review: Functions of one variable

Theorem

*Every continuous function on a closed interval, \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \), with \( a < b \in \mathbb{R} \), always has absolute extrema.*
Review: Functions of one variable

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Recall:

\( \left[ a, b \right] \) are bounded and closed sets in \( \mathbb{R} \).

\( \left[ a, b \right] \) is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.
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Recall:

- Intervals \([a, b]\) are bounded and closed sets in \( \mathbb{R} \).
- The set \([a, b]\) is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.
Recall: On open and closed sets in $\mathbb{R}^n$

**Definition**
A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called **open** iff every point in $S$ is an interior point. The set $S$ is called **closed** iff $S$ contains its boundary. A set $S$ is called **bounded** iff $S$ is contained in ball, otherwise $S$ is called **unbounded**.
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Theorem
*If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in a closed and bounded set $D$, then $f$ has an absolute maximum and an absolute minimum in $D.*
Absolute extrema on closed and bounded sets

Problem:
Find the absolute extrema of a function $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ in a closed and bounded set $D$. 

Solution:
(1) Find every critical point of $f$ in the interior of $D$ and evaluate $f$ at these points.
(2) Find the boundary points of $D$ where $f$ has local extrema, and evaluate $f$ at these points.
(3) Look at the list of values for $f$ found in the previous two steps. If $f(x_0, y_0)$ is the biggest (smallest) value of $f$ in the list above, then $(x_0, y_0)$ is the absolute maximum (minimum) of $f$ in $D$. 
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Absolute extrema on closed and bounded sets

Example
Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.
Absolute extrema on closed and bounded sets

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Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

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Absolute extrema on closed and bounded sets

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\( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

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\[ \nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \]
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(1) We find all critical points in the interior of the domain:

$\nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \Rightarrow (x_0, y_0) = (-2, 1)$. 
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Boundary I: The segment $y = 0, x \in [1, 5]$. 
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Boundary I: The segment $y = 0, x \in [1, 5]$. We select the end points $(1, 0), (5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$. 
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Absolute extrema on closed and bounded sets

Example
Find the absolute extrema of the function
\[ f(x, y) = 3 + xy - x + 2y \]
on the closed domain given in the Figure.

Solution:
(1) We find all critical points in the interior of the domain:
\[ \nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \implies (x_0, y_0) = (-2, 1). \]
Since (-2, 1) does not belong to the domain, we discard it.

(2) Three segments form the boundary of \( D \):
Boundary I: The segment \( y = 0, x \in [1, 5] \). We select the end points (1, 0), (5, 0), and we record: \( f(1, 0) = 2 \) and \( f(5, 0) = -2 \).
We look for critical point on the interior of Boundary I:
Since \( g(x) = f(x, 0) = 3 - x \), so \( g' = -1 \neq 0 \).
No critical points in the interior of Boundary I.
Example
Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.
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Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. 

```plaintext

y

II

III

x

1 1 5

II III

51

4

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$.

```
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function 
\( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

Solution: Boundary II: The segment \( x = 1, \ y \in [0, 4] \). We select 
the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).
Absolute extrema on closed and bounded sets

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We look for critical point on the interior of Boundary II:
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$. We look for critical point on the interior of Boundary II: Since $g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y$, so $g'(y) = 3 \neq 0$. 
Absolute extrema on closed and bounded sets

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We look for critical point on the interior of Boundary II:
Since \( g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y \), so \( g' = 3 \neq 0 \).

No critical points in the interior of Boundary II.

Boundary III: The segment \( y = -x + 5, x \in [1, 5] \).
Absolute extrema on closed and bounded sets

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Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

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Boundary III: The segment \( y = -x + 5, \ x \in [1, 5] \).
We look for critical point on the interior of Boundary III:
Absolute extrema on closed and bounded sets

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\( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

Solution:  **Boundary II:** The segment \( x = 1, y \in [0, 4] \). We select the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).

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We look for critical point on the interior of Boundary III:
Since \( g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5) \).
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function
\[ f(x, y) = 3 + xy - x + 2y \]
on the closed domain given in the Figure.

Solution: Boundary II: The segment \( x = 1, \ y \in [0, 4] \). We select the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).
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Since \( g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y \), so \( g'(y) = 3 \neq 0 \).
No critical points in the interior of Boundary II.

Boundary III: The segment \( y = -x + 5, \ x \in [1, 5] \).
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Since \( g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5) \).
We obtain \( g(x) = -x^2 + 2x + 13 \),
Absolute extrema on closed and bounded sets

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Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

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We look for critical point on the interior of Boundary III:

Since $g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5)$.

We obtain $g(x) = -x^2 + 2x + 13$, hence $g'(x) = -2x + 2$.
Absolute extrema on closed and bounded sets

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Find the absolute extrema of the function
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We obtain \( g(x) = -x^2 + 2x + 13 \), hence \( g'(x) = -2x + 2 = 0 \) implies \( x = 1 \).
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function
\[ f(x, y) = 3 + xy - x + 2y \]
on the closed domain given in the Figure.

Solution: Boundary II: The segment \( x = 1, \ y \in [0, 4] \). We select the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).
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Boundary III: The segment \( y = -x + 5, \ x \in [1, 5] \).
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Since \( g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5) \).
We obtain \( g(x) = -x^2 + 2x + 13 \), hence \( g'(x) = -2x + 2 = 0 \) implies \( x = 1 \). So, \( y = 4 \), and we selected the point \((1, 4)\),
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$. We look for critical point on the interior of Boundary II: Since $g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y$, so $g'(y) = 3 \neq 0$. No critical points in the interior of Boundary II.

Boundary III: The segment $y = -x + 5$, $x \in [1, 5]$. We look for critical point on the interior of Boundary III: Since $g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5)$. We obtain $g(x) = -x^2 + 2x + 13$, hence $g'(x) = -2x + 2 = 0$ implies $x = 1$. So, $y = 4$, and we selected the point $(1, 4)$, which was already in our list.
Absolute extrema on closed and bounded sets

Example

Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

Solution: Boundary II: The segment \( x = 1, y \in [0, 4] \). We select the end point \((1, 4)\) and we record: \( f(1, 4) = 14 \).

We look for critical point on the interior of Boundary II:
Since \( g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y \), so \( g'(y) = 3 \neq 0 \).
No critical points in the interior of Boundary II.

Boundary III: The segment \( y = -x + 5, x \in [1, 5] \).
We look for critical point on the interior of Boundary III:
Since \( g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5) \).
We obtain \( g(x) = -x^2 + 2x + 13 \), hence \( g'(x) = -2x + 2 = 0 \) implies \( x = 1 \). So, \( y = 4 \), and we selected the point \((1, 4)\), which was already in our list. No critical points in the interior of III.
Absolute extrema on closed and bounded sets

Example
Find the absolute extrema of the function
\[ f(x, y) = 3 + xy - x + 2y \]
on the closed domain given in the Figure.

Solution:
(3) Our list of values is:
\[ f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2. \]
Absolute extrema on closed and bounded sets

Example
Find the absolute extrema of the function \( f(x, y) = 3 + xy - x + 2y \) on the closed domain given in the Figure.

Solution:
(3) Our list of values is:
\[
\begin{align*}
f(1, 0) &= 2 & f(1, 4) &= 14 & f(5, 0) &= -2.
\end{align*}
\]

We conclude:
(a) Absolute maximum at (1, 4),
(b) Absolute minimum at (5, 0).
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$. 

Solution:
This problem can be solved by finding the local maximum of an appropriate function $f$. First, the functions volume and area of a rectangular box with vertex at $(0,0,0)$ and sides $x$, $y$ and $z$ are:

$V(x,y,z) = xyz$,

$A(x,y,z) = 2xy + 2xz + 2yz$.

Since $A(x,y,z) = A_0$, we obtain

$z = A_0 - \frac{2xy}{2(x+y)}$.

$\Rightarrow f(x,y) = A_0xy - \frac{2x^2y^2}{2(x+y)}$. 
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

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A maximization problem with a constraint

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$$z = \frac{A_0 - 2xy}{2(x + y)}.$$
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Since $A(x, y, z) = A_0$, we obtain

$$z = \frac{A_0 - 2xy}{2(x + y)} \quad \Rightarrow \quad f(x, y) = \frac{A_0 xy - 2x^2 y^2}{2(x + y)}.$$
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: Find the critical points of $f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}$. 
A maximization problem with a constraint

Example

Find the maximum volume of a closed rectangular box with a given surface area \( A_0 \).

Solution: Find the critical points of \( f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)} \).

\[
    f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x + y)^2}.
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A maximization problem with a constraint

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The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0, y \neq 0$ imply
A maximization problem with a constraint

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The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$A_0 = 2x^2 + 4xy, \quad A_0 = 2y^2 + 4xy,$$
A maximization problem with a constraint

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Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: Find the critical points of $f(x, y) = \frac{A_0 xy - 2x^2 y^2}{2(x + y)}$.

$$
\begin{align*}
    f_x &= \frac{2A_0 y^2 - 4x^2 y^2 - 8xy^3}{4(x + y)^2}, \\
    f_y &= \frac{2A_0 x^2 - 4x^2 y^2 - 8yx^3}{4(x + y)^2}.
\end{align*}
$$

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$
\begin{align*}
    A_0 &= 2x^2 + 4xy, \\
    A_0 &= 2y^2 + 4xy,
\end{align*}
\Rightarrow x = y
$$
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: Find the critical points of $f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}$.

$$f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x + y)^2}.$$  

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0, y \neq 0$ imply

$$A_0 = 2x^2 + 4xy, \quad A_0 = 2y^2 + 4xy \quad \Rightarrow \quad x = y \quad \Rightarrow \quad A_0 = 2x^2 + 4x^2.$$
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Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

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The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0, y \neq 0$ imply

$$A_0 = 2x^2 + 4xy, \quad A_0 = 2y^2 + 4xy, \quad \Rightarrow \quad x = y \quad \Rightarrow \quad A_0 = 2x^2 + 4x^2.$$

Then, $x_0 = \sqrt{\frac{A_0}{6}}$.
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: Find the critical points of $f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}$.

$$f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x + y)^2}.$$  

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0, y \neq 0$ imply

\[
\begin{align*}
A_0 &= 2x^2 + 4xy, \\
A_0 &= 2y^2 + 4xy,
\end{align*}
\]

\[\Rightarrow x = y \quad \Rightarrow \quad A_0 = 2x^2 + 4x^2.\]

Then, $x_0 = \sqrt{\frac{A_0}{6}} = y_0$.\]
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area \( A_0 \).

Solution: Find the critical points of 
\[
f(x, y) = \frac{A_0 xy - 2x^2 y^2}{2(x + y)}.
\]

\[
f_x = \frac{2A_0 y^2 - 4x^2 y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0 x^2 - 4x^2 y^2 - 8yx^3}{4(x + y)^2}.
\]

The conditions \( f_x = 0 \) and \( f_y = 0 \) and \( x \neq 0, y \neq 0 \) imply
\[
\begin{align*}
A_0 &= 2x^2 + 4xy, \\
A_0 &= 2y^2 + 4xy,
\end{align*}
\]
\[
\Rightarrow x = y \quad \Rightarrow A_0 = 2x^2 + 4x^2.
\]

Then, \( x_0 = \sqrt{\frac{A_0}{6}} = y_0 \). Since \( z = \frac{A_0 - 2xy}{2(x + y)} \),
A maximization problem with a constraint

Example
Find the maximum volume of a closed rectangular box with a given surface area $A_0$.

Solution: Find the critical points of $f(x, y) = \frac{A_0xy - 2x^2y^2}{2(x + y)}$.

$$f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x + y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x + y)^2}.$$  

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$\begin{cases} A_0 = 2x^2 + 4xy, \\ A_0 = 2y^2 + 4xy, \end{cases} \implies x = y \implies A_0 = 2x^2 + 4x^2.$$  

Then, $x_0 = \sqrt{\frac{A_0}{6}} = y_0$. Since $z = \frac{A_0 - 2xy}{2(x + y)}$, $z_0 = \sqrt{\frac{A_0}{6}}$. △
Review for Exam 2

- 50 minutes.
- 5, 6 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \)

(continued)
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2 \rangle \),
Section 14.7

Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.
(b) For each critical point of $f$, determine whether $f$ has a local maximum, local minimum, or saddle point at that point.

Solution:
(a) $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle$
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle \) = \( \langle 0, 0 \rangle \),
Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of $f$, determine whether $f$ has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle$, then,

$$x^2 = 2y, \quad y = 2x,$$
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[ x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0. \]
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[
x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0.
\]

There are two solutions,
Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of $f$, determine whether $f$ has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[ x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0. \]

There are two solutions, \( x = 0 \)
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:

(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[
x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0.
\]

There are two solutions, \( x = 0 \Rightarrow y = 0 \),
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).
(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:
(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[
x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0.
\]

There are two solutions, \( x = 0 \Rightarrow y = 0 \), and \( x = 4 \)
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

Solution:
(a) \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0, 0 \rangle \), then,

\[
x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x - 4) = 0.
\]

There are two solutions, \( x = 0 \Rightarrow y = 0 \), and \( x = 4 \Rightarrow y = 8 \).
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).
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There are two solutions, \( x = 0 \Rightarrow y = 0 \), and \( x = 4 \Rightarrow y = 8 \).
That is, there are two critical points, \((0, 0)\) and \((4, 8)\).
Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of $f$, determine whether $f$ has a local maximum, local minimum, or saddle point at that point.

Solution:

(b) Recalling $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle$, 

\[ D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144(x^2 - 1), \]

Since $D(0, 0) = -144 < 0$, the point $(0, 0)$ is a saddle point of $f$. Since $D(4, 8) = 144(2 - 1) > 0$, and $f_{xx}(4, 8) = -12 < 0$, the point $(4, 8)$ is a local maximum of $f$. ◁
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Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.
(b) For each critical point of $f$, determine whether $f$ has a local maximum, local minimum, or saddle point at that point.

Solution:
(b) Recalling $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle$, we compute

$$f_{xx} = -12x,$$
Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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(b) Recalling \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle \), we compute

\[
\begin{align*}
f_{xx} &= -12x, \\
f_{yy} &= -6,
\end{align*}
\]
### Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

(b) For each critical point of \( f \), determine whether \( f \) has a local maximum, local minimum, or saddle point at that point.

**Solution:**

(b) Recalling \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle \), we compute

\[
\begin{align*}
 f_{xx} &= -12x, \\
 f_{yy} &= -6, \\
 f_{xy} &= 12.
\end{align*}
\]
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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\[
\begin{align*}
f_{xx} &= -12x, \\
f_{yy} &= -6, \\
f_{xy} &= 12.
\end{align*}
\]

\[
D(x, y) = f_{xx}f_{yy} - (f_{xy})^2
\]
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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(b) Recalling \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle \), we compute

\[
\begin{align*}
f_{xx} &= -12x, \quad f_{yy} = -6, \quad f_{xy} = 12. \\
D(x, y) &= f_{xx}f_{yy} - (f_{xy})^2 = 144 \left( \frac{x}{2} - 1 \right),
\end{align*}
\]
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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Since \( D(0, 0) = -144 < 0 \),
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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Section 14.7

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(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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Since \( D(0, 0) = -144 < 0 \), the point \((0, 0)\) is a saddle point of \( f \).

Since \( D(4, 8) = 144(2 - 1) > 0 \),
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).

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(b) Recalling \( \nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle \), we compute

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Since \( D(4, 8) = 144(2 - 1) > 0 \), and \( f_{xx}(4, 8) = (-12)4 < 0 \),
Section 14.7

Example

(a) Find all the critical points of \( f(x, y) = 12xy - 2x^3 - 3y^2 \).
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Since \( D(0, 0) = -144 < 0 \), the point \( (0, 0) \) is a saddle point of \( f \).

Since \( D(4, 8) = 144(2 - 1) > 0 \), and \( f_{xx}(4, 8) = (-12)4 < 0 \), the point \( (4, 8) \) is a local maximum of \( f \). \( \triangle \)
Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$. 

Since $f_x(-3, 2) = 2 \cos(-6 + 6) = 2$ and $f_y(-3, 2) = 3 \cos(-6 + 6) = 3$, the linear approximation is $L(x, y) = 2(x + 3) + 3(y - 2) + 1$. 

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Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution:

(a) $L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2)$. 
Section 14.6

Example

(a) Find the linear approximation \( L(x, y) \) of the function 
\[ f(x, y) = \sin(2x + 3y) + 1 \] 
at the point \((-3, 2)\).

(b) Use the approximation above to estimate the value of 
\[ f(-2.9, 2.1). \]

Solution:

(a) \[ L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2). \]

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Section 14.6

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(a) Find the linear approximation \( L(x, y) \) of the function 
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Solution:
(a) \( L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2) \).

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Section 14.6

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(a) Find the linear approximation \( L(x, y) \) of the function
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\[ f_x(-3, 2) = 2 \cos(-6 + 6) \]
Section 14.6

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(a) Find the linear approximation \( L(x, y) \) of the function 
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\[ f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \]
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(a) Find the linear approximation \( L(x, y) \) of the function \( f(x, y) = \sin(2x + 3y) + 1 \) at the point \((-3, 2)\).

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Solution:
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Since \( f_x(x, y) = 2 \cos(2x + 3y) \) and \( f_y(x, y) = 3 \cos(2x + 3y) \),
\[
f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3 \cos(-6 + 6)
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Section 14.6

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$f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3 \cos(-6 + 6) = 3,$
Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution:

(a) $L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2)$.

Since $f_x(x, y) = 2 \cos(2x + 3y)$ and $f_y(x, y) = 3 \cos(2x + 3y)$,

$f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3 \cos(-6 + 6) = 3$,

$f(-3, 2) = \sin(-6 + 6) + 1$
Section 14.6

Example

(a) Find the linear approximation \( L(x, y) \) of the function 
\[ f(x, y) = \sin(2x + 3y) + 1 \] 
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Solution:

(a) \( L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2) \).

Since \( f_x(x, y) = 2 \cos(2x + 3y) \) and \( f_y(x, y) = 3 \cos(2x + 3y) \),
\[ f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3 \cos(-6 + 6) = 3, \]
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Section 14.6

Example

(a) Find the linear approximation \( L(x, y) \) of the function
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Solution:

(a) \( L(x, y) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) + f(-3, 2). \)

Since \( f_x(x, y) = 2 \cos(2x + 3y) \) and \( f_y(x, y) = 3 \cos(2x + 3y), \)
\[ f_x(-3, 2) = 2 \cos(-6 + 6) = 2, \quad f_y(-3, 2) = 3 \cos(-6 + 6) = 3, \]
\[ f(-3, 2) = \sin(-6 + 6) + 1 = 1. \]

the linear approximation is \( L(x, y) = 2(x + 3) + 3(y - 2) + 1. \)
Example

(a) Find the linear approximation \( L(x, y) \) of the function \( f(x, y) = \sin(2x + 3y) + 1 \) at the point \((-3, 2)\).

(b) Use the approximation above to estimate the value of \( f(-2.9, 2.1) \).

Solution: Recall: \( L(x, y) = 2(x + 3) + 3(y - 2) + 1 \).
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Example

(a) Find the linear approximation \( L(x, y) \) of the function \( f(x, y) = \sin(2x + 3y) + 1 \) at the point \((-3, 2)\).

(b) Use the approximation above to estimate the value of \( f(-2.9, 2.1) \).

Solution: Recall: \( L(x, y) = 2(x + 3) + 3(y - 2) + 1 \).

(b) We use \( L \) to find the linear approximation to \( f(-2.9, 2.1) \).
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Example

(a) Find the linear approximation \( L(x, y) \) of the function \( f(x, y) = \sin(2x + 3y) + 1 \) at the point \((-3, 2)\).

(b) Use the approximation above to estimate the value of \( f(-2.9, 2.1) \).

Solution: Recall: \( L(x, y) = 2(x + 3) + 3(y - 2) + 1 \).

(b) We use \( L \) to find the linear approximation to \( f(-2.9, 2.1) \).

We need to compute \( L(-2.9, 2.1) \).
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Example

(a) Find the linear approximation \( L(x, y) \) of the function 
\( f(x, y) = \sin(2x + 3y) + 1 \) at the point \((-3, 2)\).

(b) Use the approximation above to estimate the value of 
\( f(-2.9, 2.1) \).

Solution: Recall: \( L(x, y) = 2(x + 3) + 3(y - 2) + 1 \).

(b) We use \( L \) to find the linear approximation to \( f(-2.9, 2.1) \).

We need to compute \( L(-2.9, 2.1) \).

\[
L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 1
\]
Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution: Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

(b) We use $L$ to find the linear approximation to $f(-2.9, 2.1)$. We need to compute $L(-2.9, 2.1)$.

$$L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 1$$

$$L(-2.9, 2.1) = 2(0.1) + 3(0.1) + 1$$

Exact value is close to 1.479.
Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution: Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

(b) We use $L$ to find the a linear approximation to $f(-2.9, 2.1)$.

We need to compute $L(-2.9, 2.1)$.

$L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 1$

$L(-2.9, 2.1) = 2(0.1) + 3(0.1) + 1 \quad \Rightarrow \quad L(-2.9, 2.1) = 1.5$. △
Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.9, 2.1)$.

Solution: Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

(b) We use $L$ to find the a linear approximation to $f(-2.9, 2.1)$.

We need to compute $L(-2.9, 2.1)$.

$$L(-2.9, 2.1) = 2(-2.9 + 3) + 3(2.1 - 2) + 1$$

$$L(-2.9, 2.1) = 2(0.1) + 3(0.1) + 1 \Rightarrow L(-2.9, 2.1) = 1.5.$$  

Exact value is close to 1.479.
Section 14.5

Example

(a) Find the gradient of $f(x, y, z) = \sqrt{x + 2yz}$.

(b) Find the directional derivative of $f$ at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4 \rangle$.

(c) Find the maximum rate of change of $f$ at the point $(0, 2, 1)$. 
Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \(\langle 0, 3, 4 \rangle\).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution:

(a) \( \nabla f(x, y, z) = \frac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle. \)
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution:

(a) \( \nabla f(x, y, z) = \frac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle \).

(b) We evaluate the gradient above at \((0, 2, 1)\),
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \( (0, 2, 1) \) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \( (0, 2, 1) \).

Solution:

(a) \( \nabla f(x, y, z) = \frac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle \).

(b) We evaluate the gradient above at \( (0, 2, 1) \),

\[
\nabla f(0, 2, 1) = \frac{1}{2\sqrt{0 + 4}} \langle 1, 2, 4 \rangle
\]
Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).
(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).
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Solution:

(a) \( \nabla f(x, y, z) = \frac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle \).

(b) We evaluate the gradient above at \((0, 2, 1)\),

\[
\nabla f(0, 2, 1) = \frac{1}{2\sqrt{0 + 4}} \langle 1, 2, 4 \rangle = \frac{1}{4} \langle 1, 2, 4 \rangle.
\]
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).
Section 14.5

Example
(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).
(b) Find the directional derivative of \( f \) at \( (0, 2, 1) \) in the direction given by \( \langle 0, 3, 4 \rangle \).
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Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).
We now need a unit vector parallel to \( \langle 0, 3, 4 \rangle \),
Section 14.5

Example
(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).
(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).
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Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).
We now need a unit vector parallel to \( \langle 0, 3, 4 \rangle \),

\[
\mathbf{u} = \frac{1}{\sqrt{9 + 16}} \langle 0, 3, 4 \rangle
\]
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \( (0, 2, 1) \) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \( (0, 2, 1) \).

Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).

We now need a unit vector parallel to \( \langle 0, 3, 4 \rangle \),

\[
u = \frac{1}{\sqrt{9 + 16}} \langle 0, 3, 4 \rangle = \frac{1}{5} \langle 0, 3, 4 \rangle.
\]
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).

We now need a unit vector parallel to \( \langle 0, 3, 4 \rangle \),

\[
\mathbf{u} = \frac{1}{\sqrt{9 + 16}} \langle 0, 3, 4 \rangle = \frac{1}{5} \langle 0, 3, 4 \rangle.
\]

Then, \( (D_uf)(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \cdot \frac{1}{5} \langle 0, 3, 4 \rangle \)
Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x} + 2yz \).

(b) Find the directional derivative of \( f \) at \( (0, 2, 1) \) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \( (0, 2, 1) \).

Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).

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\]

Then, \( (D_u f)(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \cdot \frac{1}{5} \langle 0, 3, 4 \rangle = \frac{1}{20} (6 + 16) \)
Section 14.5

Example
(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).
(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).
(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution: (b) Recall: \( \nabla f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \).

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Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

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Then, \( (D_u f)(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \cdot \frac{1}{5} \langle 0, 3, 4 \rangle = \frac{1}{20} (6 + 16) = \frac{11}{10} \).

We obtain, \( (D_u f)(0, 2, 1) = \frac{11}{10} \).
Section 14.5

Example
(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).
(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).
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Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

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(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution:

(c) The maximum rate of change of \( f \) at a point is the magnitude of its gradient at that point,
Section 14.5

Example

(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution:

(c) The maximum rate of change of \( f \) at a point is the magnitude of its gradient at that point, that is,

\[
|\nabla f(0, 2, 1)| = \frac{1}{4} |\langle 1, 2, 4 \rangle|
\]
Section 14.5

Example
(a) Find the gradient of \( f(x, y, z) = \sqrt{x + 2yz} \).

(b) Find the directional derivative of \( f \) at \((0, 2, 1)\) in the direction given by \( \langle 0, 3, 4 \rangle \).

(c) Find the maximum rate of change of \( f \) at the point \((0, 2, 1)\).

Solution:
(c) The maximum rate of change of \( f \) at a point is the magnitude of its gradient at that point, that is,

\[
|\nabla f(0, 2, 1)| = \frac{1}{4} |\langle 1, 2, 4 \rangle| = \frac{1}{4} \sqrt{1 + 4 + 16} = \frac{\sqrt{21}}{4}.
\]
Example
(a) Find the gradient of $f(x, y, z) = \sqrt{x + 2yz}$.
(b) Find the directional derivative of $f$ at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4 \rangle$.
(c) Find the maximum rate of change of $f$ at the point $(0, 2, 1)$.

Solution:
(c) The maximum rate of change of $f$ at a point is the magnitude of its gradient at that point, that is,

$$|\nabla f(0, 2, 1)| = \frac{1}{4} |\langle 1, 2, 4 \rangle| = \frac{1}{4} \sqrt{1 + 4 + 16} = \frac{\sqrt{21}}{4}.$$ 

The maximum rate of change of $f$ at $(0, 2, 1)$ is

$$|\nabla f(0, 2, 1)| = \frac{\sqrt{21}}{4}. \triangle$$
Example
Find $\partial_{xy}(e^{-xy}\sin(x + yz))$. (Do not simplify your answer.)
Section 14.4

Example
Find $\partial_{xy} (e^{-xy} \sin(x + yz))$. (Do not simplify your answer.)

Solution: We first compute the $x$-derivative,

$$\partial_x (e^{-xy} \sin(x + yz)) =$$
Example
Find \( \partial_{xy}(e^{-xy} \sin(x + yz)) \). (Do not simplify your answer.)

Solution: We first compute the \( x \)-derivative,

\[
\partial_x (e^{-xy} \sin(x + yz)) = -ye^{-xy} \sin(x + yz)
\]
Example
Find $\partial_{xy}(e^{-xy} \sin(x + yz))$. (Do not simplify your answer.)

Solution: We first compute the $x$-derivative,

$$\partial_x(e^{-xy} \sin(x + yz)) = -ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz).$$
Example
Find $\partial_{xy} \left( e^{-xy} \sin(x + yz) \right)$. (Do not simplify your answer.)

Solution: We first compute the $x$-derivative,

$$\partial_x \left( e^{-xy} \sin(x + yz) \right) = -ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz).$$

The second derivative is

$$\partial_{xy} \left( e^{-xy} \sin(x + yz) \right) = \partial_y \left( -ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz) \right),$$
Example
Find $\partial_{xy}(e^{-xy}\sin(x + yz))$. (Do not simplify your answer.)

Solution: We first compute the $x$-derivative,

$$\partial_x(e^{-xy}\sin(x + yz)) = -ye^{-xy}\sin(x + yz) + e^{-xy}\cos(x + yz).$$

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$$\partial_{xy}(e^{-xy}\sin(x + yz)) = \partial_y\left(-ye^{-xy}\sin(x + yz) + e^{-xy}\cos(x + yz)\right),$$

$$= -e^{-xy}\sin(x + yz)$$
Example

Find $\partial_{xy}(e^{-xy}\sin(x + yz))$. (Do not simplify your answer.)

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\partial_x(e^{-xy}\sin(x + yz)) = -ye^{-xy}\sin(x + yz) + e^{-xy}\cos(x + yz).
$$

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$$

$$
= -e^{-xy}\sin(x + yz) + xye^{-xy}\sin(x + yz)
$$
Example

Find \( \partial_{xy} \left( e^{-xy} \sin(x + yz) \right) \). (Do not simplify your answer.)

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\partial_x \left( e^{-xy} \sin(x + yz) \right) = -ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz).
\]

The second derivative is

\[
\partial_{xy} \left( e^{-xy} \sin(x + yz) \right) = \partial_y \left( -ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz) \right),
\]

\[
= -e^{-xy} \sin(x + yz) + xye^{-xy} \sin(x + yz) - ye^{-xy} \cos(x + yz)z
\]
Example
Find $\partial_{xy}(e^{-xy} \sin(x + yz))$. (Do not simplify your answer.)

Solution: We first compute the $x$-derivative,

$$\partial_x(e^{-xy} \sin(x + yz)) = - ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz).$$

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$$\partial_{xy}(e^{-xy} \sin(x + yz)) = \partial_y \left( - ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz) \right),$$

$$= - e^{-xy} \sin(x + yz) + xye^{-xy} \sin(x + yz) - ye^{-xy} \cos(x + yz)z$$

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Section 14.4

Example
Find \( \partial_{xy}(e^{-xy} \sin(x + yz)) \). (Do not simplify your answer.)

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\partial_x (e^{-xy} \sin(x + yz)) = - ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz).
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The second derivative is

\[
\partial_{xy} (e^{-xy} \sin(x + yz)) = \partial_y \left( - ye^{-xy} \sin(x + yz) + e^{-xy} \cos(x + yz) \right),
\]

\[
= - e^{-xy} \sin(x + yz) + xye^{-xy} \sin(x + yz) - ye^{-xy} \cos(x + yz)z
\]

\[
- xe^{-xy} \cos(x + yz) - e^{-xy} \sin(x + yz)z.
\]
\[\triangleleft\]
Example

Find any value of the constant $a$ such that the function

$$f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$$

is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$. 

\[
f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x),
\]

\[
f_{yy} = -e^{-ax} \cos(y) - e^{-y} \cos(x).
\]

Therefore,

\[
f_{xx} + f_{yy} = (a^2 - 1) e^{-ax} \cos(y) + e^{-y} \cos(x).
\]

Function $f$ is solution of $f_{xx} + f_{yy} = 0$ iff $a = \pm 1$. \[\]
Section 14.3

Example

Find any value of the constant \( a \) such that the function
\[
f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)
\]
is solution of Laplace’s equation \( f_{xx} + f_{yy} = 0 \).

Solution:

\[
f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x),
\]
Example

Find any value of the constant $a$ such that the function
\[ f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x) \]
is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:
\[
\begin{align*}
f_x &= -ae^{-ax} \cos(y) + e^{-y} \sin(x), \\
f_{xx} &= a^2 e^{-ax} \cos(y) + e^{-y} \cos(x).
\end{align*}
\]
Section 14.3

Example

Find any value of the constant $a$ such that the function
$f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$ is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:

$f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x)$, \hspace{0.5cm} f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x)$.

$f_y = -e^{-ax} \sin(y) + e^{-y} \cos(x)$,
Section 14.3

Example

Find any value of the constant $a$ such that the function $f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$ is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:

$f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x)$, $f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x)$.

$f_y = -e^{-ax} \sin(y) + e^{-y} \cos(x)$, $f_{yy} = -e^{-ax} \cos(y) - e^{-y} \cos(x)$. 
Section 14.3

Example

Find any value of the constant $a$ such that the function $f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$ is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:

\[ f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x), \quad f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x). \]

\[ f_y = -e^{-ax} \sin(y) + e^{-y} \cos(x), \quad f_{yy} = -e^{-ax} \cos(y) - e^{-y} \cos(x). \]

\[ f_{xx} + f_{yy} = \left[ a^2 e^{-ax} \cos(y) + e^{-y} \cos(x) \right] \\
+ \left[ -e^{-ax} \cos(y) - e^{-y} \cos(x) \right], \]
Section 14.3

Example

Find any value of the constant $a$ such that the function

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Solution:

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$$f_{xx} + f_{yy} = \left[ a^2 e^{-ax} \cos(y) + e^{-y} \cos(x) \right]$$

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Example
Find any value of the constant $a$ such that the function $f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$ is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:

$$f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x), \quad f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x).$$

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$$f_{xx} + f_{yy} = \left[a^2 e^{-ax} \cos(y) + e^{-y} \cos(x)\right]$$
$$+ \left[-e^{-ax} \cos(y) - e^{-y} \cos(x)\right],$$

$$f_{xx} + f_{yy} = (a^2 - 1)e^{-ax} \cos(y).$$

Function $f$ is solution of $f_{xx} + f_{yy} = 0$ iff

$$a = \pm 1.$$
Example

Find any value of the constant $a$ such that the function $f(x, y) = e^{-ax} \cos(y) - e^{-y} \cos(x)$ is solution of Laplace’s equation $f_{xx} + f_{yy} = 0$.

Solution:

$f_x = -ae^{-ax} \cos(y) + e^{-y} \sin(x)$, $f_{xx} = a^2 e^{-ax} \cos(y) + e^{-y} \cos(x)$.

$f_y = -e^{-ax} \sin(y) + e^{-y} \cos(x)$, $f_{yy} = -e^{-ax} \cos(y) - e^{-y} \cos(x)$.

$$f_{xx} + f_{yy} = [a^2 e^{-ax} \cos(y) + e^{-y} \cos(x)] + [-e^{-ax} \cos(y) - e^{-y} \cos(x)],$$

$$f_{xx} + f_{yy} = (a^2 - 1)e^{-ax} \cos(y).$$

Function $f$ is solution of $f_{xx} + f_{yy} = 0$ iff $a = \pm 1$. ◁
Section 14.2

Example

Compute the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \).
Example

Compute the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2}.$$ 

Solution:

Since \( x^2 \leq 2x^2 + 3y^2 \),
Section 14.2

Example

Compute the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \).

Solution:

Since \( x^2 \leq 2x^2 + 3y^2 \), that is, \( \frac{x^2}{2x^2 + 3y^2} \leq 1 \),
Section 14.2

Example

Compute the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \).

Solution:

Since \( x^2 \leq 2x^2 + 3y^2 \), that is, \( \frac{x^2}{2x^2 + 3y^2} \leq 1 \), the non-negative function \( f(x, y) = \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \) satisfies the bounds...
Section 14.2

Example

Compute the limit
\[ \lim_{(x, y) \to (0, 0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2}. \]

Solution:

Since \( x^2 \leq 2x^2 + 3y^2 \), that is, \( \frac{x^2}{2x^2 + 3y^2} \leq 1 \), the non-negative function 
\[ f(x, y) = \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \]

satisfies the bounds
\[ 0 \leq f(x, y) \leq \sin^2(y). \]
Example

Compute the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \).

Solution:
Since \( x^2 \leq 2x^2 + 3y^2 \), that is, \( \frac{x^2}{2x^2 + 3y^2} \leq 1 \), the non-negative function \( f(x, y) = \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \) satisfies the bounds

\[
0 \leq f(x, y) \leq \sin^2(y).
\]

Since \( \lim_{y \to 0} \sin^2(y) = 0 \),
Section 14.2

Example

Compute the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \).

Solution:

Since \( x^2 \leq 2x^2 + 3y^2 \), that is, \( \frac{x^2}{2x^2 + 3y^2} \leq 1 \), the non-negative function \( f(x,y) = \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} \) satisfies the bounds

\[ 0 \leq f(x,y) \leq \sin^2(y). \]

Since \( \lim_{y \to 0} \sin^2(y) = 0 \), the Sandwich Theorem implies that

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{2x^2 + 3y^2} = 0. \]
Section 13.3

Example

Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).
Example
Reparametrize the curve \( \vec{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution:
We first compute the arc length function.
Section 13.3

Example
Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing $t$.

Solution:
We first compute the arc length function. We start with the derivative
\[
\mathbf{r}'(t) = \left\langle 3t \cos(t^2), 4t, -3 \sin(t^2) \right\rangle,
\]
Example
Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution:
We first compute the arc length function. We start with the derivative
\[
\mathbf{r}'(t) = \left\langle 3t \cos(t^2), 4t, -3 \sin(t^2) \right\rangle,
\]
We now need its magnitude,
\[
|\mathbf{r}'(t)| = \sqrt{9t^2 \cos^2(t^2) + 16t^2 + 9 \sin^2(t^2)},
\]
Example

Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

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\[ |\mathbf{r}'(t)| = \sqrt{9t^2 \cos^2(t^2) + 16t^2 + 9 \sin^2(t^2)}, \]

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Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

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\]
We now need its magnitude,
\[
|\mathbf{r}'(t)| = \sqrt{9t^2 \cos^2(t^2) + 16t^2 + 9 \sin^2(t^2)},
\]
\[
|\mathbf{r}'(t)| = \sqrt{9t^2 + 16t^2} = (\sqrt{9 + 16})t
\]
Section 13.3

Example
Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution:
We first compute the arc length function. We start with the derivative
\[
\mathbf{r}'(t) = \left\langle 3t \cos(t^2), 4t, -3 \sin(t^2) \right\rangle,
\]
We now need its magnitude,
\[
|\mathbf{r}'(t)| = \sqrt{9t^2 \cos^2(t^2) + 16t^2 + 9 \sin^2(t^2)},
\]
\[
|\mathbf{r}'(t)| = \sqrt{9t^2 + 16t^2} = (\sqrt{9 + 16}) t \quad \Rightarrow \quad |\mathbf{r}'(t)| = 5t.
\]
Example
Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution: Recall: \( |\mathbf{r}'(t)| = 5t \).
Example

Reparametrize the curve \( r(t) = \left< \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right> \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution: Recall: \( |r'(t)| = 5t \). The arc length function is

\[
s(t) = \int_{1}^{t} 5\tau \, d\tau
\]
Section 13.3

Example
Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution: Recall: \( |\mathbf{r}'(t)| = 5t \). The arc length function is

\[
s(t) = \int_{1}^{t} 5\tau \, d\tau = \frac{5}{2} \left( \tau^2 \bigg|_{1}^{t} \right)
\]
Section 13.3

Example
Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

Solution: Recall: \( |\mathbf{r}'(t)| = 5t \). The arc length function is

\[
s(t) = \int_1^t 5\tau \, d\tau = \frac{5}{2} \left( \tau^2 \bigg|_1^t \right) = \frac{5}{2}(t^2 - 1).\]
Section 13.3

Example

Reparametrize the curve \textbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \text{ with respect to its arc length measured from } t = 1 \text{ in the direction of increasing } t. \newline

Solution: \text{Recall: } |\textbf{r}'(t)| = 5t. \text{ The arc length function is}

\[
s(t) = \int_{1}^{t} 5\tau \, d\tau = \frac{5}{2} \left( \tau^2 \bigg|_{1}^{t} \right) = \frac{5}{2} (t^2 - 1).
\]

Inverting this function for \( t^2 \), we obtain \( t^2 = \frac{2}{5}s + 1. \)
Example

Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

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\]

Inverting this function for \( t^2 \), we obtain \( t^2 = \frac{2}{5}s + 1 \).

The reparametrization of \( \mathbf{r}(t) \) is given by

\[
\hat{\mathbf{r}}(s) = \left\langle \frac{3}{2} \sin\left(\frac{2}{5}s + 1\right), 2\left(\frac{2}{5}s + 1\right), \frac{3}{2} \cos\left(\frac{2}{5}s + 1\right) \right\rangle.
\]
Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- Fubini Theorem on rectangular domains.
- Examples.
Definition

The definite integral of a function $f : [a, b] \to \mathbb{R}$, in the interval $[a, b]$ is the number

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^n f(x_i^*) \Delta x.$$
Definition

The definite integral of a function \( f : [a, b] \rightarrow \mathbb{R} \), in the interval \([a, b]\) is the number

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i^*) \Delta x.
\]

where \( x_i^* \in [x_i, x_{i+1}] \) is called a sample point, while \( \{x_i\} \) is a partition in \([a, b]\), \( i = 0, \ldots, n \), and with \( x_i = a + i\Delta x \), and

\[
\Delta x = \frac{(b - a)}{n}.
\]
The integral as an area.

The sum $S_n = \sum_{i=0}^{n} f(x_i^*) \Delta x$ is called a Riemann sum. Then,

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} S_n.$$
The integral as an area.

The sum $S_n = \sum_{i=0}^{n} f(x_i^*) \Delta x$ is called a Riemann sum. Then,

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} S_n.$$  

The integral $\int_{a}^{b} f(x) \, dx$ is the area in between the graph of $f$ and the horizontal axis.
Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- **Double integral on rectangles.**
- Fubini Theorem on rectangular domains.
- Examples.
Double integrals on rectangles

Definition
The *double integral* of a function \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) in the rectangle \( R = [a, b] \times [c, d] \) is the number

\[
\int\int_{R} f(x, y) \, dx \, dy = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y.
\]
Double integrals on rectangles

Definition
The **double integral** of a function \( f : R \subset \mathbb{R}^2 \to \mathbb{R} \) in the rectangle \( R = [a, b] \times [c, d] \) is the number

\[
\int\int_R f(x, y) \, dx \, dy = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y.
\]

where \( x_i^* \in [x_i, x_{i+1}] \), \( y_j^* \in [y_j, y_{j+1}] \), are sample points, while \( \{x_i\} \) and \( \{y_j\} \), \( i, j = 0, \cdots, n \) are partitions of the intervals \([a, b]\) and \([c, d]\), and

\[
\Delta x = \frac{(b - a)}{n}, \quad \Delta y = \frac{(d - c)}{n}.
\]
The double integral as a volume

The sum
\[ S_n = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y \]
is called a Riemann sum. Then,
\[ \int \int_R f(x, y) \, dx \, dy = \lim_{n \to \infty} S_n. \]
The double integral as a volume

The sum
\[ S_n = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y \]
called a Riemann sum. Then,
\[ \int \int_R f(x, y) \, dx \, dy = \lim_{n \to \infty} S_n. \]

The integral \( \int \int_R f(x, y) \, dx \, dy \) is the volume above \( R \) and below the graph of \( f \).
Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- Fubini Theorem on rectangular domains.
- Examples.
Fubini Theorem on rectangular domains

Theorem

If \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( R = [x_0, x_1] \times [y_0, y_1] \), then

\[
\int \int_R f(x, y) \, dx \, dy = \int_{y_0}^{y_1} \left[ \int_{x_0}^{x_1} f(x, y) \, dx \right] \, dy,
\]

\[
= \int_{x_0}^{x_1} \left[ \int_{y_0}^{y_1} f(x, y) \, dy \right] \, dx.
\]
Fubini Theorem on rectangular domains

Theorem
If \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( R = [x_0, x_1] \times [y_0, y_1] \), then

\[
\int \int _R f(x, y) \, dx \, dy = \int _{y_0} ^{y_1} \left[ \int _{x_0} ^{x_1} f(x, y) \, dx \right] \, dy,
\]

\[
= \int _{x_0} ^{x_1} \left[ \int _{y_0} ^{y_1} f(x, y) \, dy \right] \, dx.
\]

Remark: Fubini’s Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.
Fubini Theorem on rectangular domains

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous in $R = [x_0, x_1] \times [y_0, y_1]$, then

$$\int\int_{R} f(x, y) \, dx \, dy = \int_{y_0}^{y_1} \left[ \int_{x_0}^{x_1} f(x, y) \, dx \right] \, dy,$$

$$= \int_{x_0}^{x_1} \left[ \int_{y_0}^{y_1} f(x, y) \, dy \right] \, dx.$$

Remark: Fubini’s Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.

Notation: The double integral is also written as

$$\int\int_{R} f(x, y) \, dx \, dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) \, dx \, dy.$$
Example

Use Fubini’s Theorem to compute the double integral
\[
\int \int_R f(x, y) \, dx \, dy,
\]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).
Example

Use Fubini’s Theorem to compute the double integral
\[ \int_0^2 \int_1^3 f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: Since \( x \in [0, 2] \) and \( y \in [1, 3] \),
Use Fubini’s Theorem to compute the double integral $$\int \int_R f(x, y) \, dx \, dy$$, where $$f(x, y) = xy^2 + 2x^2y^3$$, and $$R = [0, 2] \times [1, 3]$$. Integrate first in $$x$$, then in $$y$$.

Solution: Since $$x \in [0, 2]$$ and $$y \in [1, 3]$$,

$$I = \int \int_R f(x, y) \, dx \, dy$$
Example

Use Fubini’s Theorem to compute the double integral
\[ \int\int_{R} f(x, y) \, dx \, dy, \]  
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: Since \( x \in [0, 2] \) and \( y \in [1, 3] \),

\[ I = \int\int_{R} f(x, y) \, dx \, dy = \int_{1}^{3} \int_{0}^{2} (xy^2 + 2x^2y^3) \, dx \, dy \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int_{R} \int f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: Since \( x \in [0, 2] \) and \( y \in [1, 3] \),

\[
I = \int_{1}^{3} \left[ \int_{0}^{2} (xy^2 + 2x^2y^3) \, dx \right] \, dy.
\]
Fubini Theorem on rectangular domains

Example
Use Fubini’s Theorem to compute the double integral
\[ \int \int_R f(x, y) \, dx \, dy \], where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: Since \( x \in [0, 2] \) and \( y \in [1, 3] \),

\[ I = \int \int_R f(x, y) \, dx \, dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3) \, dx \, dy \]

\[ I = \int_1^3 \left[ \int_0^2 (xy^2 + 2x^2y^3) \, dx \right] \, dy. \]

We compute the interior integral in \( x \) first, keeping \( y \) constant. After that we compute the integral in \( y \).
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int_R \int f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We compute the integral in \( x \) first, keeping \( y \) constant.

\[ I = \int_1^3 \left[ \int_0^2 (xy^2 + 2x^2y^3) \, dx \right] \, dy, \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We compute the integral in \( x \) first, keeping \( y \) constant.

\[
I = \int \int_{R} f(x, y) \, dx \, dy = \int_{1}^{3} \left[ \int_{0}^{2} (xy^2 + 2x^2y^3) \, dx \right] \, dy,
\]

\[
I = \int_{1}^{3} \left[ \frac{y^2}{2} \left( x^2 \bigg|_{0}^{2} \right) + \frac{2y^3}{3} \left( x^3 \bigg|_{0}^{2} \right) \right] \, dy,
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy \], where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We compute the integral in \( x \) first, keeping \( y \) constant.

\[
I = \int \int_{R} f(x, y) \, dx \, dy = \int_{1}^{3} \left[ \int_{0}^{2} (xy^2 + 2x^2y^3) \, dx \right] \, dy,
\]

\[
I = \int_{1}^{3} \left[ \frac{y^2}{2} \left( x^2 \Big|_{0}^{2} \right) + \frac{2y^3}{3} \left( x^3 \Big|_{0}^{2} \right) \right] \, dy,
\]

\[
I = \int_{1}^{3} \left[ 2y^2 + \frac{16}{3} y^3 \right] \, dy.
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral\[ \int\int_R f(x, y) \, dx \, dy, \]where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We compute the integral in \( x \) first, keeping \( y \) constant.

\[
I = \int\int_R f(x, y) \, dx \, dy = \int_1^3 \left[ \int_0^2 (xy^2 + 2x^2y^3) \, dx \right] \, dy, \\
I = \int_1^3 \left[ \frac{y^2}{2} \left( x^2 \bigg|_0^2 \right) + \frac{2y^3}{3} \left( x^3 \bigg|_0^2 \right) \right] \, dy, \\
I = \int_1^3 \left[ 2y^2 + \frac{16}{3} y^3 \right] \, dy.
\]

We now compute the integral in \( y \).
Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy \], where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We now compute the integral in \( y \).

\[
I = \int_{1}^{3} \left[ 2y^2 + \frac{16}{3}y^3 \right] \, dy
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int_0^2 \int_1^3 f(x, y) \, dx \, dy, \] where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We now compute the integral in \( y \).

\[ I = \int_1^3 \left[ 2y^2 + \frac{16}{3}y^3 \right] \, dy = 2 \int_1^3 \frac{y^3}{3} \, dy + \frac{16}{3} \int_1^3 y^4 \, dy. \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We now compute the integral in \( y \).

\[
I = \int_{1}^{3} \left[ 2y^2 + \frac{16}{3}y^3 \right] \, dy = 2 \frac{y^3}{3} \Bigg|_{1}^{3} + \frac{16}{3} \frac{y^4}{4} \Bigg|_{1}^{3} 
\]

\[
I = 2 \frac{26}{3} + \frac{4}{3} \cdot 80 
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral

\[
\int \int \limits_{R} f(x, y) \, dx \, dy,
\]

where \( f(x, y) = xy^2 + 2x^2y^3 \), and

\( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We now compute the integral in \( y \).

\[
I = \int_{1}^{3} \left[ 2y^2 + \frac{16}{3}y^3 \right] \, dy = 2 \frac{y^3}{3} \bigg|_{1}^{3} + \frac{16}{3} \frac{y^4}{4} \bigg|_{1}^{3}.
\]

\[
I = 2 \frac{26}{3} + \frac{4}{3} 80 = \frac{372}{3}.
\]
Fubini Theorem on rectangular domains

Example
Use Fubini’s Theorem to compute the double integral
\[
\int\int_R f(x, y) \, dx \, dy,
\]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: We now compute the integral in \( y \).

\[
I = \int_1^3 \left[ 2y^2 + \frac{16}{3}y^3 \right] \, dy = 2 \frac{y^3}{3} \bigg|_1^3 + \frac{16}{3} \frac{y^4}{4} \bigg|_1^3.
\]

\[
I = 2 \frac{26}{3} + \frac{4}{3} \cdot 80 = \frac{372}{3}.
\]

We conclude:
\[
\int\int_R f(x, y) \, dx \, dy = \frac{372}{3}.
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[\int \int_R f(x, y) \, dx \, dy,\]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( y \), then in \( x \).
Fubini Theorem on rectangular domains

Example
Use Fubini’s Theorem to compute the double integral
\[ \int \int_R f(x, y) \, dx \, dy \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( y \), then in \( x \).

Solution:
\[ I = \int \int_R f(x, y) \, dx \, dy \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral\[ \int \int_{R} f(x, y) \, dx \, dy, \] where \( f(x, y) = xy^2 + 2x^2y^3, \) and \( R = [0, 2] \times [1, 3]. \) Integrate first in \( y, \) then in \( x. \)

Solution:

\[
I = \int \int_{R} f(x, y) \, dx \, dy = \int_{1}^{3} \int_{0}^{2} (xy^2 + 2x^2y^3) \, dx \, dy
\]
Example

Use Fubini’s Theorem to compute the double integral
\[ \int\int_R f(x, y) \, dx \, dy, \] where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( y \), then in \( x \).

Solution:

\[
I = \int\int_R f(x, y) \, dx \, dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3) \, dx \, dy
\]
\[
I = \int_0^2 \left[ \int_1^3 (xy^2 + 2x^2y^3) \, dy \right] \, dx.
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral 
\[ \int \int_R f(x, y) \, dx \, dy, \] 
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( y \), then in \( x \).

Solution:

\[ I = \int \int_R f(x, y) \, dx \, dy = \int_0^2 \int_1^3 (xy^2 + 2x^2y^3) \, dx \, dy \]

\[ I = \int_0^2 \left[ \int_1^3 (xy^2 + 2x^2y^3) \, dy \right] \, dx. \]

\[ I = \int_0^2 \left[ \frac{x}{3} (y^3 \big|_1^3) + \frac{2x^2}{4} (y^4 \big|_1^3) \right] \, dx. \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int_0^2 \int_1^3 \left( \frac{x y^2}{3} + \frac{2x^2 y^3}{4} \right) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2 y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: \( I = \int_0^2 \left[ \frac{x}{3} (y^3) \bigg|_1^3 + \frac{2x^2}{4} (y^4) \bigg|_1^3 \right] \, dx. \)
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int_{R} \int f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: \( I = \int_{0}^{2} \left[ \frac{x}{3} \left( y^3 \bigg|_{1}^{3} \right) + \frac{2x^2}{4} \left( y^4 \bigg|_{1}^{3} \right) \right] \, dx \).

\[ I = \int_{0}^{2} \left[ \frac{26}{3} x + 40 x^2 \right] \, dx \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_R f(x, y) \, dx \, dy, \]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and
\( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: \( I = \int_0^2 \left[ \frac{x}{3} \left( y^3 \bigg|_1^3 \right) + \frac{2x^2}{4} \left( y^4 \bigg|_1^3 \right) \right] \, dx. \)

\[ I = \int_0^2 \left[ \frac{26}{3} x + 40 x^2 \right] \, dx = \frac{26}{3} \left. \frac{x^2}{2} \right|_0^2 + 40 \left. \frac{x^3}{3} \right|_0^2, \]
Fubini Theorem on rectangular domains

**Example**

Use Fubini’s Theorem to compute the double integral

\[
\int \int_R f(x, y) \, dx \, dy,
\]

where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

**Solution:**

\[
I = \int_0^2 \left[ \frac{x}{3} \left( y^3 \right)_1^3 + \frac{2x^2}{4} \left( y^4 \right)_1^3 \right] \, dx.
\]

\[
I = \int_0^2 \left[ \frac{26}{3} x + 40 x^2 \right] \, dx = \frac{26}{3} \frac{x^2}{2} \bigg|_0^2 + 40 \frac{x^3}{3} \bigg|_0^2,
\]

\[
I = \frac{26}{3} (2) + 40 \frac{8}{3}
\]

We conclude:

\[
\int \int_R f(x, y) \, dx \, dy = \frac{26}{3} (2) + 40 \frac{8}{3}.
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[
\int\int_R f(x, y) \, dx \, dy,
\]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: \( I = \int_0^2 \left[ \frac{x}{3} (y^3\bigg|_1^3) + \frac{2x^2}{4} (y^4\bigg|_1^3) \right] \, dx. \)

\[
I = \int_0^2 \left[ \frac{26}{3} x + 40 x^2 \right] \, dx = \frac{26}{3} \frac{x^2}{2} \bigg|_0^2 + 40 \frac{x^3}{3} \bigg|_0^2,
\]

\[
I = \frac{26}{3} (2) + 40 \frac{8}{3} = \frac{372}{3}.
\]
Example

Use Fubini’s Theorem to compute the double integral 
\[ \int \int_R f(x, y) \, dx \, dy, \] 
where \( f(x, y) = xy^2 + 2x^2y^3 \), and 
\( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: 
\[
I = \int_0^2 \left[ \frac{x}{3} \left( y^3 \right) \right]_1^3 + \frac{2x^2}{4} \left( y^4 \right) \bigg|_1^3 \right] \, dx.
\]

\[
I = \int_0^2 \left[ \frac{26}{3} x + 40 x^2 \right] \, dx = \frac{26}{3} \left. x^2 \right|_0^2 + 40 \left. \frac{x^3}{3} \right|_0^2,
\]

\[
I = \frac{26}{3} (2) + 40 \frac{8}{3} = \frac{372}{3}.
\]

We conclude: 
\[
\int \int_R f(x, y) \, dx \, dy = \frac{372}{3}.
\]
Example

Use Fubini’s Theorem to compute the double integral
\[ \int_{R} f(x, y) \, dx \, dy, \] where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).
Example

Use Fubini’s Theorem to compute the double integral
\[
\int_{R} f(x, y) \, dx \, dy, \text{ where } f(x, y) = \frac{x}{y} + \frac{y}{x}, \text{ and } R = [1, 4] \times [1, 2].
\]

Solution: We choose to first integrate in y and then in x.
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int\int_R f(x, y) \, dx \, dy, \text{ where } f(x, y) = \frac{x}{y} + \frac{y}{x}, \text{ and } R = [1, 4] \times [1, 2]. \]

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\[ I = \int\int_R f(x, y) \, dx \, dy \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
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Solution: We choose to first integrate in \( y \) and then in \( x \).

\[ I = \int \int_R f(x, y) \, dx \, dy = \int_1^4 \int_1^2 \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \, dx, \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[
\int_{R} f(x, y) \, dx \, dy,
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where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).

Solution: We choose to first integrate in \( y \) and then in \( x \).

\[
I = \int_{R} f(x, y) \, dx \, dy = \int_{1}^{4} \int_{1}^{2} \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \, dx,
\]

\[
I = \int_{1}^{4} \left[ \int_{1}^{2} \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] \, dx.
\]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral

\[ \int_1^4 \int_1^2 (\frac{x}{y} + \frac{y}{x}) \, dy \, dx, \]

where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).

Solution: We choose to first integrate in \( y \) and then in \( x \).

\[
I = \int_1^4 \left[ \int_1^2 \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] dx
\]

\[
I = \int_1^4 \left[ \int_1^2 \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] dx
\]

\[
I = \int_1^4 \left[ x \left( \ln(y) \right)_{1}^{2} + \frac{1}{x} \left( \frac{y^2}{2} \right)_{1}^{2} \right] dx
\]
Example

Use Fubini’s Theorem to compute the double integral
\[
\int_{R} \int f(x, y) \, dx \, dy
\]
where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).

Solution: We choose to first integrate in \( y \) and then in \( x \).

\[
I = \int_{R} \int f(x, y) \, dx \, dy = \int_{1}^{4} \left[ \int_{1}^{2} \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] \, dx,
\]

\[
I = \int_{1}^{4} \left[ \int_{1}^{2} \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] \, dx = \int_{1}^{4} \left[ x \left( \ln(y) \bigg|_{1}^{2} \right) + \frac{1}{x} \left( \frac{y^2}{2} \bigg|_{1}^{2} \right) \right] \, dx
\]

\[
I = \int_{1}^{4} \left[ \ln(2) x + \frac{3}{2} \frac{1}{x} \right] \, dx.
\]
Fubini Theorem on rectangular domains

Example
Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy, \] where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).

Solution: We compute the integral in \( x \),

\[ I = \int_{1}^{4} \left[ \ln(2) x + \frac{3}{2} \frac{1}{x} \right] \, dx \]
Example

Use Fubini’s Theorem to compute the double integral
\[
\int_1^4 \int_1^2 f(x, y) \, dx \, dy,
\]
where \( f(x, y) = \frac{x}{y} + \frac{y}{x} \), and \( R = [1, 4] \times [1, 2] \).

Solution: We compute the integral in \( x \),

\[
I = \int_1^4 \left[ \ln(2) x + \frac{3}{2} \frac{1}{x} \right] \, dx = \ln(2) \left( \frac{x^2}{2} \bigg|_1^4 \right) + \frac{3}{2} \left( \ln(x) \bigg|_1^4 \right),
\]

\[
= \ln(2) \left( \frac{16}{2} - \frac{1}{2} \right) + \frac{3}{2} \left( \ln(4) - \ln(1) \right) = 7 \ln(2) + \frac{3}{2} \ln(4).
\]
Example

Use Fubini’s Theorem to compute the double integral

\[ \int_1^4 \left[ \ln(2) x + \frac{3}{2} \frac{1}{x} \right] dx = \ln(2) \left( \frac{x^2}{2} \right) \bigg|_1^4 + \frac{3}{2} \left( \ln(x) \right) \bigg|_1^4, \]

\[ I = \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4) \]
Fubini Theorem on rectangular domains

Example

Use Fubini’s Theorem to compute the double integral
\[ \int \int_{R} f(x, y) \, dx \, dy, \text{ where } f(x, y) = \frac{x}{y} + \frac{y}{x}, \text{ and } R = [1, 4] \times [1, 2]. \]

Solution: We compute the integral in \( x \),

\[
I = \int_{1}^{4} \left[ \ln(2) x + \frac{3}{2} \frac{1}{x} \right] \, dx = \ln(2) \left( \frac{x^2}{2} \bigg|_{1}^{4} \right) + \frac{3}{2} \left( \ln(x) \bigg|_{1}^{4} \right),
\]

\[
I = \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4) = \left( \frac{15}{2} + 3 \right) \ln(2).
\]
Example
Use Fubini’s Theorem to compute the double integral
\[ \int\int_R f(x, y) \, dx \, dy, \text{ where } f(x, y) = \frac{x}{y} + \frac{y}{x}, \text{ and } R = [1, 4] \times [1, 2]. \]

Solution: We compute the integral in \( x \),

\[
I = \int_1^4 \left[ \ln(2)x + \frac{3}{2} \frac{1}{x} \right] \, dx = \ln(2) \left( \frac{x^4}{2} \bigg|_1^4 \right) + \frac{3}{2} \left( \ln(x) \bigg|_1^4 \right),
\]

\[
I = \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4) = \left( \frac{15}{2} + 3 \right) \ln(2).
\]

We conclude:
\[ \int\int_R f(x, y) \, dx \, dy = \frac{21}{2} \ln(2). \]
A particular case of Fubini’s Theorem

Corollary

If the continuous function $f : R \subset R^2 \rightarrow \mathbb{R}$ satisfies that
$f(x, y) = g(x)h(y)$, then the double integral of function $f$ in the rectangle $R = [x_0, x_1] \times [y_0, y_1]$ is given by

$$
\int_{x_0}^{x_1} \int_{y_0}^{y_1} g(x)h(y) dy \, dx = \left( \int_{x_0}^{x_1} g(x) dx \right) \left( \int_{y_0}^{y_1} h(y) dy \right).
$$

Remark:
In the case that $f(x, y)$ is a product of two functions $g, h$, with $g(x)$ and $h(y)$, then the double integral of $f$ is also a product of the integral of $g$ times the integral of $h$. 
Corollary

If the continuous function \( f : R \subset R^2 \rightarrow \mathbb{R} \) satisfies that
\[ f(x, y) = g(x)h(y), \]
then the double integral of function \( f \) in the rectangle \( R = [x_0, x_1] \times [y_0, y_1] \) is given by

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} g(x)h(y)dy \, dx = \left( \int_{x_0}^{x_1} g(x)dx \right) \left( \int_{y_0}^{y_1} h(y)dy \right).
\]

Remark: In the case that \( f(x, y) \) is a product of two functions \( g, h \), with \( g(x) \) and \( h(y) \), then the double integral of \( f \) is also a product of the integral of \( g \) times the integral of \( h \).
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: \( I = \int \int_R f(x, y) \, dx \, dy \)
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: \( I = \int\int_R f(x, y) \, dx \, dy = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx \),

\[
I = \left. \left( x + \frac{1}{3} x^3 \right) \right|_0^2 \left( \arctan y \right|_0^1 = \left( 2 + \frac{8}{3} \right) \frac{\pi}{4} = \frac{14}{3} \frac{\pi}{4}.
\]

We conclude \( \int\int_R f(x, y) \, dx \, dy = \frac{7}{6} \pi \).
Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: 
\[
I = \int \int_R f(x, y) \, dx \, dy = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx,
\]

\[
I = \left[ \int_0^2 (1 + x^2) \, dx \right] \left[ \int_0^1 \frac{1}{1 + y^2} \, dy \right],
\]

We conclude \( \int \int_R f(x, y) \, dx \, dy = \frac{14}{3} \pi \). \( \triangleleft \)
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: 
\[
I = \int_{0}^{2} \int_{0}^{1} \frac{1 + x^2}{1 + y^2} \, dy \, dx,
\]

\[
I = \left[ \int_{0}^{2} (1 + x^2) \, dx \right] \left[ \int_{0}^{1} \frac{1}{1 + y^2} \, dy \right],
\]

\[
I = \left( x \Big|_{0}^{2} + \frac{1}{3} x^3 \Big|_{0}^{2} \right) \left( \arctan(y) \Big|_{0}^{1} \right)
\]

We conclude \( \int_{R} f(x, y) \, dx \, dy = \frac{7}{6} \pi \).
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: \( I = \int_R f(x, y) \, dx \, dy = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx \),

\[
I = \left[ \int_0^2 (1 + x^2) \, dx \right] \left[ \int_0^1 \frac{1}{1 + y^2} \, dy \right],
\]

\[
I = \left( x \bigg|_0^2 + \frac{1}{3} x^3 \bigg|_0^2 \right) \left( \arctan(y) \bigg|_0^1 \right) = \left( 2 + \frac{8}{3} \right) \frac{\pi}{4}
\]
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: 

\[
I = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx,
\]

\[
I = \left[ \int_0^2 (1 + x^2) \, dx \right] \left[ \int_0^1 \frac{1}{1 + y^2} \, dy \right],
\]

\[
I = \left( x \bigg|_0^2 + \frac{1}{3} x^3 \bigg|_0^2 \right) \left( \arctan(y) \bigg|_0^1 \right) = \left( 2 + \frac{8}{3} \right) \frac{\pi}{4} = \frac{14}{3} \frac{\pi}{4}
\]
A particular case of Fubini’s Theorem

Example

Compute the double integral of \( f(x, y) = \frac{1 + x^2}{1 + y^2} \), in the rectangular region \( R = [0, 2] \times [0, 1] \).

Solution: \( I = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx \),

\[
I = \left[ \int_0^2 (1 + x^2) \, dx \right] \left[ \int_0^1 \frac{1}{1 + y^2} \, dy \right],
\]

\[
I = \left( x \bigg|_0^2 + \frac{1}{3} x^3 \bigg|_0^2 \right) \left( \arctan(y) \bigg|_0^1 \right) = \left( 2 + \frac{8}{3} \right) \frac{\pi}{4} = \frac{14}{3} \frac{\pi}{4}
\]

We conclude \( \int_0^2 \int_0^1 f(x, y) \, dx \, dy = \frac{7}{6} \pi \). \( \triangleq \)