Review for MTH 234 Exam 1.

- Sections 12.1-12.6.
- 50 minutes.
- Problems similar to homework problems.
- No calculators, no notes, no books, no phones.
Example

Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point $P = (1, 1, 3)$. 

Solution:
The equation of the sphere, 

$$x^2 + y^2 + z^2 = 25.$$ 

Horizontal plane, 

$$z = z_0.$$ 

Contains $P = (1, 1, 3)$, that is, $z = 3$. 

The intersection is:

$$x^2 + y^2 + 3^2 = 25 \Rightarrow x^2 + y^2 = 25 - 9 = 16 = 4^2.$$ 

This is a circle radius $r = 4$ centered at $x = 0, y = 0, z = 3$, contained in the horizontal plane $z = 3$. 

\[\triangleright\]
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$$x^2 + y^2 + 3^2 = 25 \quad \Rightarrow \quad x^2 + y^2 = 25 - 9$$
Section 12.1

Example
Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point $P = (1, 1, 3)$.

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Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point \( P = (1, 1, 3) \).

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\begin{align*}
  x^2 + y^2 + 3^2 &= 25 \\
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Section 12.1

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Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point $P = (1, 1, 3)$.

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This is a circle radius $r = 4$. 
Section 12.1

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Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point \( P = (1, 1, 3) \).

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\[\triangleleft\]
Section 12.3

Example
Consider the vectors $\mathbf{v} = 2 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2 \mathbf{j} - \mathbf{k}$.

(a) Compute $\mathbf{v} \cdot \mathbf{w}$.
Section 12.3

Example
Consider the vectors \( \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}. \)

(a) Compute \( \mathbf{v} \cdot \mathbf{w} \).

Solution: Recall: \( \mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z. \)
Section 12.3

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\[ \mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle \]
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\[ \mathbf{v} \cdot \mathbf{w} = (2, -2, 1) \cdot (1, 2, -1) = 2 - 4 - 1 \]
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\mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \implies \mathbf{v} \cdot \mathbf{w} = -3.
\]

(b) Find the cosine of the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

Solution: Recall: \( \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \).

\[
|\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, 
|\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.
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\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{-3}{3 \sqrt{6}} \implies \cos(\theta) = -\frac{1}{\sqrt{6}}.
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Section 12.3

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**Solution:** Recall: $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.

$$\mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \Rightarrow \mathbf{v} \cdot \mathbf{w} = -3.$$ $\triangle$

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$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{-3}{3\sqrt{6}} \Rightarrow \cos(\theta) = -\frac{1}{\sqrt{6}}.$$ $\triangle$
Section 12.3

Example

(a) Find a unit vector opposite to \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).
Section 12.3

Example

(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution: The vector is $\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|}$. 
Section 12.3

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(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

**Solution:** The vector is $\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|}$. Since,

$$|\mathbf{v}| = \sqrt{1 + 4 + 1}$$
Section 12.3

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(a) Find a unit vector opposite to \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

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|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6},
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Section 12.3

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(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

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$$||\mathbf{v}|| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad \mathbf{u} = -\frac{1}{\sqrt{6}}(1, -2, 1).$$
Example

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|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad \mathbf{u} = -\frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.
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(b) Find \( |\mathbf{u} - 2\mathbf{v}| \), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \), and \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).
Section 12.3

Example

(a) Find a unit vector opposite to \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

Solution: The vector is \( \mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|} \). Since,

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(b) Find \( |\mathbf{u} - 2\mathbf{v}| \), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \), and \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

Solution: First find \( \mathbf{u} - 2\mathbf{v} \), then find \( |\mathbf{u} - 2\mathbf{v}| \).
Section 12.3

Example

(a) Find a unit vector opposite to $v = i - 2j + k$.

Solution: The vector is $u = -\frac{v}{|v|}$. Since,

$$|v| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad u = -\frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.$$

(b) Find $|u - 2v|$, where $u = 3i + 2j + k$, and $v = i - 2j + k$.

Solution: First find $u - 2v$, then find $|u - 2v|$.

$$u - 2v = \langle 3, 2, 1 \rangle - 2\langle 1, -2, 1 \rangle$$
Section 12.3

Example

(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution: The vector is $\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|}$. Since,

$$|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad \mathbf{u} = -\frac{1}{\sqrt{6}}\langle 1, -2, 1 \rangle.$$

(b) Find $|\mathbf{u} - 2\mathbf{v}|$, where $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution: First find $\mathbf{u} - 2\mathbf{v}$, then find $|\mathbf{u} - 2\mathbf{v}|$.

$$\mathbf{u} - 2\mathbf{v} = \langle 3, 2, 1 \rangle - 2\langle 1, -2, 1 \rangle = \langle 1, 6, -1 \rangle$$
Section 12.3

Example

(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

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Solution: First find $\mathbf{u} - 2\mathbf{v}$, then find $|\mathbf{u} - 2\mathbf{v}|$.

$$\mathbf{u} - 2\mathbf{v} = \langle 3, 2, 1 \rangle - 2\langle 1, -2, 1 \rangle = \langle 1, 6, -1 \rangle$$

$$|\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1}$$
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|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad \mathbf{u} = -\frac{1}{\sqrt{6}}\langle 1, -2, 1 \rangle.
\]

(b) Find \( |\mathbf{u} - 2\mathbf{v}| \), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \), and \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

Solution: First find \( \mathbf{u} - 2\mathbf{v} \), then find \( |\mathbf{u} - 2\mathbf{v}| \).

\[
\mathbf{u} - 2\mathbf{v} = \langle 3, 2, 1 \rangle - 2\langle 1, -2, 1 \rangle = \langle 1, 6, -1 \rangle
\]

\[
|\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1} \quad \Rightarrow \quad |\mathbf{u} - 2\mathbf{v}| = \sqrt{38}.
\]

\( \triangleright \)
Section 12.3

Example
Find the vector projection of vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ onto vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution:
Recall: $P_\mathbf{u}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||^2}\right)\mathbf{u}$.

$\mathbf{u} \cdot \mathbf{v} = \langle -1,3,-3 \rangle \cdot \langle 1,-1,2 \rangle = -1 - 3 - 6 = -10$.

Since $||\mathbf{u}|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$, we obtain that $P_\mathbf{u}(\mathbf{v}) = \left(\frac{-10}{\sqrt{6}}\right)\mathbf{u} = -\frac{5}{3}\mathbf{u} = -\frac{5}{3}\langle 1,-1,2 \rangle$.

We conclude that $P_\mathbf{u}(\mathbf{v}) = -\frac{5}{3}\mathbf{u}$.
Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( \mathbf{P}_u(\mathbf{v}) \)
Example

Find the vector projection of vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ onto vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution: Recall: $P_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|}$.
Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( P_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||} \right) \frac{\mathbf{u}}{||\mathbf{u}||} \).

\[
\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle
\]
Example
Find the vector projection of vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ onto vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution: Recall: $\mathbf{P}_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|}$.

$\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6$
Section 12.3

Example
Find the vector projection of vector $v = -i + 3j - 3k$ onto vector $u = i - j + 2k$.

Solution: Recall: $P_u(v) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||}\right) \frac{\mathbf{u}}{||\mathbf{u}||}$.

$\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \Rightarrow \mathbf{u} \cdot \mathbf{v} = -10.$
Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( \mathbf{P}_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} \).

\[
\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \quad \Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = -10.
\]

Since \( |\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 2^2} \)
Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( \mathbf{P}_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} \).

\[ \mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \Rightarrow \mathbf{u} \cdot \mathbf{v} = -10. \]

Since \( |\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}, \)

Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( P_u(v) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} \).

\[
\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \quad \Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = -10.
\]

Since \( |\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \), we obtain that

\[
P_u(v) = \left( \frac{-10}{\sqrt{6}} \right) \frac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle.
\]
Example
Find the vector projection of vector \( \mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \) onto vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Solution: Recall: \( \mathbf{P}_u(\mathbf{v}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} \).

\[
\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \implies \mathbf{u} \cdot \mathbf{v} = -10.
\]

Since \( |\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \), we obtain that

\[
\mathbf{P}_u(\mathbf{v}) = \left( \frac{-10}{\sqrt{6}} \right) \frac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle.
\]

We conclude that \( \mathbf{P}_u(\mathbf{v}) = -\frac{5}{3} \langle 1, -1, 2 \rangle \). \( \triangle \)
Section 12.4

Example
Find a unit vector \( \mathbf{u} \) normal to both \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \).
Example
Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

Solution: A solution is a vector proportional to $\mathbf{v} \times \mathbf{w}$. 

\[ \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2 + 6) \mathbf{i} - (6 - 6) \mathbf{j} + (12 + 4) \mathbf{k} = \langle 8, 0, 16 \rangle. \]

Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2 \rangle$ instead of $\langle 8, 0, 16 \rangle$.

$\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{\|\langle 1, 0, 2 \rangle\|} \Rightarrow \mathbf{u} = \frac{1}{\sqrt{5}} \langle 1, 0, 2 \rangle. \]
Example
Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

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$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix}$$
Section 12.4

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Find a unit vector \( \mathbf{u} \) normal to both \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \).

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\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2+6)\mathbf{i} - (6-6)\mathbf{j} + (12+4)\mathbf{k} = \langle 8, 0, 16 \rangle.
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Solution: A solution is a vector proportional to $\mathbf{v} \times \mathbf{w}$.

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\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
i & j & k \\
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\end{vmatrix} = (2+6)i - (6-6)j + (12+4)k = \langle 8, 0, 16 \rangle.
\]

Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2 \rangle$ instead of $\langle 8, 0, 16 \rangle$.
Example
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\[
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\]

Since we look for a unit vector, the calculation is simpler with \( \langle 1, 0, 2 \rangle \) instead of \( \langle 8, 0, 16 \rangle \).

\[
\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|}
\]
Section 12.4

Example
Find a unit vector \( \mathbf{u} \) normal to both \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \).

Solution: A solution is a vector proportional to \( \mathbf{v} \times \mathbf{w} \).

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
i & j & k \\
6 & 2 & -3 \\
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\]

Since we look for a unit vector, the calculation is simpler with \( \langle 1, 0, 2 \rangle \) instead of \( \langle 8, 0, 16 \rangle \).

\[
\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}} \langle 1, 0, 2 \rangle.
\]
Example
Find the area of the parallelogram formed by $v = \langle 6, 2, -3 \rangle$ and $w = \langle -2, 2, 1 \rangle$, given in the example above.
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Example
Find the area of the parallelogram formed by \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \), given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \) is

\[
A = |\mathbf{v} \times \mathbf{w}|
\]

Since \( \mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle \), then

\[
A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2} = \sqrt{8^2 (1 + 4)}.
\]

We conclude that \( A = 8\sqrt{5} \). ✷
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Example
Find the area of the parallelogram formed by \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \), given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \) is \( A = |\mathbf{v} \times \mathbf{w}| \).
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Find the area of the parallelogram formed by \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \), given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \) is \( A = |\mathbf{v} \times \mathbf{w}| \).

Since \( \mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle \),
Example
Find the area of the parallelogram formed by $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$, given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ is $A = |\mathbf{v} \times \mathbf{w}|$.

Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}|$$
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Example
Find the area of the parallelogram formed by $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$, given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ is $A = |\mathbf{v} \times \mathbf{w}|$.

Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2}$$
Example
Find the area of the parallelogram formed by $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$, given in the example above.

Solution:
Recall: The area of the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ is $A = |\mathbf{v} \times \mathbf{w}|$.

Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2} = \sqrt{8^2 + 8^2 \cdot 2^2}$$
**Example**

Find the area of the parallelogram formed by \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \), given in the example above.

**Solution:**

Recall: The area of the parallelogram formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \) is \( A = |\mathbf{v} \times \mathbf{w}| \).

Since \( \mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle \), then

\[
A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2} = \sqrt{8^2 + 8^2 \cdot 2^2} = \sqrt{8^2(1 + 4)}.
\]
Example
Find the area of the parallelogram formed by \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \), given in the example above.

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Recall: The area of the parallelogram formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \) is \( A = |\mathbf{v} \times \mathbf{w}| \).
Since \( \mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle \), then

\[
A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2} = \sqrt{8^2 + 8^2 2^2} = \sqrt{8^2(1 + 4)}.
\]

We conclude that

\[
A = 8\sqrt{5}.
\]
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).
Example

Find the volume of the parallelepiped determined by the vectors $u = \langle 6, 3, -1 \rangle$, $v = \langle 0, 1, 2 \rangle$, and $w = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $u \cdot (v \times w)$. 

\[
\begin{vmatrix}
9 & 8 & -4 \\
6 & 3 & -1 \\
0 & 1 & 2 \\
\end{vmatrix}
\] 

Since $V = |u \cdot (v \times w)|$, we obtain $V = 82$. 

\[\]
Example

Find the volume of the parallelepiped determined by the vectors
\( u = \langle 6, 3, -1 \rangle \), \( v = \langle 0, 1, 2 \rangle \), and \( w = \langle 4, -2, 5 \rangle \).

**Solution:** We need to compute the triple product \( u \cdot (v \times w) \).
We must start with the cross product.
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    0 & 1 & 2 \\
    4 & -2 & 5 \\
\end{vmatrix}
\]

Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.
\]
Triple product (Only if covered by your instructor.)

Example
Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = \langle 6, 3, -1 \rangle$, $\mathbf{v} = \langle 0, 1, 2 \rangle$, and $\mathbf{w} = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 1 & 2 \\
4 & -2 & 5
\end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.
\]

We obtain $\mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle$. 
Example

Find the volume of the parallelepiped determined by the vectors \( u = \langle 6, 3, -1 \rangle \), \( v = \langle 0, 1, 2 \rangle \), and \( w = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( u \cdot (v \times w) \). We must start with the cross product.

\[
v \times w = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.
\]

We obtain \( v \times w = \langle 9, 8, -4 \rangle \). The triple product is

\[
u \cdot (v \times w)
\]
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 1 & 2 \\
4 & -2 & 5 \\
\end{vmatrix} = \langle 5 + 4, -(0 - 8), (0 - 4) \rangle.
\]

We obtain \( \mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle \). The triple product is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle
\]
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

**Solution:** We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle 5 + 4, -(0 - 8), (0 - 4) \rangle.
\]

We obtain \( \mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle \). The triple product is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4
\]
Example
Find the volume of the parallelepiped determined by the vectors $u = \langle 6, 3, -1 \rangle$, $v = \langle 0, 1, 2 \rangle$, and $w = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $u \cdot (v \times w)$. We must start with the cross product.

$$v \times w = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.$$ 

We obtain $v \times w = \langle 9, 8, -4 \rangle$. The triple product is

$$u \cdot (v \times w) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.$$
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
i & j & k \\
0 & 1 & 2 \\
4 & -2 & 5 \\
\end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.
\]

We obtain \( \mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle \). The triple product is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.
\]

Since \( V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \),
Example

Find the volume of the parallelepiped determined by the vectors $u = \langle 6, 3, -1 \rangle$, $v = \langle 0, 1, 2 \rangle$, and $w = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $u \cdot (v \times w)$. We must start with the cross product.

\[
v \times w = \begin{vmatrix}
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4 & -2 & 5 \\
\end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.
\]

We obtain $v \times w = \langle 9, 8, -4 \rangle$. The triple product is

\[
u \cdot (v \times w) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.
\]

Since $V = |u \cdot (v \times w)|$, we obtain $V = 82$. △
Example
Does the line given by \( \mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t \) intersects the plane \( 2x + y - z = 1 \)? If “yes”, then find the intersection point.

Solution:
First, find the parametric equation of the line,
\[
\begin{align*}
x(t) &= t, \\
y(t) &= 1 + 2t, \\
z(t) &= 1 + 3t.
\end{align*}
\]
Then replace \( x(t) \), \( y(t) \), and \( z(t) \) above in the equation of the plane \( 2x + y - z = 1 \).
If there is a solution for \( t \), then there is an intersection between the line and the plane.
Let us find that out,
\[
2t + (1 + 2t) - (1 + 3t) = 1
\]
\( \Rightarrow \)
\( t = 1 \).
So, the intersection of the line and the plane is the point with coordinates \( x = 1, y = 3, z = 4 \), that is, \( P = (1, 3, 4) \).
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Example

Does the line given by \( \mathbf{r}(t) = \langle 0,1,1 \rangle + \langle 1,2,3 \rangle t \) intersects the plane \( 2x + y - z = 1 \)? If “yes”, then find the intersection point.

Solution: First, find the parametric equation of the line,
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If there is a solution for \( t \), then there is an intersection between the line and the plane.
Example
Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t$ intersects the plane $2x + y - z = 1$? If “yes”, then find the intersection point.

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If there is a solution for $t$, then there is an intersection between the line and the plane. Let us find that out,
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Example
Does the line given by \( \mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t \) intersects the plane \( 7x - 2y - z = 1 \)? If “yes”, then find the intersection point.

Solution:
First, find the parametric equation of the line,
\[
\begin{align*}
x(t) &= t, \\
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z(t) &= 1 + 3t.
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\]
Then replace \( x(t), y(t), \) and \( z(t) \) above in the equation of the plane \( 7x - 2y - z = 1 \).
In this case we get
\[
7t - 2(1 + 2t) - (1 + 3t) = 1
\]
\[
\Rightarrow -3 = 1,
\]
No solution.
So, there is no intersection between the line and the plane.

Notice:
The line is parallel and not contained in the plane.
\[
\langle 1, 2, 3 \rangle \cdot \langle 7, -2, -3 \rangle = 0,
\]
and
\[
\mathbf{P} = (0, 1, 1) \in \text{The Plane}.
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Notice: The line is parallel and not contained in the plane.
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Does the line given by \( r(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t \) intersects the plane \( 7x - 2y - z = 1 \)? If “yes”, then find the intersection point.

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Then replace \( x(t) \), \( y(t) \), and \( z(t) \) above in the equation of the plane \( 7x - 2y - z = 1 \). In this case we get

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So, there is no intersection between the line and the plane. \( \triangleleft \)

Notice: The line is parallel and not contained in the plane.

\[
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Section 12.5

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Does the line given by \( \mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t \) intersects the plane \( 7x - 2y - z = 1 \)? If “yes”, then find the intersection point.

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x(t) = t, \quad y(t) = 1 + 2t, \quad z(t) = 1 + 3t.
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Notice: The line is parallel and not contained in the plane.
\[
\langle 1, 2, 3 \rangle \cdot \langle 7, -2, -3 \rangle = 0, \quad \text{and} \quad P = (0, 1, 1) \notin \text{The Plane.}
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Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$. 
Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $r(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$. 
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$.

A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. 

Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$.

Solution:

The vector equation of the line is $r(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$. A vector tangent to the line, and so to the plane, is $v = \langle 1, 1, 2 \rangle$. 
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Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

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The vector equation of the line is $r(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$. A vector tangent to the line, and so to the plane, is $v = \langle 1, 1, 2 \rangle$.

The point $P_0 = (1, 2, 3)$ is in the plane.
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$.
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The point $P_0 = (1, 2, 3)$ is in the plane. The line is in the plane,
Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:
The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle$.
A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$.

The point $P_0 = (1, 2, 3)$ is in the plane. The line is in the plane, hence $P_1 = (-2, 0, -1)$ is in the plane.
Example

Find the equation for the plane that contains the point \( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, \ y = t, \ z = -1 + 2t \).

Solution:

The vector equation of the line is \( \mathbf{r}(t) = \langle -2, 0, -1 \rangle + t \langle 1, 1, 2 \rangle \).
A vector tangent to the line, and so to the plane, is \( \mathbf{v} = \langle 1, 1, 2 \rangle \).

The point \( P_0 = (1, 2, 3) \) is in the plane. The line is in the plane, hence \( P_1 = (-2, 0, -1) \) is in the plane.
Then a second vector tangent to the plane is \( \overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle \).
Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$. 

Solution:

Recall:

$v = \langle 1, 1, 2 \rangle, \quad \vec{P}_1P_0 = \langle 3, 2, 4 \rangle$.

The normal to the plane is $n = v \times \vec{P}_1P_0$.

\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & 1 & 2 \\
  3 & 2 & 4 \\
\end{vmatrix} 
\] 

$\Rightarrow n = \langle 0, 2, -1 \rangle$.

The plane normal to $\langle 0, 2, -1 \rangle$ containing $P_0 = (1, 2, 3)$ is given by

$0(x - 1) + 2(y - 2) - (z - 3) = 0 \Rightarrow 2y - z = 1$. 

$\triangleleft$
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$.

Solution:

Recall: $v = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$. 

⇒ $2y - z = 1$. \(\Box\)
Example

Find the equation for the plane that contains the point \( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, \ y = t, \ z = -1 + 2t \).

Solution:

Recall: \( \mathbf{v} = \langle 1, 1, 2 \rangle \), \( \overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle \).

The normal to the plane is \( \mathbf{n} = \mathbf{v} \times \overrightarrow{P_1P_0} \).
Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$.

Solution:

Recall: $\mathbf{v} = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

The normal to the plane is $\mathbf{n} = \mathbf{v} \times \overrightarrow{P_1P_0}$.

$$
\mathbf{n} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 1 & 2 \\
3 & 2 & 4 \\
\end{vmatrix} = \langle 0, 2, -1 \rangle.
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The plane normal to $\langle 0, 2, -1 \rangle$ containing $P_0 = (1, 2, 3)$ is given by

$$0 (x - 1) + 2(y - 2) - (z - 3) = 0 \Rightarrow 2y - z = 1.$$
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$.

Solution:

Recall: $v = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

The normal to the plane is $n = v \times \overrightarrow{P_1P_0}$.

$$n = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle (4 - 4), -(4 - 6), (2 - 3) \rangle = \langle 0, 2, -1 \rangle.$$
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

Recall: $\mathbf{v} = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

The normal to the plane is $\mathbf{n} = \mathbf{v} \times \overrightarrow{P_1P_0}$.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle (4 - 4), -(4 - 6), (2 - 3) \rangle \quad \Rightarrow \quad \mathbf{n} = \langle 0, 2, -1 \rangle.$$
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Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$.

Solution:

Recall: $v = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

The normal to the plane is $n = v \times \overrightarrow{P_1P_0}$.

\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & 1 & 2 \\
  3 & 2 & 4 \\
\end{vmatrix} = \langle (4 - 4), -(4 - 6), (2 - 3) \rangle \implies n = \langle 0, 2, -1 \rangle.
\]

The plane normal to $\langle 0, 2, -1 \rangle$ containing $P_0 = (1, 2, 3)$ is given by
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Example

Find the equation for the plane that contains the point
$P_0 = (1, 2, 3)$ and the line $x = −2 + t, y = t, z = −1 + 2t$.

Solution:

Recall: $v = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

The normal to the plane is $n = v \times \overrightarrow{P_1P_0}$.

\[
\begin{vmatrix}
i & j & k \\
1 & 1 & 2 \\
3 & 2 & 4
\end{vmatrix} = \langle (4 − 4), −(4 − 6), (2 − 3) \rangle \quad \Rightarrow \quad n = \langle 0, 2, −1 \rangle.
\]

The plane normal to $\langle 0, 2, −1 \rangle$ containing $P_0 = (1, 2, 3)$ is given by

$0(x − 1) + 2(y − 2) − (z − 3) = 0$
Example

Find the equation for the plane that contains the point
\( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, \ y = t, \ z = -1 + 2t \).

Solution:

Recall: \( \mathbf{v} = \langle 1, 1, 2 \rangle, \ \overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle \).

The normal to the plane is
\( \mathbf{n} = \mathbf{v} \times \overrightarrow{P_1P_0} \).

\[
\mathbf{n} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 1 & 2 \\
3 & 2 & 4
\end{vmatrix}
= \langle (4 - 4), -(4 - 6), (2 - 3) \rangle \quad \Rightarrow \quad \mathbf{n} = \langle 0, 2, -1 \rangle.
\]

The plane normal to \( \langle 0, 2, -1 \rangle \) containing \( P_0 = (1, 2, 3) \) is given by
\[
0(x - 1) + 2(y - 2) - (z - 3) = 0 \quad \Rightarrow \quad 2y - z = 1.
\]
Example

Find the equation of the plane that containing the points $P = (1, 1, 1)$, $Q = (1, -1, 1)$, and $R = (0, 0, 2)$. 

Solution:

Find two vectors tangent to the plane: $\vec{PQ}$, $\vec{PR}$.

$\vec{PQ} = \langle 0, -2, 0 \rangle$, $\vec{PR} = \langle -1, -1, 1 \rangle$.

The normal vector to the plane is $\vec{n} = \vec{PQ} \times \vec{PR}$.

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = (2i + 2j - 2k)$, that is, $\vec{n} = \langle -2, 0, -2 \rangle$.

A point in the plane is $R = (0, 0, 2)$.

The equation of the plane is $-2(x - 0) + 0(y - 0) - 2(z - 2) = 0$.

$\Rightarrow x + z = 2$. \(\blacksquare\)
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Example

Find the equation of the plane that containing the points \( P = (1, 1, 1), \ Q = (1, -1, 1), \) and \( R = (0, 0, 2). \)

Solution: Find two vectors tangent to the plane:

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\vec{PQ} = \langle 0, -2, 0 \rangle, \quad \vec{PR} = \langle -1, -1, 1 \rangle.
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\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & -2 & 0 \\
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Find the equation of the plane parallel to $x - 2y + 3z = 1$ and containing the center of the sphere $x^2 + 2x + y^2 + z^2 - 2z = 0$. 
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Solution: Recall: Planes are parallel iff their normal are parallel.
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Example

Find the equation of the plane parallel to $x - 2y + 3z = 1$ and containing the center of the sphere $x^2 + 2x + y^2 + z^2 - 2z = 0$.

Solution: Recall: Planes are parallel iff their normal are parallel. We choose the normal vector $\mathbf{n} = \langle 1, -2, 3 \rangle$. 

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Therefore, the center of the sphere is at \( P_0 = (-1, 0, 1) \).
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Therefore, the center of the sphere is at $P_0 = (-1, 0, 1)$. The equation of the plane is

$(x + 1) - 2(y - 0) + 3(z - 1) = 0 \implies x - 2y + 3z = 2$. \triangleleft
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Example
Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$. 

Solution:
The angle between planes is the angle between their normal vectors. The normal vectors are $n = \langle 2, -3, 2 \rangle$, $N = \langle 1, 2, 2 \rangle$. 

Use the dot product to find the cosine of the angle $\theta$ between these vectors; $\cos(\theta) = \frac{n \cdot N}{|n||N|}$. 

But $n \cdot N = 2 - 6 + 4 = 0$, we conclude that $n \perp N$.

The planes are perpendicular, the angle is $\theta = \frac{\pi}{2}$.

$\triangleq$
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Example
Find the vector equation for the line of intersection of the planes
$2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$. 

Solution:
First, find a vector $v$ tangent to both planes.
Then, find a point in the intersection.
Since vector $v$ must belong to both planes, $v \perp n = \langle 2, -3, 2 \rangle$ and $v \perp N = \langle 1, 2, 2 \rangle$.
We choose $v = n \times N = \begin{vmatrix} i & j & k \\ 2 & -3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \langle -10, -2, 7 \rangle$.
So, $v = \langle -10, -2, 7 \rangle$. 

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Example
Find the vector equation for the line of intersection of the planes
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\]

So, \(v = \langle −10, −2, 7 \rangle\).
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Example

Find the vector equation for the line of intersection of the planes

$2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: Recall $\mathbf{v} = \langle -10, -2, 7 \rangle$. 
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Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: Recall $\mathbf{v} = \langle -10, -2, 7 \rangle$. Now find a point in the intersection of the planes.
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Solution: Recall \(\mathbf{v} = \langle -10, -2, 7 \rangle\). Now find a point in the intersection of the planes.

From the first plane we compute \(z\) as follows: \(2z = 1 - 2x + 3y\).
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We need just one solution. Choose: \(y = 0\),
Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: Recall $\mathbf{v} = \langle -10, -2, 7 \rangle$. Now find a point in the intersection of the planes.

From the first plane we compute $z$ as follows: $2z = 1 - 2x + 3y$. Introduce this equation for $2z$ into the second plane:

$$x + 2y + (1 - 2x + 3y) = 5 \quad \Rightarrow \quad -x + 5y = 4.$$ 

We need just one solution. Choose: $y = 0$, then $x = -4$,
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Section 12.5

Example

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The vector equation of the line is:

$$\mathbf{r}(t) = \langle -4, -0, 9/2 \rangle + t \langle -10, -2, 7 \rangle.$$
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Example

Sketch the surface $36x^2 + 4y^2 + 9z^2 = 36$. 

Solution: We first rewrite the equation above in the standard form 

$$x^2 + 3y^2 + 2z^2 = 1.$$ 

This is the equation of an ellipsoid with principal radius of length 1, 3, and 2 on the $x$, $y$, and $z$ axis, respectively.
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Vector functions (Sect. 13.1)

- Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Motion on a sphere.
Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \)

**Definition**

A *vector function* is function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), and the function *domain* is the interval \( I \subset \mathbb{R} \).
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(a) Motion in space motivates to define vector functions.
**Definition of vector functions: \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \)**

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**Remarks:**
(a) Motion in space motivates to define vector functions.
(b) Given Cartesian coordinates in \( \mathbb{R}^3 \), the values of a vector function can be written in components
Definition of vector functions: $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$

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A *vector function* is function $\mathbf{r} : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, and the function *domain* is the interval $I \subset \mathbb{R}$.

**Remarks:**

(a) Motion in space motivates to define vector functions.

(b) Given Cartesian coordinates in $\mathbb{R}^3$, the values of a vector function can be written in components as follows:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in I,$$

where $x(t)$, $y(t)$, and $z(t)$ are the values of three scalar functions.
Definition of vector functions: \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \)

Remarks:

- There is a natural association between a curve in \( \mathbb{R}^n \) and the vector function values \( r(t) \).

The curve is determined by the terminal points of the vector function values \( r(t) \).

The independent variable \( t \) is called the parameter of the curve.
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Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \)

Example
Graph the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

\[
\begin{align*}
\text{Definition of vector functions: } & \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \\
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**Example**

Graph the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

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The curve given by \( \mathbf{r}(t) \) lies on a vertical cylinder with radius one,
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The curve given by \( r(t) \) lies on a vertical cylinder with radius one, since

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The $z(t)$ coordinate of the curve increases with $t$, 

\[ r(0) \quad 1 \quad z \quad y \quad x \quad r(t) \]
Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \)

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The \( z(t) \) coordinate of the curve increases with \( t \), so the terminal point \( \mathbf{r}(t) \) moves up on the cylinder surface when \( t \) increases.
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Graph the vector function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$.

The curve given by $\mathbf{r}(t)$ lies on a horizontal cylinder with radius one, since $x^2 + z^2 = \sin^2(t) + \cos^2(t) = 1$.

The $y(t)$ coordinate of the curve $\mathbf{r}(t)$ increases with $t$, so the terminal point $\mathbf{r}(t)$ moves to the right on the cylinder surface when $t$ increases.

$\triangleright$
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Vector functions (Sect. 13.1)

- Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- **Limits and continuity of vector functions**.
- Derivatives and motion.
- Differentiation rules.
- Motion on a sphere.
Limits and continuity of vector functions

Definition
The vector function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), has a limit given by the vector \( L \) when \( t \) approaches \( t_0 \), denoted as \( \lim_{t \to t_0} r(t) = L \), iff:

For every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that

\[
0 < |t - t_0| < \delta \quad \Rightarrow \quad |r(t) - L| < \epsilon.
\]
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Remark:
- The limit of \( r(t) = \langle x(t), y(t), z(t) \rangle \) as \( t \to t_0 \) is the limit of its components \( x(t), y(t), z(t) \) in Cartesian coordinates.
Limits and continuity of vector functions

Definition
The vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), has a \textit{limit} given by the vector \( \mathbf{L} \) when \( t \) approaches \( t_0 \), denoted as \( \lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L} \), iff:
For every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that
\[
0 < |t - t_0| < \delta \quad \Rightarrow \quad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.
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Remark:
- The limit of \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) as \( t \to t_0 \) is the limit of its components \( x(t), y(t), z(t) \) in Cartesian coordinates.
- That is: \( \lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle. \)
\[ \lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle \]

Example

Given \( \mathbf{r}(t) = \langle \cos(t), \sin(t)/t, t^2 + 2 \rangle \), compute \( \lim_{t \to 0} \mathbf{r}(t) \).
\[
\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle
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Example
Given \( \mathbf{r}(t) = \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle \), compute \( \lim_{t \to 0} \mathbf{r}(t) \).

Solution:
Notice that the vector function \( \mathbf{r} \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists.
\[
\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle
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**Solution:**

Notice that the vector function \( \mathbf{r} \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists. Indeed,

\[
\lim_{t \to 0} \mathbf{r}(t) = \lim_{t \to 0} \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle
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\[
\lim_{t \to t_0} r(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle
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\]

\[
\lim_{t \to 0} \mathbf{r}(t) = \langle 1, 1, 2 \rangle.
\]
\[
\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle
\]

**Example**

Given \( \mathbf{r}(t) = \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle \), compute \( \lim_{t \to 0} \mathbf{r}(t) \).

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Notice that the vector function \( \mathbf{r} \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists. Indeed,

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\]

We conclude that \( \lim_{t \to 0} \mathbf{r}(t) = \left\langle 1, 1, 2 \right\rangle \).
Limits and continuity of vector functions.

Definition
A vector function $\mathbf{r} : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, is **continuous at** $t = t_0 \in I$ iff holds $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function $\mathbf{r} : I \rightarrow \mathbb{R}^n$ is **continuous** if it is continuous at every $t$ in its domain interval $I$. 

Remark:
Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.
Limits and continuity of vector functions.

**Definition**
A vector function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is **continuous at** \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} r(t) = r(t_0) \). The function \( r : I \rightarrow \mathbb{R}^n \) is **continuous** if it is continuous at every \( t \) in its domain interval \( I \).

**Remark:** A vector function with Cartesian components \( r = \langle x, y, z \rangle \) is continuous iff each component is continuous.
Limits and continuity of vector functions.

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A vector function \( r : I \to \mathbb{R}^n \), with \( n = 2, 3 \), is \textit{continuous at} \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} r(t) = r(t_0) \). The function \( r : I \to \mathbb{R}^n \) is \textit{continuous} if it is continuous at every \( t \) in its domain interval \( I \).

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Example
The function \( r(t) = \langle \sin(t), t, \cos(t) \rangle \) is continuous for \( t \in \mathbb{R} \). \( \triangle \)
Limits and continuity of vector functions.

**Definition**
A vector function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is *continuous at* \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} r(t) = r(t_0) \). The function \( r : I \rightarrow \mathbb{R}^n \) is *continuous* if it is continuous at every \( t \) in its domain interval \( I \).

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Vector functions (Sect. 13.1)

- Definition of vector functions: \( \mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- **Derivatives and motion.**
- Differentiation rules.
- Motion on a sphere.
Derivatives and motion

Definition
The vector function $r : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, is differentiable at $t = t_0$, denoted as $r'(t)$ or $\frac{dr}{dt}$, iff the following limit exists,

$$r'(t) = \lim_{h \to 0} \frac{r(t + h) - r(t)}{h}.$$
Derivatives and motion

Definition
The vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is differentiable at \( t = t_0 \), denoted as \( \mathbf{r}'(t) \) or \( \frac{d\mathbf{r}}{dt} \), iff the following limit exists,

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}.
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Remarks:
- A vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \) is differentiable if it is differentiable for each \( t \in I \).
Derivatives and motion

Definition
The vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is \textit{differentiable at} \( t = t_0 \), denoted as \( \mathbf{r}'(t) \) or \( \frac{d\mathbf{r}}{dt} \), iff the following limit exists,

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}.
\]

Remarks:

- A vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \) is \textit{differentiable} if it is differentiable for each \( t \in I \).
- If a vector function with values \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) in Cartesian components is differentiable, then

\[
\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.
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Derivatives and motion.

**Theorem**

*If a vector function with values \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) in Cartesian components is differentiable, then*

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Theorem

If a vector function with values \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) in Cartesian components is differentiable, then

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Proof:

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}
\]
Derivatives and motion.

Theorem

If a vector function with values \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) in Cartesian components is differentiable, then

\[
\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.
\]

Proof:

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} = \lim_{h \to 0} \left\langle \frac{x(t + h) - x(t)}{h}, \frac{y(t + h) - y(t)}{h}, \frac{z(t + h) - z(t)}{h} \right\rangle.
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\[
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Derivatives and motion.

Example
Find the derivative of the vector function
\[ r(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle. \]
Derivatives and motion.

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Find the derivative of the vector function \( r(t) = \langle \cos(2t), e^{3t}, 1/t \rangle \).
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Derivatives and motion.

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\[
  \mathbf{r}'(t) = \langle -2 \sin(2t), 3e^{3t}, -1/t^2 \rangle.
\]
Remark: The vector $r'(t)$ is tangent to the curve given by the vector function $r$ at the end point of $r(t)$. 
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Remark: The vector $\mathbf{r}'(t)$ is tangent to the curve given by the vector function $\mathbf{r}$ at the end point of $\mathbf{r}(t)$.

Remark: If $\mathbf{r}(t)$ represents the vector position of a particle, then:

- The derivative of the position function is the velocity function, $\mathbf{v}(t) = \mathbf{r}'(t)$.
- The speed is $|\mathbf{v}(t)|$.
- The derivative of the velocity function is the acceleration function, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$. 
Geometrical property of the derivative

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Derivatives and motion.

Example

Compute the derivative of the position function \( r(t) = \langle \cos(t), \sin(t), 0 \rangle \). Graph the curve given by \( r \), and explicitly show the position vector \( r(0) \) and velocity vector \( v(0) \).

Solution:
The derivative of \( r \) is computed component by component, \( v(t) = \langle -\sin(t), \cos(t), 0 \rangle \).

\( r(0) = \langle 1, 0, 0 \rangle, \quad v(0) = \langle 0, 1, 0 \rangle \).

\( \blacksquare \)
Example

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Derivatives and motion.

**Example**

Compute the derivative of the position function \( r(t) = \left\langle \cos(t), \sin(t), 0 \right\rangle \). Graph the curve given by \( r \), and explicitly show the position vector \( r(0) \) and velocity vector \( v(0) \).

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Example

Compute the derivative of the position function $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Graph the curve given by $\mathbf{r}$, and explicitly show the position vector $\mathbf{r}(0)$ and velocity vector $\mathbf{v}(0)$.

Solution:

The derivative of $\mathbf{r}$ is computed component by component,

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$

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Vector functions (Sect. 13.1)

- Definition of vector functions: \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- **Differentiation rules.**
- Motion on a sphere.
Differentiation rules are the similar as for scalar functions

**Theorem**

If \( \mathbf{v} \) and \( \mathbf{w} \) are differentiable vector functions, then holds:

- \([\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)\), \((\text{addition})\);
- \([c\mathbf{v}(t)]' = c\mathbf{v}'(t)\), \((\text{product rule})\);
- \([\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)\), \((\text{chain rule})\);
- \([f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)\), \((\text{product rule})\);
- \([\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)\), \((\text{dot product})\);
- \([\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)\), \((\text{cross product})\).
Higher derivatives can also be computed.

Definition
The \textit{m-derivative} of a vector function \(r\) is denoted as \(r^{(m)}\) and is given by the expression \(r^{(m)} = [r^{(m-1)}]'\).

Example
Compute the third derivative of \(r(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle\).

Solution:
\[
\begin{align*}
r'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
r''(t) &= \langle -\cos(t), -\sin(t), 2 \rangle, \\
r'''(t) &= \langle \sin(t), -\cos(t), 0 \rangle.
\end{align*}
\]

Recall: If \(r(t)\) is the position of a particle, then \(v(t) = r'(t)\) is the velocity and \(a(t) = r''(t)\) is the acceleration of the particle.
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**Definition**
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\]
\[
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\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
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Recall: If \( \mathbf{r}(t) \) is the position of a particle, then \( \mathbf{v}(t) = \mathbf{r}'(t) \) is the velocity and \( \mathbf{a}(t) = \mathbf{r}^{(2)}(t) \) is the acceleration of the particle.
Vector functions (Sect. 13.1)

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$.
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- **Motion on a sphere.**
Motion in a sphere

**Remark:** A particle with position function $\mathbf{r}$ moves on the surface of a sphere iff the vector function $\mathbf{r}$ has constant magnitude, that is, $|\mathbf{r}(t)| = r_0$ for every $t$ in the function domain.
Motion in a sphere

Remark: A particle with position function $r$ moves on the surface of a sphere iff the vector function $r$ has constant magnitude, that is, $|r(t)| = r_0$ for every $t$ in the function domain.
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**Remark:** A particle with position function \( \mathbf{r} \) moves on the surface of a sphere iff the vector function \( \mathbf{r} \) has constant magnitude, that is, \( |\mathbf{r}(t)| = r_0 \) for every \( t \) in the function domain.

**Theorem**

*If a differentiable vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^3 \) has constant length, then for all \( t \in I \) holds*

\[
\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.
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**Remark:** A motion on a sphere satisfies that \( \mathbf{r} \perp \mathbf{v} \).
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\]

**Proof:** Since \( |\mathbf{r}(t)| = r_0 \), constant for all \( t \in I \),

\[
\mathbf{r} \cdot \mathbf{r} = r_0^2.
\]

Derivate on both sides above, use the derivative properties,

\[
(\mathbf{r} \cdot \mathbf{r})' = (r_0^2)' = 0 \Rightarrow \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.
\]

Since the dot product is symmetric and \( \mathbf{r}' = \mathbf{v} \), we obtain that

\[
\mathbf{r} \cdot \mathbf{r}' = 0 \iff \mathbf{r} \cdot \mathbf{v} = 0.
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Motion in a sphere

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If a differentiable vector function \( r : I \to \mathbb{R}^3 \) has constant length, then for all \( t \in I \) holds

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Derivate on both sides above, use the derivative properties,
Motion in a sphere

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$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$ 

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$$\mathbf{r} \cdot \mathbf{r} = r_0^2.$$ 

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(r \cdot r)' = (r_0^2)' = 0 \quad \Rightarrow \quad \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.
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Motion in a sphere

Theorem
If a differentiable vector function \( r : I \rightarrow \mathbb{R}^3 \) has constant length, then for all \( t \in I \) holds
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Proof: Since \( |r(t)| = r_0 \), constant for all \( t \in I \),
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r \cdot r = r_0^2.
\]
Derivate on both sides above, use the derivative properties,
\[
(r \cdot r)' = (r_0^2)' = 0 \quad \Rightarrow \quad r' \cdot r + r \cdot r' = 0.
\]
Since the dot product is symmetric and \( r' = v \), we obtain that
\[
r \cdot r' = 0 \quad \Leftrightarrow \quad r \cdot v = 0.
\]
\[\square\]
Example

Show that the position vector \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) of a particle moving in a circle is perpendicular to its velocity for \( t \in \mathbb{R} \).
Motion in a sphere

Example
Show that the position vector $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ of a particle moving in a circle is perpendicular to its velocity for $t \in \mathbb{R}$.

Solution:
We compute its velocity vector,

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$
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Motion in a sphere

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Show that the position vector \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) of a particle moving in a circle is perpendicular to its velocity for \( t \in \mathbb{R} \).

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Then we compute \( \mathbf{r}(t) \cdot \mathbf{v}(t) \), that is,

\[ \mathbf{r}(t) \cdot \mathbf{v}(t) = -\cos(t) \sin(t) + \sin(t) \cos(t) \]

\( \triangleright \)
Motion in a sphere

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\[ \mathbf{r}(t) \cdot \mathbf{v}(t) = -\cos(t) \sin(t) + \sin(t) \cos(t) = 0. \]
Integration and projectile motion (Sect. 13.2)

- Integration of vector functions.
- Application: Projectile motion.
  - Equations of a projectile motion.
  - Range, Height, Flight Time.
Integration of vector functions

Definition
An *antiderivative* of a vector function $\mathbf{v}$ is any vector valued function $\mathbf{V}$ such that $\mathbf{V}' = \mathbf{v}$.

Remark: Antiderivatives are also called *indefinite integrals*, or *primitives*, they are denoted as $\int \mathbf{v}(t) \, dt$, that is, $\int \mathbf{v}(t) \, dt = \mathbf{V}(t) + \mathbf{C}$, where $\mathbf{C}$ is a constant vector in Cartesian coordinates.

Example
Verify that $\mathbf{V} = \langle -\cos(3t)/3 + 1, \sin(t) - 2, e^{2t}/2 + 2 \rangle$ is an antiderivative of $\mathbf{v} = \langle \sin(3t), \cos(t), e^{2t} \rangle$.

Solution:
$\mathbf{V}' = \langle -\cos(3t)/3 + 1, \sin(t) - 2, e^{2t}/2 + 2 \rangle = \mathbf{v}$. 

Integration of vector functions

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An *antiderivative* of a vector function \( \mathbf{v} \) is any vector valued function \( \mathbf{V} \) such that \( \mathbf{V}' = \mathbf{v} \).

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\[
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\]

where \( \mathbf{C} \) is a constant vector in Cartesian coordinates.

Example: Verify that \( \mathbf{V} = \langle -\cos(3t)/3 + 1, \sin(t) - 2, e^{2t}/2 + 2 \rangle \) is an antiderivative of \( \mathbf{v} = \langle \sin(3t), \cos(t), e^{2t} \rangle \).

Solution: \( \mathbf{V}' = \langle -\cos(3t)/3 + 1, \sin(t) - 2, e^{2t}/2 + 2 \rangle \) = \( \mathbf{v} \).
Integration of vector functions

Definition
An *antiderivative* of a vector function \( \mathbf{v} \) is any vector valued function \( \mathbf{V} \) such that \( \mathbf{V}' = \mathbf{v} \).

Remark: Antiderivatives are also called *indefinite integrals*, or *primitives*, they are denoted as \( \int \mathbf{v}(t) \, dt \), that is,

\[
\int \mathbf{v}(t) \, dt = \mathbf{V}(t) + \mathbf{C},
\]

where \( \mathbf{C} \) is a constant vector in Cartesian coordinates.

Example
Verify that \( \mathbf{V} = \langle (\cos(3t)/3 + 1), (\sin(t) - 2), (e^{2t}/2 + 2) \rangle \) is an antiderivative of \( \mathbf{v} = \langle \sin(3t), \cos(t), e^{2t} \rangle \).
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Solution: \( \mathbf{V}' \)
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Example
Verify that \( \mathbf{V} = \langle \frac{-\cos(3t)}{3 + 1}, (\sin(t) - 2), \frac{e^{2t}}{2 + 2} \rangle \) is an antiderivative of \( \mathbf{v} = \langle \sin(3t), \cos(t), e^{2t} \rangle \).

Solution: \( \mathbf{V}' = \langle \frac{-\cos(3t)}{3 + 1}', (\sin(t) - 2)', \frac{e^{2t}}{2 + 2}' \rangle \)
Integration of vector functions

Definition
An antiderivative of a vector function $\mathbf{v}$ is any vector valued function $\mathbf{V}$ such that $\mathbf{V}' = \mathbf{v}$.

Remark: Antiderivatives are also called indefinite integrals, or primitives, they are denoted as $\int \mathbf{v}(t) \, dt$, that is,

$$\int \mathbf{v}(t) \, dt = \mathbf{V}(t) + \mathbf{C},$$

where $\mathbf{C}$ is a constant vector in Cartesian coordinates.

Example
Verify that $\mathbf{V} = \langle (-\cos(3t)/3 + 1), (\sin(t) - 2), (e^{2t}/2 + 2) \rangle$ is an antiderivative of $\mathbf{v} = \langle \sin(3t), \cos(t), e^{2t} \rangle$.

Solution: $\mathbf{V}' = \langle (-\cos(3t)/3 + 1)', (\sin(t) - 2)', (e^{2t}/2 + 2)' \rangle = \mathbf{v}$. 
Integrals of vector functions.

**Example**

Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).
Integrals of vector functions.

Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

Solution: The position function is a primitive of the velocity,

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C}
\]
Integrals of vector functions.

Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,$$

that is, $c_x = 1, c_y = 1, c_z = 2$.

The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2 \rangle$. \[\square\]
Integrals of vector functions.

**Example**

Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

**Solution:** The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

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Integrals of vector functions.

Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:
Example
Find the position function $r$ knowing that the velocity function is $v(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $r(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$r(t) = \int v(t) \, dt + C = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $C = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $C$:

$$\langle 1, 1, 1 \rangle = r(0)$$
Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,$$
Integrals of vector functions.

Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

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that is, \( c_x = 1, c_y = 1, c_z = 2 \).
Integrals of vector functions.

Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,
$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

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$$

that is, $c_x = 1$, $c_y = 1$, $c_z = 2$.

The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2 \rangle$. ◯
Example

Find the position function of a particle with acceleration \( a(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( v(0) = \langle 0, 1, 1 \rangle \) and initial position \( r(0) = \langle 1, 0, 1 \rangle \).
Example
Find the position function of a particle with acceleration \( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

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Integrals of vector functions.

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Solution: The velocity is the antiderivative of the acceleration:

\[
\mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle,
\]

\[
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Integrals of vector functions.

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\[
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Integrals of vector functions.

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Solution: The velocity is the antiderivative of the acceleration:
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\mathbf{v}(t) = \langle v_0, v_0, (-10t + v_0) \rangle,
\]
where \( \mathbf{v}_0 = \langle v_0, v_0, v_0 \rangle \) is fixed by the initial condition.
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\mathbf{v}(0) = \langle 0, 1, 1 \rangle = \langle v_{0x}, v_{0y}, v_{0z} \rangle
\]

The velocity function is \( \mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle \).

The position is \( \mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle \),
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\mathbf{v}(0) = \langle 0, 1, 1 \rangle = \langle v_{0x}, v_{0y}, v_{0z} \rangle
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Integrals of vector functions.

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Solution: The velocity is the antiderivative of the acceleration:
$$\mathbf{v}(t) = \langle v_0x, v_0y, (-10t + v_0z) \rangle,$$
where $\mathbf{v}_0 = \langle v_0x, v_0y, v_0z \rangle$ is fixed by the initial condition.
$$\mathbf{v}(0) = \langle 0, 1, 1 \rangle = \langle v_0x, v_0y, v_0z \rangle$$

The velocity function is $\mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle$.
The position is $\mathbf{r}(t) = \langle r_0x, (t + r_0y), (-5t^2 + t + r_0z) \rangle$, and $\mathbf{r}(0) = \langle 1, 0, 1 \rangle = \langle r_0x, r_0y, r_0z \rangle$.\end{document}
Integrals of vector functions.

Example
Find the position function of a particle with acceleration \( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

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\mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle,
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The position is \( \mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle \), and

\[
\mathbf{r}(0) = \langle 1, 0, 1 \rangle = \langle r_{0x}, r_{0y}, r_{0z} \rangle,
\]

The obtain that \( \mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1) \rangle \).\( \triangleright \)
Integrals of vector functions.

Definition

The *definite integral* of an integrable vector function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) on the interval \([a, b]\) is given by

\[
\int_{a}^{b} \mathbf{r}(t) \, dt = \left\langle \int_{a}^{b} x(t) \, dt, \int_{a}^{b} y(t) \, dt, \int_{a}^{b} z(t) \, dt \right\rangle.
\]
Integrals of vector functions.

Definition
The \textit{definite integral} of an integrable vector function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) on the interval \([a, b]\) is given by

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\]

Example
Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).
Integrals of vector functions.

Definition
The definite integral of an integrable vector function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) on the interval \([a, b]\) is given by

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\]

Example
Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution: We compute an antiderivative and we evaluate the result,
Integrals of vector functions.

Definition

The \textit{definite integral} of an integrable vector function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) on the interval \([a, b]\) is given by

\[
\int_a^b \mathbf{r}(t) dt = \langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \rangle.
\]

Example

Compute \( \int_0^\pi \mathbf{r}(t) dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution: We compute an antiderivative and we evaluate the result,

\[
\mathbf{I} = \int_0^\pi \mathbf{r}(t) dt.
\]
Integrals of vector functions.

Definition
The definite integral of an integrable vector function $r(t) = \langle x(t), y(t), z(t) \rangle$ on the interval $[a, b]$ is given by

$$\int_a^b r(t) \, dt = \left\langle \int_a^b x(t) \, dt, \int_a^b y(t) \, dt, \int_a^b z(t) \, dt \right\rangle.$$

Example
Compute $\int_0^\pi r(t) \, dt$ for the function $r(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution: We compute an antiderivative and we evaluate the result,

$$I = \int_0^\pi r(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt.$$
Integrals of vector functions.

Example

Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution:

\[
\mathbf{l} = \int_0^\pi \mathbf{r}(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt.
\]
Integrals of vector functions.

Example

Compute $\int_0^\pi \mathbf{r}(t) \, dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

\[
\mathbf{l} = \int_0^\pi \mathbf{r}(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt.
\]

\[
\mathbf{l} = \left\langle \int_0^\pi \cos(t) \, dt, \int_0^\pi \sin(t) \, dt, \int_0^\pi t \, dt \right\rangle,
\]
Integrals of vector functions.

Example
Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution:

\[
\mathbf{I} = \int_0^\pi \mathbf{r}(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt.
\]

\[
\mathbf{I} = \left\langle \int_0^\pi \cos(t) \, dt, \int_0^\pi \sin(t) \, dt, \int_0^\pi t \, dt \right\rangle,
\]

\[
\mathbf{I} = \left\langle \sin(t) \bigg|_0^\pi, -\cos(t) \bigg|_0^\pi, \frac{t^2}{2} \bigg|_0^\pi \right\rangle
\]
Example
Compute \( \int_{0}^{\pi} r(t) \, dt \) for the function \( r(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution:

\[
I = \int_{0}^{\pi} r(t) \, dt = \int_{0}^{\pi} \langle \cos(t), \sin(t), t \rangle \, dt.
\]

\[
I = \langle \int_{0}^{\pi} \cos(t) \, dt, \int_{0}^{\pi} \sin(t) \, dt, \int_{0}^{\pi} t \, dt \rangle,
\]

\[
I = \langle \sin(t) \big|_{0}^{\pi}, -\cos(t) \big|_{0}^{\pi}, \frac{t^2}{2} \big|_{0}^{\pi} \rangle
\]

\[
I = \langle 0, 2, \frac{\pi^2}{2} \rangle
\]
Integrals of vector functions.

Example

Compute $\int_{0}^{\pi} r(t) \, dt$ for the function $r(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

\[
\begin{align*}
\mathbf{l} &= \int_{0}^{\pi} r(t) \, dt = \int_{0}^{\pi} \langle \cos(t), \sin(t), t \rangle \, dt. \\
\mathbf{l} &= \left( \int_{0}^{\pi} \cos(t) \, dt, \int_{0}^{\pi} \sin(t) \, dt, \int_{0}^{\pi} t \, dt \right), \\
\mathbf{l} &= \left( \left. \sin(t) \right|_{0}^{\pi}, - \left. \cos(t) \right|_{0}^{\pi}, \left. \frac{t^2}{2} \right|_{0}^{\pi} \right) \\
\mathbf{l} &= \left( 0, 2, \frac{\pi^2}{2} \right) \Rightarrow \int_{0}^{\pi} r(t) \, dt = \left( 0, 2, \frac{\pi^2}{2} \right).
\end{align*}
\]
Integration and projectile motion (Sect. 13.2)

- Integration of vector functions.
- **Application: Projectile motion.**
  - Equations of a projectile motion.
  - Range, Height, Flight Time.
Equations of a projectile motion

**Remark:** Projectile motion is the position of a point particle moving near the Earth surface subject to gravitational attraction.

Theorem

The motion of a particle with initial velocity $v_0$ and position $r_0$ subject to an acceleration $a = -g k$, where $g$ is a constant, is

$$r(t) = -\frac{g}{2}t^2 k + v_0 t + r_0.$$

Remarks:

(a) The equation above in vector components is

$$r(t) = \langle v_0 x t + r_0 x \rangle, \langle v_0 y t + r_0 y \rangle, \langle -\frac{g}{2}t^2 z + v_0 z t + r_0 z \rangle \rangle,$$

where $v_0 = \langle v_0 x, v_0 y, v_0 z \rangle$ and $r_0 = \langle r_0 x, r_0 y, r_0 z \rangle$.

(b) The motion occurs in a plane. We describe it with vectors in the plane $\mathbb{R}^2$. We use the coordinates $x, y$, only.
Equations of a projectile motion

**Remark:** Projectile motion is the position of a point particle moving near the Earth surface subject to gravitational attraction.

**Theorem**

*The motion of a particle with initial velocity \( \mathbf{v}_0 \) and position \( \mathbf{r}_0 \) subject to an acceleration \( \mathbf{a} = -g\mathbf{k} \), where \( g \) is a constant, is*

\[
\mathbf{r}(t) = -\frac{g}{2}t^2\mathbf{k} + \mathbf{v}_0 t + \mathbf{r}_0.
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Equations of a projectile motion

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Remarks:
(a) The equation above in vector components is

$$\mathbf{r}(t) = \langle (v_{0x} t + r_{0x}), (v_{0y} t + r_{0y}), \left(-\frac{g}{2} t^2 + v_{0z} t + r_{0z}\right) \rangle,$$
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\]

where \( \mathbf{v}_0 = \langle v_{0x}, v_{0y}, v_{0z} \rangle \) and \( \mathbf{r}_0 = \langle r_{0x}, r_{0y}, r_{0z} \rangle \).
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(a) The equation above in vector components is

\[
\mathbf{r}(t) = \langle \langle v_{0x} t + r_{0x}, v_{0y} t + r_{0y}, -\frac{g}{2} t^2 + v_{0z} t + r_{0z} \rangle, \mathbf{r}_0 \rangle,
\]

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(b) The motion occurs in a plane.
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The motion of a particle with initial velocity $v_0$ and position $r_0$ subject to an acceleration $a = -g \mathbf{k}$, where $g$ is a constant, is

$$r(t) = -\frac{g}{2} t^2 \mathbf{k} + v_0 t + r_0.$$  

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(b) The motion occurs in a plane. We describe it with vectors in the plane $\mathbb{R}^2$. 
Equations of a projectile motion

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Theorem
The motion of a particle with initial velocity \( \mathbf{v}_0 \) and position \( \mathbf{r}_0 \) subject to an acceleration \( \mathbf{a} = -g\mathbf{k} \), where \( g \) is a constant, is

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\mathbf{r}(t) = -\frac{g}{2}t^2\mathbf{k} + \mathbf{v}_0 t + \mathbf{r}_0.
\]

Remarks:
(a) The equation above in vector components is

\[
\mathbf{r}(t) = \begin{pmatrix} (v_{0x}t + r_{0x}) \\ (v_{0y}t + r_{0y}) \\ \left(-\frac{g}{2}t^2 + v_{0z}t + r_{0z}\right) \end{pmatrix},
\]

where \( \mathbf{v}_0 = \begin{pmatrix} v_{0x} \\ v_{0y} \\ v_{0z} \end{pmatrix} \) and \( \mathbf{r}_0 = \begin{pmatrix} r_{0x} \\ r_{0y} \\ r_{0z} \end{pmatrix} \).

(b) The motion occurs in a plane. We describe it with vectors in the plane \( \mathbb{R}^2 \). We use the coordinates \( x, y \), only.
Equations of a projectile motion

Remark: Same Theorem, written in $x$, $y$ coordinates in $\mathbb{R}^2$. 

Proof:

Since $r''(t) = -g j$, then $r'(t) = v_0 x i + (-gt + v_0 y) j$. 

$r'(0) = v_0 x i + v_0 y j = \Rightarrow r'(t) = v_0 x i + (-gt + v_0 y) j$. 

One more integration, $r(t) = (dx + v_0 x t) i + (dy + v_0 y t - \frac{1}{2} g t^2) j$. 

The initial condition $r(0) = r_0 x i + r_0 y j = \Rightarrow r(t) = (v_0 x t + r_0 x) i + (-\frac{1}{2} g t^2 + v_0 y t + r_0 y) j$. 

Equations of a projectile motion

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Theorem

The motion of a particle with initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$ and position $\mathbf{r}_0 = r_{0x}\mathbf{i} + r_{0y}\mathbf{j}$ subject to the acceleration $\mathbf{a} = -g\mathbf{j}$, where $g$ is a constant, is

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\mathbf{r}(t) = -\frac{g}{2}\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0,
$$
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\mathbf{r}(t) = (v_{0x} t + r_{0x})\mathbf{i} + \left(-\frac{g}{2} t^2 + v_{0y} t + r_{0y}\right)\mathbf{j}.
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r(t) = -\frac{g}{2}j + v_0 t + r_0,
$$

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$$\mathbf{r}(t) = -\frac{g}{2} \mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0,$$

equivalently, $\mathbf{r}(t) = (v_{0x} t + r_{0x})\mathbf{i} + \left(-\frac{g}{2} t^2 + v_{0y} t + r_{0y}\right)\mathbf{j}$.

Proof: Since $\mathbf{r}''(t) = -g\mathbf{j}$, then $\mathbf{r}'(t) = c_x\mathbf{i} + (-gt + c_y)\mathbf{j}$. 
Equations of a projectile motion

Remark: Same Theorem, written in $x$, $y$ coordinates in $\mathbb{R}^2$.

Theorem

The motion of a particle with initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$ and position $\mathbf{r}_0 = r_{0x}\mathbf{i} + r_{0y}\mathbf{j}$ subject to the acceleration $\mathbf{a} = -g\mathbf{j}$, where $g$ is a constant, is

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$$\mathbf{r}(t) = -\frac{g}{2t^2} + \mathbf{v}_0 t + \mathbf{r}_0,$$

equivalently,

$$\mathbf{r}(t) = (v_{0x} t + r_{0x})\mathbf{i} + \left(-\frac{g}{2}t^2 + v_{0y} t + r_{0y}\right)\mathbf{j}.$$ 

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$$\mathbf{r}'(0) = v_{0x}\mathbf{i} + v_{0y}\mathbf{j} = c_x\mathbf{i} + c_y\mathbf{j}.$$
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The motion of a particle with initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$ and position $\mathbf{r}_0 = r_{0x}\mathbf{i} + r_{0y}\mathbf{j}$ subject to the acceleration $\mathbf{a} = -g\mathbf{j}$, where $g$ is a constant, is

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$$\mathbf{r}'(0) = v_{0x}\mathbf{i} + v_{0y}\mathbf{j} = c_x\mathbf{i} + c_y\mathbf{j} \Rightarrow \mathbf{r}'(t) = v_{0x}\mathbf{i} + (-gt + v_{0y})\mathbf{j}.$$
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The motion of a particle with initial velocity $v_0 = v_{0x}i + v_{0y}j$ and position $r_0 = r_{0x}i + r_{0y}j$ subject to the acceleration $a = -gj$, where $g$ is a constant, is

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$$r'(0) = v_{0x}i + v_{0y}j = c_x i + c_y j \Rightarrow r'(t) = v_{0x}i + (-gt + v_{0y})j.$$  

One more integration, $r(t) = (d_x + v_{0x} t)i + (d_y + v_{0y} t - \frac{g}{2} t^2)j$. 

Equations of a projectile motion

Remark: Same Theorem, written in \( x, y \) coordinates in \( \mathbb{R}^2 \).

Theorem

The motion of a particle with initial velocity \( \mathbf{v}_0 = v_{0x} \mathbf{i} + v_{0y} \mathbf{j} \) and position \( \mathbf{r}_0 = r_{0x} \mathbf{i} + r_{0y} \mathbf{j} \) subject to the acceleration \( \mathbf{a} = -g \mathbf{j} \), where \( g \) is a constant, is

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\mathbf{r}(t) = (v_{0x} t + r_{0x}) \mathbf{i} + \left(-\frac{g}{2} t^2 + v_{0y} t + r_{0y}\right) \mathbf{j}.
\]

Proof: Since \( \mathbf{r}''(t) = -g \mathbf{j} \), then \( \mathbf{r}'(t) = c_x \mathbf{i} + (-gt + c_y) \mathbf{j} \).

\[
\mathbf{r}'(0) = v_{0x} \mathbf{i} + v_{0y} \mathbf{j} = c_x \mathbf{i} + c_y \mathbf{j} \Rightarrow \mathbf{r}'(t) = v_{0x} \mathbf{i} + (-gt + v_{0y}) \mathbf{j}.
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equivalently, $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \left(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\right)\mathbf{j}$.

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The initial condition $\mathbf{r}(0) = r_{0x}\mathbf{i} + r_{0y}\mathbf{j} = d_x\mathbf{i} + d_y\mathbf{j}$, implies that $\mathbf{r}(t) = (v_{0x}t + r_{0x})\mathbf{i} + \left(-\frac{g}{2}t^2 + v_{0y}t + r_{0y}\right)\mathbf{j}$. $\square$
Equations of a projectile motion

Example
Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4 \text{ m/s}$, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$. 

Solution:
The projectile acceleration is $a = -g \mathbf{j}$, with $g = 10 \text{ m/s}^2$. Therefore, 

$$v(t) = (-10t + |v_0| \sin(\theta)) \mathbf{j} + |v_0| \cos(\theta) \mathbf{i},$$

where $v_0 \sin(\theta) = 4 \sqrt{3}$ and $v_0 \cos(\theta) = 4 \frac{1}{2} = 2$. 

Since $v(t) = (-10t + 2\sqrt{3}) \mathbf{j} + 2 \mathbf{i}$ and $r_0 = 0$, then

$$r(t) = (-5t^2 + 2\sqrt{3}t) \mathbf{j} + 2t \mathbf{i}.$$

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -\frac{5}{4}x^2 + \sqrt{3}x.$$
Equations of a projectile motion

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Equations of a projectile motion

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\[
\begin{align*}
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\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.
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Find the position function and the trajectory of a projectile with initial speed $|\mathbf{v}_0| = 4$ m/s, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

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$$\mathbf{r}(t) = (-5t^2 + 2\sqrt{3}t)\mathbf{j} + 2t\mathbf{i}.$$  

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -5 \left( \frac{x^2}{4} \right) + 2\sqrt{3} \frac{x}{2}.$$
Equations of a projectile motion

Example
Find the position function and the trajectory of a projectile with initial speed $|v_0| = 4 \text{ m/s}$, launched from the coordinate system origin with an elevation angle of $\theta = \pi/3$.

Solution: The projectile acceleration is $a = -g \mathbf{j}$, with $g = 10 \text{ m/s}$. Therefore, $v(t) = (-10t + v_{0y}) \mathbf{j} + v_{0x} \mathbf{i}$, where

$$v_{0y} = |v_0| \sin(\theta) = 4 \frac{\sqrt{3}}{2} = 2\sqrt{3}, \quad v_{0x} = |v_0| \cos(\theta) = 4 \frac{1}{2} = 2.$$

Since $v(t) = (-10t + 2\sqrt{3}) \mathbf{j} + 2\mathbf{i}$ and $r_0 = \mathbf{0}$, then

$$r(t) = (-5t^2 + 2\sqrt{3}t) \mathbf{j} + 2t \mathbf{i}.$$ 

Since $y(t) = -5t^2 + 2\sqrt{3}t$ and $x(t) = 2t$, the trajectory is

$$y(x) = -5 \left( \frac{x^2}{4} \right) + 2\sqrt{3} \frac{x}{2} \quad \Rightarrow \quad y(x) = -\frac{5}{4} x^2 + \sqrt{3} x.$$ \[\Box\]
Integration and projectile motion (Sect. 13.2)

- Integration of vector functions.
- Application: Projectile motion.
  - Equations of a projectile motion.
  - Range, Height, Flight Time.
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
$$
Range, Height, Flight Time

Theorem

The the range $x_r$, height $y_h$, and the fight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y} \mathbf{j} + v_{0x} \mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.$$ 

Remark: Since the initial speed $|\mathbf{v}_0|$ and the elevation angle $\theta$ determine $v_{0y}$ and $v_{0x}$.
Range, Height, Flight Time

Theorem
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.$$ 

Remark: Since the initial speed $|\mathbf{v}_0|$ and the elevation angle $\theta$ determine $v_{0y}$ and $v_{0x}$ by the equations

$$v_{0y} = |\mathbf{v}_0| \sin(\theta), \quad v_{0x} = |\mathbf{v}_0| \cos(\theta).$$
Range, Height, Flight Time

**Theorem**

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

\[ x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}. \]

**Remark:** Since the initial speed $|\mathbf{v}_0|$ and the elevation angle $\theta$ determine $v_{0y}$ and $v_{0x}$ by the equations

\[ v_{0y} = |\mathbf{v}_0| \sin(\theta), \quad v_{0x} = |\mathbf{v}_0| \cos(\theta), \]

then holds

\[ x_r = \frac{|\mathbf{v}_0|^2 \sin(2\theta)}{g}. \]
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Remark: Since the initial speed \( |\mathbf{v}_0| \) and the elevation angle \( \theta \) determine \( v_{0y} \) and \( v_{0x} \) by the equations

\[
v_{0y} = |\mathbf{v}_0| \sin(\theta), \quad v_{0x} = |\mathbf{v}_0| \cos(\theta),
\]

then holds

\[
x_r = \frac{|\mathbf{v}_0|^2 \sin(2\theta)}{g}, \quad y_h = \frac{|\mathbf{v}_0|^2 \sin^2(\theta)}{2g},
\]
Range, Height, Flight Time

Theorem
The the range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$  

Remark: Since the initial speed $|\mathbf{v}_0|$ and the elevation angle $\theta$ determine $v_{0y}$ and $v_{0x}$ by the equations

$$v_{0y} = |\mathbf{v}_0| \sin(\theta), \quad v_{0x} = |\mathbf{v}_0| \cos(\theta),$$

then holds

$$x_r = \frac{|\mathbf{v}_0|^2 \sin(2\theta)}{g}, \quad y_h = \frac{|\mathbf{v}_0|^2 \sin^2(\theta)}{2g}, \quad t_r = \frac{2|\mathbf{v}_0| \sin(\theta)}{g}.$$
Range, Height, Flight Time

Theorem
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$

Proof:
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0j + v_0i$ are

$$x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$

Proof: Since $\mathbf{r}_0 = 0$, 

...
Range, Height, Flight Time

Theorem

The the range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y \mathbf{j} + v_0x \mathbf{i} \) are

\[
 x_r = \frac{2v_0x v_0y}{g} , \quad y_h = \frac{(v_0y)^2}{2g} , \quad t_r = \frac{2v_0y}{g} .
\]

Proof: Since \( \mathbf{r}_0 = \mathbf{0} \), the expression for the projectile position function \( \mathbf{r}(t) = y(t) \mathbf{j} + x(t) \mathbf{i} \) is
Range, Height, Flight Time

Theorem
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$ 

Proof: Since $\mathbf{r}_0 = \mathbf{0}$, the expression for the projectile position function $\mathbf{r}(t) = y(t)\mathbf{j} + x(t)\mathbf{i}$ is

$$y(t) = -\frac{g}{2} t^2 + v_0y \, t.$$
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Since \( \mathbf{r}_0 = \mathbf{0} \), the expression for the projectile position function \( \mathbf{r}(t) = y(t)\mathbf{j} + x(t)\mathbf{i} \) is

\[
y(t) = -\frac{g}{2} t^2 + v_0 y \ t, \quad x(t) = v_0 x \ t.
\]
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0 y \mathbf{j} + v_0 x \mathbf{i}$ are

$$x_r = \frac{2 v_0 x v_0 y}{g}, \quad y_h = \frac{(v_0 y)^2}{2g}, \quad t_r = \frac{2v_0 y}{g}.$$ 

Proof: Since $r_0 = 0$, the expression for the projectile position function $\mathbf{r}(t) = y(t) \mathbf{j} + x(t) \mathbf{i}$ is

$$y(t) = -\frac{g}{2} t^2 + v_0 y \ t, \quad x(t) = v_0 x \ t.$$ 

Using $t = x/v_0 x$ we get the trajectory
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$

Proof: Since $\mathbf{r}_0 = 0$, the expression for the projectile position function $\mathbf{r}(t) = y(t)\mathbf{j} + x(t)\mathbf{i}$ is

$$y(t) = -\frac{g}{2} t^2 + v_0y \ t, \quad x(t) = v_0x \ t.$$

Using $t = x/v_0x$ we get the trajectory

$$y(x) = -\frac{g}{2v_0^2x} x^2 + \frac{v_0y}{v_0x} x.$$
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y} \mathbf{j} + v_{0x} \mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.$$

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_{0y}}{v_{0x}} x.$
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_{0y} \mathbf{j} + v_{0x} \mathbf{i} \) are

\[
x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_{0y}}{v_{0x}} x \). The range is given by the condition \( y(x_r) = 0 \) and \( x_r \neq 0 \),
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y} \mathbf{j} + v_{0x} \mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.$$ 

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_{0y}}{v_{0x}} x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

$$-\frac{g}{2v_{0x}} x_r + v_{0y} = 0$$
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the fight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_{0y}\mathbf{j} + v_{0x}\mathbf{i}$ are

$$x_r = \frac{2v_{0x}v_{0y}}{g}, \quad y_h = \frac{(v_{0y})^2}{2g}, \quad t_r = \frac{2v_{0y}}{g}.$$ 

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$$-\frac{g}{2v_{0x}^2}x_r + v_{0y} = 0 \quad \Rightarrow \quad x_r = \frac{2v_{0x}v_{0y}}{g}.$$
Range, Height, Flight Time

Theorem
The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$

Proof: Recall: $y(x) = -\frac{g}{2v_0^2} x^2 + \frac{v_0y}{v_0x} x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

$$-\frac{g}{2v_0^2} x_r + \frac{v_0y}{v_0x} x_r = 0 \quad \Rightarrow \quad x_r = \frac{2v_0xv_0y}{g}.$$

The maximum height occurs where $y'(x) = 0$, 

...
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$ 

Proof: Recall: $y(x) = -\frac{g}{2v_0^2} x^2 + \frac{v_0y}{v_0x} x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

$$-\frac{g}{2v_0} x_r + v_0y = 0 \quad \Rightarrow \quad x_r = \frac{2v_0x v_0y}{g}.$$ 

The maximum height occurs where $y'(x) = 0$, that is,

$$-\frac{g}{v_0^2} x_h + \frac{v_0y}{v_0x} = 0$$
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y \mathbf{j} + v_0x \mathbf{i}$ are

\[
x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_0y}{v_0x} x$. The range is given by the condition $y(x_r) = 0$ and $x_r \neq 0$, that is,

\[
-\frac{g}{2v_{0x}} x_r + v_0y = 0 \quad \Rightarrow \quad x_r = \frac{2v_0x v_0y}{g}.
\]

The maximum height occurs where $y'(x) = 0$, that is,

\[
-\frac{g}{v_{0x}^2} x_h + \frac{v_0y}{v_0x} = 0 \quad \Rightarrow \quad x_h = \frac{v_0x v_0y}{g} \quad \Rightarrow \quad x_h = \frac{x_r}{2}.
\]
Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$ 

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2}x^2 + \frac{v_0y}{v_0x}x$, and $x_h = \frac{v_0xv_0y}{g}$. 
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the fight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$  

Proof: Recall: $y(x) = -\frac{g}{2v_0^2}x^2 + \frac{v_0y}{v_0x}x$, and $x_h = \frac{v_0xv_0y}{g}$.

Then, the maximum height $y_h = y(x_h)$ is
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the fight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v_0^2} x^2 + \frac{v_0y}{v_0x} x \), and \( x_h = \frac{v_0xv_0y}{g} \).

Then, the maximum height \( y_h = y(x_h) \) is

\[
y_h = -\frac{g}{2v_0^2} \frac{v_0^2}{g} \frac{v_0^2}{g} + \frac{v_0y}{v_0x} \frac{v_0xv_0y}{g}
\]
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_0y}{v_0x} x \), and \( x_h = \frac{v_0xv_0y}{g} \).

Then, the maximum height \( y_h = y(x_h) \) is

\[
y_h = -\frac{g}{2v_{0x}^2} \frac{v_0x^2v_0y^2}{g^2} + \frac{v_0y}{v_0x} \frac{v_0xv_0y}{g} = -\frac{v_0y^2}{2g} + \frac{v_0y^2}{g}.
\]
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_{0y}}{v_{0x}} x \), and \( x_h = \frac{v_{0x}v_{0y}}{g} \).

Then, the maximum height \( y_h = y(x_h) \) is

\[
y_h = -\frac{g}{2v_{0x}^2} \frac{v_{0x}^2 v_{0y}^2}{g^2} + \frac{v_{0y}}{v_{0x}} \frac{v_{0x} v_{0y}}{g} = -\frac{v_{0y}^2}{2g} + \frac{v_{0y}^2}{g} \quad \Rightarrow \quad y_h = \frac{v_{0y}^2}{2g}.
\]
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0 y \mathbf{j} + v_0 x \mathbf{i} \) are

\[
  x_r = \frac{2v_0 x v_0 y}{g}, \quad y_h = \frac{(v_0 y)^2}{2g}, \quad t_r = \frac{2v_0 y}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v_0 x} x^2 + \frac{v_0 y}{v_0 x} x \), and \( x_h = \frac{v_0 x v_0 y}{g} \).

Then, the maximum height \( y_h = y(x_h) \) is

\[
  y_h = -\frac{g}{2v_0 x} \frac{v_0^2 x v_0 y^2}{g^2} + \frac{v_0 y}{v_0 x} \frac{v_0 x v_0 y}{g} = -\frac{v_0^2 y}{2g} + \frac{v_0^2 y}{g} \quad \Rightarrow \quad y_h = \frac{v_0^2 y}{2g}.
\]

Recalling that \( x(t) = v_0 x t \),
Range, Height, Flight Time

Theorem

The range $x_r$, height $y_h$, and the flight time $t_r$ of a projectile launched from the origin with initial velocity $\mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i}$ are

$$x_r = \frac{2v_0x v_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.$$ 

Proof: Recall: $y(x) = -\frac{g}{2v_{0x}^2} x^2 + \frac{v_{0y}}{v_{0x}} x$, and $x_h = \frac{v_{0x}v_{0y}}{g}$.

Then, the maximum height $y_h = y(x_h)$ is

$$y_h = -\frac{g}{2v_{0x}^2} \frac{v_{0x}^2 v_{0y}^2}{g^2} + \frac{v_{0y}}{v_{0x}} \frac{v_{0x} v_{0y}}{g} = -\frac{v_{0y}^2}{2g} + \frac{v_{0y}^2}{g} \Rightarrow y_h = \frac{v_{0y}^2}{2g}.$$ 

Recalling that $x(t) = v_{0x} t$, then the flight time $t_r$ is

$$t_r = \frac{x_r}{v_{0x}}$$
Range, Height, Flight Time

Theorem

The range \( x_r \), height \( y_h \), and the flight time \( t_r \) of a projectile launched from the origin with initial velocity \( \mathbf{v} = v_0y\mathbf{j} + v_0x\mathbf{i} \) are

\[
x_r = \frac{2v_0xv_0y}{g}, \quad y_h = \frac{(v_0y)^2}{2g}, \quad t_r = \frac{2v_0y}{g}.
\]

Proof: Recall: \( y(x) = -\frac{g}{2v^2_{0x}}x^2 + \frac{v_{0y}}{v_{0x}}x \), and \( x_h = \frac{v_0xv_0y}{g} \).

Then, the maximum height \( y_h = y(x_h) \) is

\[
y_h = -\frac{g}{2v^2_{0x}} \frac{v^2_{0x}v^2_{0y}}{g^2} + \frac{v_{0y}}{v_{0x}} \frac{v_{0x}v_{0y}}{g} = -\frac{v^2_{0y}}{2g} + \frac{v^2_{0y}}{g} \Rightarrow y_h = \frac{v^2_{0y}}{2g}.
\]

Recalling that \( x(t) = v_0x t \), then the flight time \( t_r \) is

\[
t_r = \frac{x_r}{v_0x} \frac{2}{2} \Rightarrow t_r = \frac{2v_0y}{g}.
\]
Range, Height, Flight Time

Example
Find the range, height and flight time of the projectile with initial velocity $v_0 = 3j + i$. 

Solution:
We could use the formulas from the Theorem. However, we compute them following the Theorem proof.

From $a = -10j$ we get the projectile position function, $y(t) = -5t^2 + 3t$, $x(t) = t$.

The trajectory is $y(x) = -5x^2 + 3x$.

The range is $y(x_r) = 0 = -5x_r^2 + 3x_r \Rightarrow x_r = \frac{3}{5}$.

The height is $y_h = y(x_r^2)$, so, $y_h = -\frac{5}{2} \left(\frac{3}{5}\right)^2 + 3 \left(\frac{3}{5}\right)$, so $y_h = \frac{9}{20}$.

The time flight is $t_r = \frac{x_r}{v_0 x}$, that is, $t_r = \frac{3}{5}$. 

$\Rightarrow$
Range, Height, Flight Time

Example
Find the range, height and flight time of the projectile with initial velocity \( \mathbf{v}_0 = 3\mathbf{j} + \mathbf{i} \).

Solution: We could use the formulas from the Theorem.
Range, Height, Flight Time

Example
Find the range, height and flight time of the projectile with initial velocity $v_0 = 3j + i$.

Solution: We could use the formulas from the Theorem. However, we compute them following the Theorem proof.
Range, Height, Flight Time

Example
Find the range, height and flight time of the projectile with initial velocity $v_0 = 3j + i$.

Solution: We could use the formulas from the Theorem. However, we compute them following the Theorem proof.

From $a = -10j$ we get the projectile position function,
Range, Height, Flight Time

Example
Find the range, height and flight time of the projectile with initial velocity \( \mathbf{v}_0 = 3\mathbf{j} + \mathbf{i} \).

Solution: We could use the formulas from the Theorem. However, we compute them following the Theorem proof.

From \( \mathbf{a} = -10\mathbf{j} \) we get the projectile position function,

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y(t) = -5t^2 + 3t,
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$$y(x_r) = 0 = -5x_r + 3 \quad \Rightarrow \quad x_r = \frac{3}{5}.$$
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\[
y_h = -5 \left(\frac{3}{10}\right)^2 + 3 \left(\frac{3}{10}\right),
\]

\[
y_h = -5 \left(\frac{9}{100}\right) + \frac{9}{10} = -\frac{9}{2} + \frac{9}{10} = -\frac{18}{10} + \frac{9}{10} = -\frac{9}{10}.
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The height is $y_h = y\left(\frac{x_r}{2}\right)$, so, $y_h = -5 \cdot \frac{3^2}{10^2} + 3 \cdot \frac{3}{10}$, so $y_h = \frac{9}{20}$.
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