Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.
Two main ways to introduce the dot product

Geometrical definition
Two main ways to introduce the dot product

Geometrical definition $\rightarrow$ Properties

We choose the first way, the textbook chooses the second way.
Two main ways to introduce the dot product

Geometrical definition $\rightarrow$ Properties $\rightarrow$ Expression in components.

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Two main ways to introduce the dot product

- Geometrical definition → Properties → Expression in components.

- Definition in components
Two main ways to introduce the dot product

Geometrical definition → Properties → Expression in components.

Definition in components → Properties
Two main ways to introduce the dot product

Geometrical definition → Properties → Expression in components.

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The dot product of two vectors is a scalar

Definition
The *dot product* of the vectors \( \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^n \), with \( n = 2, 3 \), having magnitudes \( |\mathbf{v}|, |\mathbf{w}| \) and angle in between \( \theta \), where \( 0 \leq \theta \leq \pi \), is denoted by \( \mathbf{v} \cdot \mathbf{w} \) and given by

\[
\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta).
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Initial points together.
The dot product of two vectors is a scalar

Example

Compute \( \mathbf{v} \cdot \mathbf{w} \) knowing that \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \), with \( |\mathbf{v}| = 2 \), \( \mathbf{w} = \langle 1, 2, 3 \rangle \) and the angle in between is \( \theta = \pi/4 \).

Solution:

We first compute \( |\mathbf{w}| \), that is, \( |\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \Rightarrow |\mathbf{w}| = \sqrt{14} \).

We now use the definition of dot product:

\[
\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) = (2) \sqrt{14} \sqrt{2/2} \Rightarrow \mathbf{v} \cdot \mathbf{w} = 2 \sqrt{7}.
\]

▶

The angle between two vectors usually is not known in applications.

It is useful to have a formula for the dot product involving the vector components.
The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

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Perpendicular vectors have zero dot product.

**Definition**
Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$. 

![Diagram of perpendicular vectors](image-url)
Perpendicular vectors have zero dot product.

Definition
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Theorem
*The non-zero vectors \( \mathbf{v} \) and \( \mathbf{w} \) are perpendicular iff \( \mathbf{v} \cdot \mathbf{w} = 0 \).*
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Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.

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*The non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular iff $\mathbf{v} \cdot \mathbf{w} = 0$.***

**Proof.**
\[
0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta)
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$0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta)$

$|\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0$
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Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi / 2$.

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\[
0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \\
\begin{cases}
|\mathbf{v}| \neq 0, & |\mathbf{w}| \neq 0 \\
\cos(\theta) = 0
\end{cases}
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\cos(\theta) = 0 \\
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0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \quad \iff \quad \left\{ \begin{array}{l} \cos(\theta) = 0 \\ 0 \leq \theta \leq \pi \end{array} \right. \iff \theta = \frac{\pi}{2}.
\]
The dot product of $i$, $j$, and $k$ is simple to compute.

**Example**

Compute all dot products involving the vectors $i$, $j$, and $k$.

Solution:

Recall:

- $i = \langle 1, 0, 0 \rangle$
- $j = \langle 0, 1, 0 \rangle$
- $k = \langle 0, 0, 1 \rangle$.

- $i \cdot i = 1$
- $j \cdot j = 1$
- $k \cdot k = 1$
- $i \cdot j = 0$
- $j \cdot i = 0$
- $k \cdot i = 0$
- $i \cdot k = 0$
- $j \cdot k = 0$
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$\blacksquare$
The dot product of \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) is simple to compute

**Example**

Compute all dot products involving the vectors \( \mathbf{i} \), \( \mathbf{j} \), and \( \mathbf{k} \).

**Solution:** Recall: \( \mathbf{i} = \langle 1, 0, 0 \rangle \), \( \mathbf{j} = \langle 0, 1, 0 \rangle \), \( \mathbf{k} = \langle 0, 0, 1 \rangle \).
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Example
Compute all dot products involving the vectors $i$, $j$, and $k$.

Solution: Recall: $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$.

\[
\begin{align*}
    i \cdot i &= 1, & j \cdot j &= 1, & k \cdot k &= 1, \\
    i \cdot j &= 0, & j \cdot i &= 0, & k \cdot i &= 0, \\
    i \cdot k &= 0, & j \cdot k &= 0, & k \cdot j &= 0.
\end{align*}
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**Remark:** The dot product is closely related to orthogonal projections of one vector onto the other.
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![Diagram of vectors and dot product](attachment:vector_diagram.png)
The dot product and orthogonal projections.

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Recall: \( \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \).

\[ |v| \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}. \]
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**Remark:** If \( |\mathbf{u}| = 1 \), then \( \mathbf{v} \cdot \mathbf{u} \) is the projection of \( \mathbf{v} \) along \( \mathbf{u} \).
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Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, \hspace{1cm} (symmetric);
(b) $\mathbf{v} \cdot (a\mathbf{w}) = a(\mathbf{v} \cdot \mathbf{w})$, \hspace{1cm} (linear);
(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, \hspace{1cm} (linear);
(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geq 0$, \hspace{0.5cm} and \hspace{0.5cm} $\mathbf{v} \cdot \mathbf{v} = 0 \iff \mathbf{v} = \mathbf{0}$, \hspace{1cm} (positive);
(e) $0 \cdot \mathbf{v} = 0$. 

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cdot \cos(\theta)$. For example, the proof of (b) for $a > 0$: $\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}| \cdot |a\mathbf{w}| \cdot \cos(\theta) = a |\mathbf{v}| \cdot |\mathbf{w}| \cdot \cos(\theta) = a (\mathbf{v} \cdot \mathbf{w})$. 


Properties of the dot product.

Theorem

(a) \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \), \hspace{1cm} (\text{symmetric});
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Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product \( \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cos(\theta) \).
Properties of the dot product.

Theorem

(a) $v \cdot w = w \cdot v$, \hspace{1em} (symmetric);
(b) $v \cdot (aw) = a(v \cdot w)$, \hspace{1em} (linear);
(c) $u \cdot (v + w) = u \cdot v + u \cdot w$, \hspace{1em} (linear);
(d) $v \cdot v = |v|^2 \geq 0$, \textit{and} $v \cdot v = 0 \iff v = 0$, \hspace{1em} (positive);
(e) $0 \cdot v = 0$.

Proof.
Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $v \cdot w = |v||w|\cos(\theta)$.
For example, the proof of (b) for $a > 0$: 
Properties of the dot product.

Theorem

(a) \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \), \hspace{1cm} (symmetric);
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(d) \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geq 0 \), and \( \mathbf{v} \cdot \mathbf{v} = 0 \iff \mathbf{v} = \mathbf{0} \), \hspace{1cm} (positive);
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Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product \( \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \).

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**Theorem**

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(c) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \), \hspace{1cm} \text{(linear)};

(d) \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geq 0 \), and \( \mathbf{v} \cdot \mathbf{v} = 0 \iff \mathbf{v} = \mathbf{0} \), \hspace{1cm} \text{(positive)};

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**Proof.**

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product \( \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \).

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\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}| |a\mathbf{w}| \cos(\theta) = a |\mathbf{v}| |\mathbf{w}| \cos(\theta) = a (\mathbf{v} \cdot \mathbf{w}).
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Properties of the dot product.

(c) $u \cdot (v + w) = u \cdot v + u \cdot w$, is non-trivial.
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(c) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \), is non-trivial. The proof is:
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(c) \( u \cdot (v + w) = u \cdot v + u \cdot w \), is non-trivial. The proof is:

\[
|v + w| \cos(\theta) = \frac{u \cdot (v + w)}{|u|},
\]

\[
|V+W| \cos(\theta) = |V| \cos(\theta_v) + |W| \cos(\theta_w).
\]
Properties of the dot product.

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$$|\mathbf{v} + \mathbf{w}| \cos(\theta) = \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|},$$

$$|\mathbf{w}| \cos(\theta_w) = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|},$$
Properties of the dot product.

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|\mathbf{v} + \mathbf{w}| \cos(\theta) = \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|},

|\mathbf{w}| \cos(\theta_w) = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|},

|\mathbf{v}| \cos(\theta_v) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|},
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Properties of the dot product.

(c) \( u \cdot (v + w) = u \cdot v + u \cdot w \), is non-trivial. The proof is:

\[
\begin{align*}
|v + w| \cos(\theta) &= \frac{u \cdot (v + w)}{|u|}, \\
|w| \cos(\theta_w) &= \frac{u \cdot w}{|u|}, \\
|v| \cos(\theta_v) &= \frac{u \cdot v}{|u|},
\end{align*}
\]

\[\Rightarrow u \cdot (v + w) = u \cdot v + u \cdot w\]
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The dot product in vector components (Case $\mathbb{R}^2$)

**Theorem**

If $\mathbf{v} = \langle v_x, v_y \rangle$ and $\mathbf{w} = \langle w_x, w_y \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

$$v \cdot w = v_x w_x + v_y w_y.$$
The dot product in vector components (Case $\mathbb{R}^2$)

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$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$  

Proof.

Recall: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j}$. 

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$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$ 

**Proof.**

Recall: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x \mathbf{i} + v_y \mathbf{j}) \cdot (w_x \mathbf{i} + w_y \mathbf{j})$$
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$$v \cdot w = (v_x \mathbf{i} + v_y \mathbf{j}) \cdot (w_x \mathbf{i} + w_y \mathbf{j})$$

$$v \cdot w = v_x w_x \mathbf{i} \cdot \mathbf{i} + v_x w_y \mathbf{i} \cdot \mathbf{j} + v_y w_x \mathbf{j} \cdot \mathbf{i} + v_y w_y \mathbf{j} \cdot \mathbf{j}.$$
The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ and $\mathbf{w} = \langle w_x, w_y \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$ 

Proof.

Recall: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x \mathbf{i} + v_y \mathbf{j}) \cdot (w_x \mathbf{i} + w_y \mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x \mathbf{i} \cdot \mathbf{i} + v_x w_y \mathbf{i} \cdot \mathbf{j} + v_y w_x \mathbf{j} \cdot \mathbf{i} + v_y w_y \mathbf{j} \cdot \mathbf{j}.$$ 

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. 

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ and $\mathbf{w} = \langle w_x, w_y \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

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Proof.

Recall: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x \mathbf{i} + v_y \mathbf{j}) \cdot (w_x \mathbf{i} + w_y \mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x \mathbf{i} \cdot \mathbf{i} + v_x w_y \mathbf{i} \cdot \mathbf{j} + v_y w_x \mathbf{j} \cdot \mathbf{i} + v_y w_y \mathbf{j} \cdot \mathbf{j}.$$ 

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. We conclude that

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$
Theorem

If \( \mathbf{v} = \langle v_x, v_y, v_z \rangle \) and \( \mathbf{w} = \langle w_x, w_y, w_z \rangle \), then \( \mathbf{v} \cdot \mathbf{w} \) is given by

\[
\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.
\]
The dot product in vector components (Case $\mathbb{R}^3$)

**Theorem**

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ and $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$  

▶ The proof is similar to the case in $\mathbb{R}^2$. 

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem
If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ and $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$ 

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$. 

The geometrical meaning of the dot product is simple to see from the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta)$. 
The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ and $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, then $\mathbf{v} \cdot \mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$ 

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.
- The geometrical meaning of the dot product is simple to see from the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cdot \cos(\theta)$. 
Example

Find the cosine of the angle between \( \mathbf{v} = \langle 1, 2 \rangle \) and \( \mathbf{w} = \langle 2, 1 \rangle \).
Example

Find the cosine of the angle between \( \mathbf{v} = \langle 1, 2 \rangle \) and \( \mathbf{w} = \langle 2, 1 \rangle \)

Solution:

\[
\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta)
\]
Example
Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

\[
\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}.
\]
Example
Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}.$$ 

Furthermore,

$$\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(1)$$
Example
Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \implies \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}.$$  

Furthermore,

$$\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(1)$$

$$|\mathbf{v}| = \sqrt{1^2 + 2^2} = \sqrt{5},$$
Example
Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}. $$

Furthermore,

$$\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(1)$$

$$|\mathbf{v}| = \sqrt{1^2 + 2^2} = \sqrt{5},$$

$$|\mathbf{w}| = \sqrt{2^2 + 1^2} = \sqrt{5},$$
Example
Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos(\theta) \Rightarrow \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|}.$$

Furthermore,

$$\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(1) \quad \Rightarrow \quad \cos(\theta) = \frac{4}{5}.$$
Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- **Scalar and vector projection formulas.**
Scalar and vector projection formulas.

Theorem
The scalar projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the number \( p_{\mathbf{w}}(\mathbf{v}) \),

\[
p_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.
\]

The vector projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the vector \( \mathbf{p}_{\mathbf{w}}(\mathbf{v}) \),

\[
\mathbf{p}_{\mathbf{w}}(\mathbf{v}) = \left( \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \right) \frac{\mathbf{w}}{|\mathbf{w}|}.
\]
Scalar and vector projection formulas.

**Theorem**

*The scalar projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the number \( p_w(v) \),*

\[
p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.
\]

*The vector projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the vector \( p_w(v) \),*

\[
p_w(v) = \left( \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \right) \frac{\mathbf{w}}{|\mathbf{w}|}.
\]
Scalar and vector projection formulas.

Theorem

*The scalar projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the number \( p_w(v) \),*

\[
p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.
\]

*The vector projection of \( \mathbf{v} \) along \( \mathbf{w} \) is the vector \( p_w(v) \),*

\[
p_w(v) = \left( \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \right) \frac{\mathbf{w}}{|\mathbf{w}|}.
\]
Example
Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$. 

Solution:
The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number 

$$p_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{a}||} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$ 

We therefore obtain 

$$p_{\mathbf{a}}(\mathbf{b}) = -2 \sqrt{5}.$$
Example
Find the scalar projection of \( \mathbf{b} = \langle -4, 1 \rangle \) onto \( \mathbf{a} = \langle 1, 2 \rangle \).

Solution: The scalar projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is the number

\[
p_a(b) = |\mathbf{b}| \cos(\theta)
\]
Example
Find the scalar projection of \( \mathbf{b} = \langle -4, 1 \rangle \) onto \( \mathbf{a} = \langle 1, 2 \rangle \).

Solution: The scalar projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is the number

\[
p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}
\]
Example
Find the scalar projection of $b = \langle -4, 1 \rangle$ onto $a = \langle 1, 2 \rangle$.

Solution: The scalar projection of $b$ onto $a$ is the number

$$p_a(b) = |b| \cos(\theta) = \frac{b \cdot a}{|a|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$
Example
Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

We therefore obtain $p_a(b) = -\frac{2}{\sqrt{5}}$. 
Example
Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$ 

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Example

Find the vector projection of \( \mathbf{b} = \langle -4, 1 \rangle \) onto \( \mathbf{a} = \langle 1, 2 \rangle \).
Example
Find the vector projection of $b = \langle -4, 1 \rangle$ onto $a = \langle 1, 2 \rangle$.

Solution: The vector projection of $b$ onto $a$ is the vector

$$p_a(b) = \left( \frac{b \cdot a}{|a|} \right) \frac{a}{|a|}$$
Example
Find the vector projection of \( \mathbf{b} = \langle -4, 1 \rangle \) onto \( \mathbf{a} = \langle 1, 2 \rangle \).

Solution: The vector projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is the vector

\[
p_a(b) = \left( \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left( -\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.
\]
Example
Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$p_a(b) = \left( \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left( -\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

We therefore obtain $p_a(b) = -\langle \frac{2}{5}, \frac{4}{5} \rangle$. 

Example
Find the vector projection of $\mathbf{b} = (-4, 1)$ onto $\mathbf{a} = (1, 2)$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$p_a(b) = \left( \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left( -\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} (1, 2).$$

We therefore obtain $p_a(b) = -\left( \frac{2}{5}, \frac{4}{5} \right)$. 

![Diagram showing vector projection](attachment:image.png)
Example
Find the vector projection of \( \mathbf{a} = \langle 1, 2 \rangle \) onto \( \mathbf{b} = \langle -4, 1 \rangle \).
Example
Find the vector projection of \( \mathbf{a} = \langle 1, 2 \rangle \) onto \( \mathbf{b} = \langle -4, 1 \rangle \).

Solution: The vector projection of \( \mathbf{a} \) onto \( \mathbf{b} \) is the vector

\[
p_b(a) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|}
\]
Example
Find the vector projection of \( \mathbf{a} = \langle 1, 2 \rangle \) onto \( \mathbf{b} = \langle -4, 1 \rangle \).

Solution: The vector projection of \( \mathbf{a} \) onto \( \mathbf{b} \) is the vector

\[
\mathbf{p}_{b}(a) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \right) \frac{\mathbf{b}}{||\mathbf{b}||} = \left( - \frac{2}{\sqrt{17}} \right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.
\]
Example
Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is the vector

$$
\mathbf{p}_b(a) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \right) \frac{\mathbf{b}}{||\mathbf{b}||} = \left( -\frac{2}{\sqrt{17}} \right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.
$$

We therefore obtain $\mathbf{p}_a(b) = \langle \frac{8}{17}, -\frac{2}{17} \rangle$. 
Example
Find the vector projection of \( \mathbf{a} = \langle 1, 2 \rangle \) onto \( \mathbf{b} = \langle -4, 1 \rangle \).

Solution: The vector projection of \( \mathbf{a} \) onto \( \mathbf{b} \) is the vector

\[
\mathbf{p}_b(a) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\| \mathbf{b} \|} \right) \frac{\mathbf{b}}{\| \mathbf{b} \|} = \left( -\frac{2}{\sqrt{17}} \right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.
\]

We therefore obtain \( \mathbf{p}_a(b) = \langle \frac{8}{17}, -\frac{2}{17} \rangle \).
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.
Two main ways to introduce the cross product

Geometrical definition
Two main ways to introduce the cross product

Geometrical definition \rightarrow \text{Properties}
Two main ways to introduce the cross product

- Geometrical definition → Properties → Expression in components.

We choose the first way, like the textbook.
Two main ways to introduce the cross product

Geometrical definition → Properties → Expression in components.

Definition in components
Two main ways to introduce the cross product

- Geometrical definition → Properties → Expression in components.

- Definition in components → Properties
Two main ways to introduce the cross product

Geometrical definition → Properties → Expression in components.

Definition in components → Properties → Geometrical expression.
Two main ways to introduce the cross product

Geometrical definition → Properties → Expression in components.

Definition in components → Properties → Geometrical expression.

We choose the first way, like the textbook.
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
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- Triple product and volumes.
Geometric definition of cross product

Definition
The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between $\theta$, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin(\theta).$$
Geometric definition of cross product

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The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between $\theta$, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

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Geometric definition of cross product

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The \textit{cross product} of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between $\theta$, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin(\theta).$$

Remark: Cross product of two vectors is another vector; which is perpendicular to the original vectors.
Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$.
Geometric definition of cross product

Theorem
\[ |\mathbf{v} \times \mathbf{w}| \text{ is the area of the parallelogram formed by vectors } \mathbf{v} \text{ and } \mathbf{w}. \]

Proof.

The area \( A \) of the parallelogram formed by \( \mathbf{v} \) and \( \mathbf{w} \) is
\[
A = |\mathbf{w}| (|\mathbf{v}| \sin(\theta)) = |\mathbf{v} \times \mathbf{w}|.
\]
Geometric definition of cross product

**Theorem**

\[ |v \times w| \text{ is the area of the parallelogram formed by vectors } v \text{ and } w. \]

**Proof.**

The area \( A \) of the parallelogram formed by \( v \) and \( w \) is

\[ A = |w| \left( |v| \sin(\theta) \right) \]
**Geometric definition of cross product**

**Theorem**

$$|\mathbf{v} \times \mathbf{w}|$$ is the area of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$.

**Proof.**

The area $A$ of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}| (|\mathbf{v}| \sin(\theta)) = |\mathbf{v} \times \mathbf{w}|.$$
Geometric definition of cross product

**Theorem**

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The area $A$ of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}| (|\mathbf{v}| \sin(\theta)) = |\mathbf{v} \times \mathbf{w}|.$$

**Definition**

Two vectors are *parallel* iff the angle in between them is $\theta = 0$. 
Geometric definition of cross product

**Theorem**

\[ |\mathbf{v} \times \mathbf{w}| \text{ is the area of the parallelogram formed by vectors } \mathbf{v} \text{ and } \mathbf{w}. \]

**Proof.**

The area \( A \) of the parallelogram formed by \( \mathbf{v} \) and \( \mathbf{w} \) is

\[
A = |\mathbf{w}| (|\mathbf{v}| \sin(\theta)) = |\mathbf{v} \times \mathbf{w}|.
\]

**Definition**

Two vectors are *parallel* iff the angle in between them is \( \theta = 0 \).
Geometric definition of cross product

Theorem

\[ |\mathbf{v} \times \mathbf{w}| \text{ is the area of the parallelogram formed by vectors } \mathbf{v} \text{ and } \mathbf{w}. \]

Proof.

The area \( A \) of the parallelogram formed by \( \mathbf{v} \) and \( \mathbf{w} \) is

\[ A = |\mathbf{w}| (|\mathbf{v}| \sin(\theta)) = |\mathbf{v} \times \mathbf{w}|. \]

Definition

Two vectors are \textit{parallel} iff the angle in between them is \( \theta = 0 \).

Theorem

\textit{The non-zero vectors } \mathbf{v} \text{ and } \mathbf{w} \text{ are parallel iff } \mathbf{v} \times \mathbf{w} = \mathbf{0}. \]
**Geometric definition of cross product**

*Recall:* $|v \times w|$ is the area of a parallelogram.
Geometric definition of cross product

Recall: $|v \times w|$ is the area of a parallelogram.

Example
The closer the vectors $v, w$ are to be parallel, the smaller is the area of the parallelogram they form,
Geometric definition of cross product

Recall: \(|v \times w|\) is the area of a parallelogram.

Example
The closer the vectors \(v, w\) are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector \(v \times w\).
Geometric definition of cross product

**Recall:** $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

**Example**

The closer the vectors $\mathbf{v}, \mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$. 
Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example
The closer the vectors $\mathbf{v}, \mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.
Geometric definition of cross product

Example

Compute all cross products involving the vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k}. \)

\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}, \\
\mathbf{j} \times \mathbf{k} = \mathbf{i}, \\
\mathbf{k} \times \mathbf{i} = \mathbf{j}, \\
\mathbf{i} \times \mathbf{i} = 0, \\
\mathbf{j} \times \mathbf{j} = 0, \\
\mathbf{k} \times \mathbf{k} = 0, \\
\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \\
\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \\
\mathbf{k} \times \mathbf{j} = -\mathbf{i}.
\]
Geometric definition of cross product

Example
Compute all cross products involving the vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k}. \)

Solution: Recall: \( \mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle. \)
Geometric definition of cross product

**Example**
Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

**Solution:** Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

\[
\begin{align*}
\mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{i} &= \mathbf{j}, \\
\mathbf{i} \times \mathbf{i} &= \mathbf{0}, & \mathbf{j} \times \mathbf{j} &= \mathbf{0}, & \mathbf{k} \times \mathbf{k} &= \mathbf{0}, \\
\mathbf{i} \times \mathbf{k} &= -\mathbf{j}, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}.
\end{align*}
\]
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- **Properties of the cross product.**
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.
Properties of the cross product

Theorem

(a) \( \mathbf{v} \times \mathbf{w} = - (\mathbf{w} \times \mathbf{v}) \), \hspace{1cm} (skew-symmetric);
(b) \( \mathbf{v} \times \mathbf{v} = 0 \);
(c) \( (a \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a \mathbf{w}) = a(\mathbf{v} \times \mathbf{w}) \), \hspace{1cm} (linear);
(d) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \), \hspace{1cm} (linear);
(e) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \), \hspace{1cm} (not associative).
Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = - (\mathbf{w} \times \mathbf{v})$, (skew-symmetric);
(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;
(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a (\mathbf{v} \times \mathbf{w})$, (linear);
(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, (linear);
(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, (not associative).

Proof.

Part (a) results from the right-hand rule
Properties of the cross product

Theorem

(a) \( \mathbf{v} \times \mathbf{w} = - (\mathbf{w} \times \mathbf{v}) \), \hspace{1cm} (skew-symmetric);
(b) \( \mathbf{v} \times \mathbf{v} = 0 \);
(c) \( (a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w}) \), \hspace{1cm} (linear);
(d) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \), \hspace{1cm} (linear);
(e) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \), \hspace{1cm} (not associative).

Proof.
Part (a) results from the right-hand rule and (b) from part (a).
Properties of the cross product

Theorem

(a) \( \mathbf{v} \times \mathbf{w} = - (\mathbf{w} \times \mathbf{v}) \), \hspace{2cm} (skew-symmetric);
(b) \( \mathbf{v} \times \mathbf{v} = 0 \);
(c) \( (a \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a \mathbf{w}) = a (\mathbf{v} \times \mathbf{w}) \), \hspace{2cm} (linear);
(d) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \), \hspace{2cm} (linear);
(e) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \), \hspace{2cm} (not associative).

Proof.

Part (a) results from the right-hand rule and (b) from part (a).
Parts (b) and (c) are proven in a similar ways as the linear property of the dot product.
Properties of the cross product

Theorem

(a) \( \mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v}) \), \quad (\text{skew-symmetric});
(b) \( \mathbf{v} \times \mathbf{v} = 0 \);
(c) \( (a \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a \mathbf{w}) = a(\mathbf{v} \times \mathbf{w}) \), \quad (\text{linear});
(d) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \), \quad (\text{linear});
(e) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \), \quad (\text{not associative}).

Proof.

Part (a) results from the right-hand rule and (b) from part (a). Parts (b) and (c) are proven in a similar ways as the linear property of the dot product. Part (d) is proven by giving an example. \( \square \)
Properties of the cross product

Example

Show that the cross product is \textit{not associative}, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]
Properties of the cross product

Example

Show that the cross product is *not* associative, that is, \( u \times (v \times w) \neq (u \times v) \times w \).

Solution: We prove this statement giving an example.

Recall: The cross product of parallel vectors vanishes.
Properties of the cross product

Example
Show that the cross product is *not associative*, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]

Solution: We prove this statement giving an example. We now show that 
\[ i \times (i \times k) \neq (i \times i) \times k = 0. \]
Properties of the cross product

Example
Show that the cross product is \textit{not associative}, that is,
\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}. \]

Solution: We prove this statement giving an example. We now show that \( \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}. \) Indeed,

\[ \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \]
Properties of the cross product

Example
Show that the cross product is \textit{not associative}, that is,
$$u \times (v \times w) \neq (u \times v) \times w.$$  

Solution: We prove this statement giving an example. We now show that $i \times (i \times k) \neq (i \times i) \times k = 0$. Indeed,

$$i \times (i \times k) = i \times (-j)$$
Properties of the cross product

Example
Show that the cross product is not associative, that is, $u \times (v \times w) \neq (u \times v) \times w$.

Solution: We prove this statement giving an example. We now show that $i \times (i \times k) \neq (i \times i) \times k = 0$. Indeed,

\[ i \times (i \times k) = i \times (-j) = -(i \times j) \]
Properties of the cross product

Example
Show that the cross product is \textit{not associative}, that is, \(u \times (v \times w) \neq (u \times v) \times w\).

Solution: We prove this statement giving an example. We now show that \(i \times (i \times k) \neq (i \times i) \times k = 0\). Indeed,

\[
i \times (i \times k) = i \times (-j) = -(i \times j) = -k
\]
Properties of the cross product

Example

Show that the cross product is not associative, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]

Solution: We prove this statement giving an example. We now show that \( i \times (i \times k) \neq (i \times i) \times k = 0 \). Indeed,

\[ i \times (i \times k) = i \times (-j) = -(i \times j) = -k \quad \Rightarrow \quad i \times (i \times k) = -k, \]
Properties of the cross product

Example
Show that the cross product is *not associative*, that is, \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \).

Solution: We prove this statement giving an example. We now show that \( \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \). Indeed,

\[
\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = - (\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},
\]

\[
(\mathbf{i} \times \mathbf{i}) \times \mathbf{k}
\]

Recall: The cross product of parallel vectors vanishes.
Properties of the cross product

Example
Show that the cross product is *not associative*, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]

Solution: We prove this statement giving an example. We now show that \( i \times (i \times k) \neq (i \times i) \times k = 0 \). Indeed,

\[
\begin{align*}
i \times (i \times k) &= i \times (-j) = -(i \times j) = -k \\
&\Rightarrow \quad i \times (i \times k) = -k,
\end{align*}
\]

\[
(i \times i) \times k = 0 \times j
\]
Properties of the cross product

Example
Show that the cross product is not associative, that is, $u \times (v \times w) \neq (u \times v) \times w$.

Solution: We prove this statement giving an example. We now show that $i \times (i \times k) \neq (i \times i) \times k = 0$. Indeed,

$$i \times (i \times k) = i \times (-j) = -(i \times j) = -k \quad \Rightarrow \quad i \times (i \times k) = -k,$$

$$i \times i) \times k = 0 \times j = 0$$
Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]

Solution: We prove this statement giving an example. We now show that \( i \times (i \times k) \neq (i \times i) \times k = 0. \) Indeed,

\[ i \times (i \times k) = i \times (-j) = -(i \times j) = -k \quad \Rightarrow \quad i \times (i \times k) = -k, \]

\[ (i \times i) \times k = 0 \times j = 0 \quad \Rightarrow \quad (i \times i) \times k = 0. \]
Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
\[ u \times (v \times w) \neq (u \times v) \times w. \]

**Solution:** We prove this statement giving an example. We now show that \( i \times (i \times k) \neq (i \times i) \times k = 0 \). Indeed,

\[
\begin{align*}
  i \times (i \times k) &= i \times (-j) = -(i \times j) = -k \\
  \Rightarrow \quad i \times (i \times k) &= -k, \\
  (i \times i) \times k &= 0 \times j = 0 \\
  \Rightarrow \quad (i \times i) \times k &= 0.
\end{align*}
\]

We conclude that \( i \times (i \times k) \neq (i \times i) \times k = 0. \)  

\( \blacksquare \)
Properties of the cross product

Example
Show that the cross product is not associative, that is, \( u \times (v \times w) \neq (u \times v) \times w \).

Solution: We prove this statement giving an example. We now show that \( i \times (i \times k) \neq (i \times i) \times k = 0 \). Indeed,

\[
    i \times (i \times k) = i \times (-j) = -(i \times j) = -k \quad \Rightarrow \quad i \times (i \times k) = -k,
\]

\[
    (i \times i) \times k = 0 \times j = 0 \quad \Rightarrow \quad (i \times i) \times k = 0.
\]

We conclude that \( i \times (i \times k) \neq (i \times i) \times k = 0 \).

Recall: The cross product of parallel vectors vanishes.
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.
Cross product in vector components

Theorem

The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \)
is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]
Cross product in vector components

Theorem

The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]

Proof: Use the cross product properties
Cross product in vector components

Theorem
The cross product of vectors $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$ is given by

$$v \times w = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$ 

Proof: Use the cross product properties and recall the non-zero cross products $i \times j = k$, and $j \times k = i$, and $k \times i = j$. 
Cross product in vector components

Theorem
The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]

Proof: Use the cross product properties and recall the non-zero cross products \( \mathbf{i} \times \mathbf{j} = \mathbf{k}, \) and \( \mathbf{j} \times \mathbf{k} = \mathbf{i}, \) and \( \mathbf{k} \times \mathbf{i} = \mathbf{j}. \)
Express \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}, \)
Cross product in vector components

Theorem

The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]

Proof: Use the cross product properties and recall the non-zero cross products \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \), and \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \).

Express \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k} \), then

\[
\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).
\]
Cross product in vector components

Theorem
The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is given by

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$ 

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).$$

Use the linearity property.
Cross product in vector components

Theorem

The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]

Proof: Use the cross product properties and recall the non-zero cross products \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \), and \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \).

Express \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k} \), then

\[
\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).
\]

Use the linearity property. The only non-zero terms involve \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \), and \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \) and the symmetric analogues.
Cross product in vector components

Theorem
The cross product of vectors \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is given by

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.
\]

Proof: Use the cross product properties and recall the non-zero cross products \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \), and \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \).
Express \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k} \), then

\[
\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).
\]

Use the linearity property. The only non-zero terms involve \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \), and \( \mathbf{k} \times \mathbf{i} = \mathbf{j} \) and the symmetric analogues. The result is

\[
\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \mathbf{i} + (v_3 w_1 - v_1 w_3) \mathbf{j} + (v_1 w_2 - v_2 w_1) \mathbf{k}.
\]
Cross product in vector components.

Example
Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution:
We use the formula

$$\mathbf{v} \times \mathbf{w} = \langle (v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1) \rangle$$

$$\Rightarrow \mathbf{v} \times \mathbf{w} = \langle (2)(1) - (0)(2), (0)(3) - (1)(1), (1)(2) - (2)(3) \rangle$$

$$\Rightarrow \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$
Cross product in vector components.

Example
Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$.

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle$$

⇒ $\mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle$.

Exercise: Find the angle between $\mathbf{v}$ and $\mathbf{w}$ above, using both the cross and the dot products. Verify that you get the same answer.
Cross product in vector components.

Example
Find \( \mathbf{v} \times \mathbf{w} \) for \( \mathbf{v} = \langle 1, 2, 0 \rangle \) and \( \mathbf{w} = \langle 3, 2, 1 \rangle \),

Solution: We use the formula

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1) \rangle
\]

\[
\mathbf{v} \times \mathbf{w} = \langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \rangle
\]
Cross product in vector components.

Example
Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \langle (v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1) \rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle$$

Exercise: Find the angle between $\mathbf{v}$ and $\mathbf{w}$ above, using both the cross and the dot products. Verify that you get the same answer.
Cross product in vector components.

Example
Find \( \mathbf{v} \times \mathbf{w} \) for \( \mathbf{v} = \langle 1, 2, 0 \rangle \) and \( \mathbf{w} = \langle 3, 2, 1 \rangle \),

Solution: We use the formula

\[
\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle
\]

\[
\mathbf{v} \times \mathbf{w} = \langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \rangle
\]

\[
\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \Rightarrow \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.
\]
Cross product in vector components.

Example
Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2)(1) - (0)(2), (0)(3) - (1)(1), (1)(2) - (2)(3) \rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \Rightarrow \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$  

Exercise: Find the angle between $\mathbf{v}$ and $\mathbf{w}$ above, using both the cross and the dot products. Verify that you get the same answer.
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- **Determinants to compute cross products.**
- Triple product and volumes.
Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.
Determinants to compute cross products.

**Remark**: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

**Recall**:

(a) The determinant of a $2 \times 2$ matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$
Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a $2 \times 2$ matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$ 

(b) The determinant of a $3 \times 3$ matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$
Determinants to compute cross products.

**Remark:** Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

**Recall:**

(a) The determinant of a $2 \times 2$ matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) The determinant of a $3 \times 3$ matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

$2 \times 2$ determinants are used to find $3 \times 3$ determinants.
Determinants to compute cross products.

Theorem

*The formula to compute determinants of $3 \times 3$ matrices can be used to find the cross product $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, as follows*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3
\end{vmatrix}$$
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  i & j & k \\
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\end{vmatrix}
$$

**Proof:** Indeed, a straightforward computation shows that

$$
\begin{vmatrix}
  i & j & k \\
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$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left( v_2 w_3 - v_3 w_2 \right) i$$
Determinants to compute cross products.

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\begin{vmatrix}
    i & j & k \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
\end{vmatrix} = (v_2 w_3 - v_3 w_2) i - (v_1 w_3 - v_3 w_1) j
\]
Determinants to compute cross products.

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\end{vmatrix} = (v_2 w_3 - v_3 w_2)\mathbf{i} - (v_1 w_3 - v_3 w_1)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}.
\]
Determinants to compute cross products.

Example
Given the vectors \( \mathbf{v} = \langle 1, 2, 3 \rangle \) and \( \mathbf{w} = \langle -2, 3, 1 \rangle \), compute both \( \mathbf{w} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{w} \).
Determinants to compute cross products.

Example
Given the vectors \( \mathbf{v} = \langle 1, 2, 3 \rangle \) and \( \mathbf{w} = \langle -2, 3, 1 \rangle \), compute both \( \mathbf{w} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{w} \).

Solution: We need to compute the following determinant:

\[
\mathbf{w} \times \mathbf{v} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    w_1 & w_2 & w_3 \\
    v_1 & v_2 & v_3 
\end{vmatrix}
\]

The result is \( \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle \).

The properties of the determinant imply \( \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \).

Hence, \( \mathbf{v} \times \mathbf{w} = \langle -7, -7, 7 \rangle \).
Determinants to compute cross products.

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Given the vectors \( \mathbf{v} = \langle 1, 2, 3 \rangle \) and \( \mathbf{w} = \langle -2, 3, 1 \rangle \), compute both \( \mathbf{w} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{w} \).

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\end{vmatrix}
= \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  -2 & 3 & 1 \\
  1 & 2 & 3
\end{vmatrix}
\]

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Solution: We need to compute the following determinant:

\[
\mathbf{w} \times \mathbf{v} = \begin{vmatrix}
1 & 2 & 3 \\
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The result is

\( \mathbf{w} \times \mathbf{v} \)
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\mathbf{w} \times \mathbf{v} = \begin{vmatrix}
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w_1 & w_2 & w_3 \\
v_1 & v_2 & v_3 \\
\end{vmatrix} = \begin{vmatrix}
i & j & k \\
-2 & 3 & 1 \\
1 & 2 & 3 \\
\end{vmatrix}
\]

The result is

\[
\mathbf{w} \times \mathbf{v} = (9 - 2) \mathbf{i}
\]
Determinants to compute cross products.

Example
Given the vectors \( \mathbf{v} = \langle 1, 2, 3 \rangle \) and \( \mathbf{w} = \langle -2, 3, 1 \rangle \), compute both \( \mathbf{w} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{w} \).

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\mathbf{w} \times \mathbf{v} = \begin{vmatrix}
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The result is

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\mathbf{w} \times \mathbf{v} = (9 - 2)i - (-6 - 1)j
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Given the vectors \( \mathbf{v} = \langle 1, 2, 3 \rangle \) and \( \mathbf{w} = \langle -2, 3, 1 \rangle \), compute both \( \mathbf{w} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{w} \).

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-2 & 3 & 1 \\
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\]

The result is

\[
\mathbf{w} \times \mathbf{v} = (9 - 2) \mathbf{i} - (-6 - 1) \mathbf{j} + (-4 - 3) \mathbf{k}
\]

\[
\Rightarrow \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.
\]

The properties of the determinant imply

\[
\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}.
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The result is

\[
\mathbf{w} \times \mathbf{v} = (9-2)\mathbf{i} - (-6-1)\mathbf{j} + (-4-3)\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.
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Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9 - 2) \mathbf{i} - (-6 - 1) \mathbf{j} + (-4 - 3) \mathbf{k} \Rightarrow \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$  

The properties of the determinant imply $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. 
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The properties of the determinant imply \( \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \). Hence, \( \mathbf{v} \times \mathbf{w} = \langle -7, -7, 7 \rangle \).
Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- **Triple product and volumes.**
Definition
The *triple product* of the vectors $u, v, w$, is the scalar $u \cdot (v \times w)$. 

Remarks:
(a) The triple product of three vectors is a scalar.
(b) The parentheses are important. First do the cross product, and only then dot the resulting vector with the first vector.

Theorem (Cyclic rotation formula for triple product)
$$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u).$$
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Triple product and volumes

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Triple product and volumes

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Theorem (Cyclic rotation formula for triple product)

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$
Theorem

The number $|u \cdot (v \times w)|$ is the volume of the parallelepiped determined by the vectors $u, v, w$. 

Proof:

Recall the dot product:

$$x \cdot y = |x| |y| \cos(\theta).$$

Then,

$$|u \cdot (v \times w)| = |u||v \times w|\cos(\theta) = h |v \times w|.$$

$|v \times w|$ is the area $A$ of the parallelogram formed by $v$ and $w$.

So,

$$|u \cdot (v \times w)| = h A,$$

which is the volume of the parallelepiped formed by $u, v, w$. 


Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. 

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Recall the dot product: 

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$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}| |\mathbf{v} \times \mathbf{w}| |\cos(\theta)| = h |\mathbf{v} \times \mathbf{w}|$.

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$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h A$, which is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. 


Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos(\theta)$. 
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The number $|u \cdot (v \times w)|$ is the volume of the parallelepiped determined by the vectors $u$, $v$, $w$.

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Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped determined by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos(\theta)$. Then,

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$|\mathbf{v} \times \mathbf{w}|$ is the area $A$ of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h A,$$
Theorem

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which is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. □
The triple product and volumes

Example
Compute the volume of the parallelepiped formed by the vectors \( \mathbf{u} = \langle 1, 2, 3 \rangle \), \( \mathbf{v} = \langle 3, 2, 1 \rangle \), \( \mathbf{w} = \langle 1, -2, 1 \rangle \).
The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.
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Solution: We use the formula \( V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \). We must compute the cross product first:
The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $u = \langle 1, 2, 3 \rangle$, $v = \langle 3, 2, 1 \rangle$, $w = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |u \cdot (v \times w)|$. We must compute the cross product first:

$v \times w$


The triple product and volumes

Example
Compute the volume of the parallelepiped formed by the vectors \( u = \langle 1, 2, 3 \rangle, \ v = \langle 3, 2, 1 \rangle, \ w = \langle 1, -2, 1 \rangle. \)

Solution: We use the formula \( V = |u \cdot (v \times w)|. \) We must compute the cross product first:

\[
v \times w = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}
\]

that is, \( v \times w = \langle 4, -2, -8 \rangle. \) Now compute the dot product,

\[
u \cdot (v \times w) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle \]

that is, \( u \cdot (v \times w) = -24. \) We conclude that \( V = 24. \) \( \ll \)
The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2 + 2)\mathbf{i} - (3 - 1)\mathbf{j} + (-6 - 2)\mathbf{k},
\]

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,
\]

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. Therefore, we conclude that $V = 24$.\hfill \Box
The triple product and volumes

Example
Compute the volume of the parallelepiped formed by the vectors \( \mathbf{u} = \langle 1, 2, 3 \rangle, \mathbf{v} = \langle 3, 2, 1 \rangle, \mathbf{w} = \langle 1, -2, 1 \rangle. \)

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The triple product and volumes

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Compute the volume of the parallelepiped formed by the vectors \( u = \langle 1, 2, 3 \rangle \), \( v = \langle 3, 2, 1 \rangle \), \( w = \langle 1, -2, 1 \rangle \).

Solution: We use the formula \( V = |u \cdot (v \times w)| \). We must compute the cross product first:

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\end{vmatrix} = (2 + 2)i - (3 - 1)j + (-6 - 2)k,
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The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors \( \mathbf{u} = \langle 1, 2, 3 \rangle, \mathbf{v} = \langle 3, 2, 1 \rangle, \mathbf{w} = \langle 1, -2, 1 \rangle. \)

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that is, \( \mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle. \) Now compute the dot product,

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\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle
\]
The triple product and volumes

Example
Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2 + 2) \mathbf{i} - (3 - 1) \mathbf{j} + (-6 - 2) \mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$
The triple product and volumes

Example
Compute the volume of the parallelepiped formed by the vectors $u = \langle 1, 2, 3 \rangle$, $v = \langle 3, 2, 1 \rangle$, $w = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |u \cdot (v \times w)|$. We must compute the cross product first:

$$v \times w = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2 + 2) i - (3 - 1) j + (-6 - 2) k,$$

that is, $v \times w = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$u \cdot (v \times w) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

that is, $u \cdot (v \times w) = -24$. 

The triple product and volumes

Example

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that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. We conclude that $V = 24$. △
The triple product and volumes

**Remark:** The triple product can be computed with a determinant.

**Theorem**

If \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \), \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \), and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \), then

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \left| \begin{array}{ccc}
u_1 & \nu_2 & \nu_3 \\
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3 \end{array} \right|.
\]

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\]

that is,

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24 = -24.
\]

Hence \( V = 24 \).
The triple product and volumes

Remark: The triple product can be computed with a determinant.

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Solution:
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Solution:
\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
  1 & 2 & 3 \\
  3 & 2 & 1 \\
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\]

Hence, \( V = 24 \). \( \Box \)
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that is, \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24 = -24 \). Hence \( V = 24 \). \( \triangleq \)